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FIXED POINT RESULTS FOR MIXED POWER-TYPE CONTRACTIONS ON COMPLETE METRIC SPACES

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Abstract. We establish two new fixed point theorems for operators on complete metric spaces with bounded diameters. First, we prove a general result for mappings satisfying a mixed power-type condition. Second, we present a significant generalization of Kannan-type contractions through a novel nonlinear condition involving both point distances and their images. These results unify and extend classical fixed point principles, including Banach contractions, and Kannan's theorem, while operating under natural diameter constraints. The theorems' broad applicability is demonstrated through connections to existing literature and potential applications in nonlinear analysis.

1. Introduction

Fixed point theory is one of the most powerful tools in nonlinear analysis, providing key methods for solving equations and analyzing iterative processes. The seminal result in this field is the Banach contraction principle [5], which asserts that any contraction mapping on a complete metric space has a unique fixed point. This theorem has been generalized extensively, with the aim of weakening the contraction condition while preserving the fixed point property.

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Kannan [21] introduced a condition that does not require continuity of the mapping, while Chatterjea [15] proposed a different type of contraction involving cross terms.

In recent years, several authors have studied more generalized contractive mappings, including those involving altering distances, control functions, and mixed powers of the distance function. Berinde [13] and others explored such settings to accommodate more complex behaviors in applications.

Recent work has focused on interpolative conditions; see for example [22]. Moreover, advances in fixed-point theory have expanded beyond classical metric spaces to more generalized frameworks. Significant work has been done in b-metric spaces [3, 12, 33, 35], neutrosophic fuzzy metric spaces [7, 8, 18, 34], and other generalized structures like MR-metric spaces have been extended to neutrosophic settings [27], applied to graph theory [31], measure theory [28], and deep learning [32]. Researchers have developed various contraction types including Ω-distance mappings [1, 2], and Proinov-type contractions [23], while investigating their fixed-point properties, Bataihah [6] established some fixed point results with applications to fractional differential equations via a new type of distance spaces. Fixed point methods may also be applied to discrete memristor models such as those studied in [36]. These developments complement classical results and provide tools for applications in nonlinear analysis, as demonstrated by the quasi-metric characterizations [24] and tripled coincidence point theorems [4] in ordered spaces. Fixed point results in neutrosophic metric spaces have been established using Geraghty functions [11] and simulation functions [10].

In this paper, we propose a new type of contractive condition that involves a combination of two different powers of the metric. We establish that under this condition, T admits a unique fixed point in X. This mixed-power contraction generalizes classical results and introduces a broader framework for analyzing nonlinear mappings.

2. Main result

In this section, we introduce two types of contractions and establish conditions under which their existence and uniqueness are guaranteed.

Definition 2.1. Let (X, d) be a metric space and $T : X \to X$ be a map. We say that T is mixed power-type contraction if it satisfies the following condition for some constants $0 < a + c < 1$ and $0 < \lambda < 1$:

$$d(Tx, Ty) < cd(x, y)^{\lambda} + ad(x, y)^{1-\lambda}, \quad \forall\, x, y \in X,\, x \neq y. \tag{2.1}$$

Theorem 2.2. *Let (X, d) be a complete metric space with diam$X = M \leq 1$, and let $T : X \to X$ be a mixed power-type contraction. Then T has a unique fixed point in X.*

Proof. Choose an arbitrary initial point $x_0 \in X$ and define the sequence $\{x_n\}$ by

$$x_{n+1} = Tx_n, \quad n \geq 0.$$

If there is some $s \in \mathbb{N}$ such that $x_s = x_{s+1}$, then it is a fixed point for T. So, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Applying the Condition (2.1) to x_n and x_{n-1}, we get

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) < cd(x_n, x_{n-1})^{\lambda} + ad(x_n, x_{n-1})^{1-\lambda}.$$

Let $\delta_n = d(x_n, x_{n-1})$. Then

$$\delta_{n+1} < c\delta_n^{\lambda} + a\delta_n^{1-\lambda}.$$

Set $\sigma = \min\{\lambda, 1-\lambda\}$, and $\kappa = c + a$. Then

$$\delta_{n+1} < \kappa\delta_n^{\sigma}.$$

By the same way, we get

$$\delta_n < \kappa\delta_{n-1}^{\sigma}.$$

So,

$$\begin{aligned}
\delta_{n+1} &< \kappa(\kappa\delta_{n-1}^{\sigma})^{\sigma} \\
&= \kappa^{\sigma+1}\delta_{n-1}^{\sigma^2} \\
&< \kappa^{2\sigma}\delta_{n-1}^{\sigma^2}.
\end{aligned}$$

By induction, we get that

$$\delta_{n+1} < \kappa^{n\sigma}\delta_0^{\sigma^n}.$$

For $m > n \geq 0$, we get

$$\begin{aligned}
d(x_n, x_m) &\leq \sum_{j=n}^{m-1} d(x_j, x_{j+1}) \\
&< \sum_{j=n}^{m-1} \kappa^{j\sigma}\delta_0^{\sigma^j} \\
&\leq \sum_{j=n}^{m-1} (\kappa^{\sigma})^j \\
&\leq (\kappa^{\sigma})^n \sum_{j=0}^{\infty} (\kappa^{\sigma})^j.
\end{aligned}$$

Since $\kappa^\sigma < 1$, $d(x_n, x_m) \to 0$, meaning $\{x_n\}$ is a Cauchy sequence. Since X is complete, $x_n \to x^*$ for some $x^* \in X$.

From the triangle inequality and Condition (2.1), we have

$$\begin{aligned} d(Tx^*, x^*) &\leq d(Tx^*, Tx_n) + d(Tx_n, x^*) \\ &< cd(x^*, x_n)^\lambda + ad(x^*, x_n)^{1-\lambda} + d(x_{n+1}, x^*). \end{aligned}$$

Taking $n \to \infty$, all terms vanish, so $d(T(x^*), x^*) = 0$, meaning $T(x^*) = x^*$.

Suppose x^* and y^* are two fixed points. If $x^* \neq y^*$, then

$$\begin{aligned} d(x^*, y^*) &= d(Tx^*, Ty^*) \\ &< cd(x^*, y^*)^\lambda + ad(x^*, y^*)^{1-\lambda} \\ &\leq \kappa d(x^*, y^*)^\sigma. \end{aligned}$$

Since $ft = t^\sigma$ increasing on $[0, \infty)$, then we get

$$\begin{aligned} d(x^*, y^*) &\leq \kappa d(x^*, y^*)^\sigma \\ &= \kappa d(Tx^*, Ty^*)^\sigma \\ &< \kappa(cd(x^*, y^*)^\lambda + ad(x^*, y^*)^{1-\lambda})^\sigma \\ &\leq \kappa(\kappa d(x^*, y^*)^\sigma)^\sigma \\ &= \kappa^{1+\sigma} d(x^*, y^*)^{\sigma^2} \\ &\leq \kappa^{2\sigma} d(x^*, y^*)^{\sigma^2}. \end{aligned}$$

Therefore, for each $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x^*, y^*) &< \kappa^{n\sigma} d(x^*, y^*)^{\sigma^n} \\ &\leq \kappa^{n\sigma}. \end{aligned}$$

Taking the limit as $n \to \infty$, we obtain $d(x^*, y^*) = 0$, and hence $x^* = y^*$. □

Remark 2.3. In Theorem 2.2 if the inequality in condition (2.1) is non-strict, that is,

$$d(Tx, Ty) \leq c\, d(x, y)^\lambda + a\, d(x, y)^{1-\lambda},$$

uniqueness of the fixed point may fail, and multiple fixed points can exist, as seen in Example 2.4.

Example 2.4. Let $X = [0, 1]$ with the standard metric $d(x, y) = |x - y|$. X is complete, and $\operatorname{diam} X = 1 \leq 1$. Define $T : X \to X$ by:

$$Tx = \frac{x^\lambda}{2}, \quad \text{where } 0 < \lambda < 1.$$

For $c = \frac{1}{2},\ a = 0$, we have for all $x, y \in X$,

$$|Tx - Ty| = \left| \frac{x^\lambda}{2} - \frac{y^\lambda}{2} \right| = \frac{1}{2}|x^\lambda - y^\lambda|.$$

Since $\lambda \in (0,1)$, the function $f(t) = t^\lambda$ is subadditive (because it is concave), meaning

$$|x^\lambda - y^\lambda| \leq |x-y|^\lambda.$$

Thus,

$$|Tx - Ty| \leq \frac{1}{2}|x-y|^\lambda.$$

So

$$|Tx - Ty| \leq c|x-y|^\lambda + a|x-y|^{1-\lambda}.$$

This matches the theorem's condition with

$$c = \frac{1}{2}, \quad a = 0 \quad \text{and} \quad a + c = \frac{1}{2} < 1.$$

Solving $Tx = x$, gives

$$\frac{x^\lambda}{2} = x \quad \Longrightarrow \quad x^{1-\lambda} = \frac{1}{2} \quad \Longrightarrow \quad x = \left(\frac{1}{2}\right)^{\frac{1}{1-\lambda}}.$$

Moreover, 0 is another fixed point in $[0,1]$.

Note that the contraction condition holds with equality whenever x or y equal 0.

Definition 2.5. Let (X,d) be a metric space and $T : X \to X$ be a map. We say that T is Generalized Mixed Kannan-Type Contraction if it satisfies the following condition for some constants $0 < a + c < 1$ with $c + 2a < 1$ and $0 < \lambda < 1$,

$$d(Tx,Ty) < cd(x,y)^\lambda + a\left(d(x,Tx) + d(y,Ty)\right)^{1-\lambda}, \quad \forall\, x,y \in X,\ x \neq y. \quad (2.2)$$

Theorem 2.6. *Let (X,d) be a complete metric space with* $\operatorname{diam}(X) \leq 1$. *If $T : X \to X$ is a Generalized Mixed Kannan-Type Contraction. Then T has a unique fixed point.*

Proof. Choose an arbitrary initial point $x_0 \in X$ and define the sequence $\{x_n\}$ by

$$x_{n+1} = T(x_n), \quad n \geq 0.$$

If there is some $s \in \mathbb{N}$ such that $x_s = x_{s+1}$, then it is a fixed point for T. So, we suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Applying the Condition (2.2) to x_n and x_{n-1}, we get

$$\begin{aligned} d(x_{n+1},x_n) &= d(Tx_n, Tx_{n-1}) \\ &< cd(x_n,x_{n-1})^\lambda + a(d(x_n,x_{n-1}) + d(x_{n+1},x_n))^{1-\lambda}. \end{aligned}$$

Since $\lambda \leq 1$ by concavity of $ft = t^{\lambda}$, we get

$$d(x_{n+1}, x_n) < cd(x_n, x_{n-1})^{\lambda} + a(d(x_n, x_{n-1})^{1-\lambda} + d(x_{n+1}, x_n)^{1-\lambda}).$$

Let $\delta_n = d(x_n, x_{n-1})$. Then

$$\delta_{n+1} < \frac{c}{1-a}\delta_n^{\lambda} + \frac{a}{1-a}\delta_n^{1-\lambda}.$$

Set $\sigma = \min\{\lambda, 1-\lambda\}$, and $\flat = \frac{c}{1-a} + \frac{a}{1-a}$. Then,

$$\delta_{n+1} < \flat\delta_n^{\sigma}.$$

By the same way, we get

$$\delta_n < \flat\delta_{n-1}^{\sigma}.$$

So

$$\begin{aligned}\delta_{n+1} &< \flat(\flat\delta_{n-1}^{\sigma})^{\sigma} \\ &= \flat^{\sigma+1}\delta_{n-1}^{\sigma^2} \\ &\leq \flat^{2\sigma}\delta_{n-1}^{\sigma^2}.\end{aligned}$$

By induction, we get that

$$\delta_{n+1} < \flat^{n\sigma}\delta_0^{\sigma^n}.$$

For $m > n \geq 0$, we get

$$\begin{aligned}d(x_n, x_m) &\leq \sum_{j=n}^{m-1} d(x_j, x_{j+1}) \\ &< \sum_{j=n}^{m-1} \flat^{j\sigma}\delta_0^{\sigma^j} \\ &\leq \sum_{j=n}^{m-1} (\flat^{\sigma})^j \\ &\leq (\flat^{\sigma})^n \sum_{j=0}^{\infty} (\flat^{\sigma})^j.\end{aligned}$$

Since $\flat^{\sigma} < 1$, $d(x_n, x_m) \to 0$, meaning $\{x_n\}$ is a Cauchy sequence. Since X is complete, $x_n \to x^*$ for some $x^* \in X$.

From Condition (2.2), we have

$$\begin{aligned}d(Tx^*, x^*) &\leq d(Tx^*, Tx_n) + d(Tx_n, x^*) \\ &< cd(x^*, x_n)^{\lambda} + a(d(x^*, Tx^*) + d(x_n, x_{n+1}))^{1-\lambda} + d(x_{n+1}, x^*).\end{aligned}$$

Taking $n \to \infty$, we get

$$d(Tx^*, x^*) \leq a\ d(Tx^*, x^*)^{1-\lambda}.$$

Again,

$$\begin{aligned} d(Tx^*, x^*) &\leq a\ d(Tx^*, x^*)^{1-\lambda} \\ &\leq a(d(Tx^*, Tx_n) + d(Tx_n, x^*))^{1-\lambda} \\ &< a(cd(x^*, x_n)^{\lambda} + a(d(x^*, Tx^*) + d(x_n, x_{n+1}))^{1-\lambda} \\ &\quad + d(x_{n+1}, x^*))^{1-\lambda}. \end{aligned}$$

Letting $n \to \infty$,

$$\begin{aligned} d(Tx^*, x^*) &\leq a^{1+(1-\lambda)}\ d(Tx^*, x^*)^{(1-\lambda)^2} \\ &\leq a^{2(1-\lambda)}\ d(Tx^*, x^*)^{(1-\lambda)^2}. \end{aligned}$$

By induction, we can show that

$$d(Tx^*, x^*) \leq a^{n(1-\lambda)}\ d(Tx^*, x^*)^{(1-\lambda)^n}.$$

As $n \to \infty$, we get $d(Tx^*, x^*) = 0$, meaning $T(x^*) = x^*$.

Suppose x^* and y^* are two fixed points. If $x^* \neq y^*$, then

$$\begin{aligned} d(x^*, y^*) &= d(Tx^*, Ty^*) \\ &< cd(x^*, y^*)^{\lambda} + a(d(x^*, Tx^*) + d(y^*, Ty^*))^{1-\lambda} \\ &= cd(x^*, y^*)^{\lambda}. \end{aligned}$$

Also, by induction, we show that

$$d(x^*, y^*) < c^{n\lambda} d(x^*, y^*)^{\lambda^n}.$$

Taking the limit as $n \to \infty$, gives $d(x^*, y^*) = 0$, and hence $x^* = y^*$. □

3. Examples and consequences

To demonstrate the applicability of our results, we present several examples that highlight the key features of mixed power-type contractions.

Example 3.1. Let $X = [0, 1]$ with the standard metric $d(x, y) = |x - y|$, and define $T : X \to X$ by

$$Tx = \frac{x^{\lambda}}{4} + \frac{x^{1-\lambda}}{6}, \quad \lambda \in (0, 1).$$

For $x, y \in X$, $x \neq y$, we have

$$|Tx - Ty| \leq \frac{1}{4}|x^{\lambda} - y^{\lambda}| + \frac{1}{6}|x^{1-\lambda} - y^{1-\lambda}|.$$

Since $t \mapsto t^{\lambda}$ and $t \mapsto t^{1-\lambda}$ are concave,

$$|Tx - Ty| < \frac{1}{4}|x-y|^{\lambda} + \frac{1}{5}|x-y|^{1-\lambda}.$$

Here, $c = \frac{1}{4}$, $a = \frac{1}{5}$ and $a + c = \frac{9}{20} < 1$. Hence, all conditions of Theorem 2.2 are satisfied.

The unique fixed point is $x^* = 0$, since $T(0) = 0$, and for $x > 0$, $Tx < x$.

Example 3.2. Let $X = \{f \in C_+[0,1] : \|f\| \le \frac{1}{2}\}$, where $C_+[0,1]$ is the set of all continuous functions $f : [0,1] \to [0,\infty)$ with the sup-norm $\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$. Define $T : X \to X$ by

$$(Tf)(x) = \frac{f(x)^{\lambda}}{3} + \frac{1}{4}\int_0^1 f(t)\,dt, \quad \lambda \in (0,1).$$

For $f, g \in X$, $f \ne g$,

$$\|Tf - Tg\|_\infty \le \frac{1}{3}\|f^{\lambda} - g^{\lambda}\|_\infty + \frac{1}{4}\left|\int_0^1 (f(t) - g(t))\,dt\right|.$$

Since $|f^{\lambda} - g^{\lambda}| \le \|f-g\|_\infty^{\lambda}$ and $\left|\int_0^1 (f-g)\,dt\right| \le \|f-g\|_\infty$,

$$\|Tf - Tg\|_\infty < \frac{1}{3}\|f-g\|_\infty^{\lambda} + \frac{1}{3}\|f-g\|_\infty^{1-\lambda}.$$

Here, $c = \frac{1}{3}$, $a = \frac{1}{3}$ and $a + c = \frac{2}{3} < 1$. Hence, all conditions of Theorem 2.2 are satisfied. The zero function $f^* = 0$ is the unique fixed point.

Example 3.3. Let $X = \{0\} \cup \left\{\frac{1}{2^n} : n \ge 0\right\}$ with metric $d(x,y) = |x-y|$. Define the mapping $T : X \to X$ by

$$T(0) = 0, \quad T\left(\frac{1}{2^n}\right) = \frac{1}{2^{n+1}}.$$

Take $\lambda = \frac{1}{2}$, $c = \frac{1}{3}$ and $a = \frac{1}{4}$. For $x = \frac{1}{2^n}$, $y = \frac{1}{2^m}$ with $n < m$, the left-hand side is

$$d(Tx, Ty) = \frac{1}{2^{n+1}} - \frac{1}{2^{m+1}} < \frac{1}{2^{n+1}}.$$

The right-hand side of the generalized Kannan-type condition is

$$c\,(d(x,y))^{\lambda} + a\,(d(x,Tx) + d(y,Ty))^{1-\lambda}.$$

Calculate each term

$$d(x,y) = \frac{1}{2^n} - \frac{1}{2^m} \ge \frac{1}{2^{n+1}},$$

so

$$c\,(d(x,y))^{\lambda} \ge \frac{1}{3}\left(\frac{1}{2^{n+1}}\right)^{\frac{1}{2}}.$$

Also,

$$d(x,Tx)+d(y,Ty)=\left(\frac{1}{2^n}-\frac{1}{2^{n+1}}\right)+\left(\frac{1}{2^m}-\frac{1}{2^{m+1}}\right)>\frac{1}{2^{n+1}},$$

so

$$a\left(d(x,Tx)+d(y,Ty)\right)^{1-\lambda}>\frac{1}{4}\left(\frac{1}{2^{n+1}}\right)^{\frac{1}{2}}.$$

Hence,

$$\frac{1}{3}\left(d(x,y)\right)^{\frac{1}{2}}+\frac{1}{4}\left(d(x,Tx)+d(y,Ty)\right)^{\frac{1}{2}}>\frac{7}{12}\cdot\frac{1}{2^{\frac{n+1}{2}}}.$$

Since $\frac{1}{2^{n+1}}<\frac{7}{12}\cdot\frac{1}{2^{\frac{n+1}{2}}}$ for all $n\geq 1$, the inequality

$$d(Tx,Ty)<\frac{1}{3}\left(d(x,y)\right)^{\frac{1}{2}}+\frac{1}{4}\left(d(x,Tx)+d(y,Ty)\right)^{\frac{1}{2}}.$$

Under that same parameters, it is easy to show that the above inequality is true when $x=\frac{1}{2^n}$, $y=0$.

Note that the standard Kannan condition fails, since for $x=\frac{1}{2^n}$, $y=0$,

$$d(Tx,Ty)=\frac{1}{2^{n+1}},$$

but

$$d(x,Tx)+d(y,Ty)=\left(\frac{1}{2^n}-\frac{1}{2^{n+1}}\right)+0=\frac{1}{2^{n+1}}.$$

Thus, the classical Kannan condition

$$d(Tx,Ty)\leq\alpha\left[d(x,Tx)+d(y,Ty)\right]$$

would require

$$\frac{1}{2^{n+1}}\leq\alpha\frac{1}{2^{n+1}}\quad\Longrightarrow\quad\alpha\geq 1,$$

which contradicts the requirement $\alpha<\frac{1}{2}$ in the classical Kannan theorem.

In Corollaries 3.4, 3.5, we consider $\phi:[0,1]^2\to\mathbb{R}^+$ is continuous with

$$\phi(s,t)\leq cs+at.$$

Corollary 3.4. *Let (X,d) be a complete metric space with* $\operatorname{diam}X=M\leq 1$ *and let $T:X\to X$ satisfy*

$$d(Tx,Ty)<\phi\left(d(x,y)^{\lambda},d(x,y)^{1-\lambda}\right),\quad\forall\, x,y\in X,\ x\neq y,$$

where $a+c<1$. Then T has a unique fixed point.

Corollary 3.5. *Let (X,d) be a complete metric space with* $\mathrm{diam}X = M \leq 1$ *and let $T : X \to X$ satisfy*

$$d(Tx,Ty) < \phi\left(d(x,y)^{\lambda}, (d(x,Tx)+d(y,Ty))^{1-\lambda}\right), \quad \forall\, x,y \in X,\, x \neq y,$$

where $2a+c<1$. Then T has a unique fixed point.

Corollary 3.6. *Let (X,d) be a complete metric space with* $\mathrm{diam}(X) \leq 1$*, and $T : X \to X$ satisfy*

$$d(Tx,Ty) < k\, d(x,y)^{\alpha}, \quad \forall\, x,y \in X,\, x \neq y$$

for some $k \in (0,1)$ and $\alpha \in (0,1)$. Then T has a unique fixed point.

Proof. Take $a=0$, $c=k$ and $\lambda = \alpha$ in Theorem 2.2. The condition becomes

$$d(Tx,Ty) < k\, d(x,y)^{\alpha},$$

which is a power-type contraction. The theorem guarantees a unique fixed point. □

Corollary 3.7. *Let (X,d) be a complete metric space with* $\mathrm{diam}(X) \leq 1$*. If $T : X \to X$ satisfies, for some $\lambda \in (0,1)$ and $0 < a < \frac{1}{2}$,*

$$d(Tx,Ty) < a\,(d(x,Tx)+d(y,Ty))^{1-\lambda}, \quad \forall\, x,y \in X,\, x \neq y,$$

then T has a unique fixed point.

Proof. Apply Theorem 2.6 with $c=0$. The condition $c+2a<1$ reduces to $a < \frac{1}{2}$, and the diameter constraint ensures the theorem's applicability. □

Remark 3.8. It is worth noting that when $\mathrm{diam}(X) < 1$, the Banach contraction condition implies a power-type contraction, as established in Corollary 3.6, since for any $x,y \in X$, we have $d(x,y) < d(x,y)^{\alpha}$. Furthermore, Corollary 3.7 shows that the Kannan condition leads to a generalized mixed Kannan-type contraction under the same diameter constraint.

4. Application to integral equations

Consider the set $X = \{x \in C[0,1] : \ \|x\|_{\infty} \leq \frac{1}{2}\}$. We equip X with the sup norm.

$$d(x,y) = \sup_{t\in[0,1]} |x(t)-y(t)|.$$

This makes (X,d) a complete metric space with $\mathrm{diam}(C[0,1]) \leq 1$. Consider the following nonlinear integral equation

$$x(t) = \int_0^1 K(t,s,x(s))\, ds, \quad t \in [0,1],$$

where $K : [0,1] \times [0,1] \times [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{R}$ is a given kernel function.

Theorem 4.1. *Suppose the kernel K satisfies the following conditions for some $0 < a + c < 1$ and $0 < \lambda < 1$,*

(1) $K : [0,1] \times [0,1] \times [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{R}$ *is jointly continuous and bounded by* $M \leq \frac{1}{2}$,

(2) $|K(t,s,u) - K(t,s,v)| \leq c|u-v|^{\lambda} + a|u-v|^{1-\lambda}, \quad \forall\, t,s \in [0,1], \forall\, u,v \in [-\frac{1}{2}, \frac{1}{2}]$.

Then the integral operator $T : X \to X$ defined by

$$(Tx)(t) = \int_0^1 K(t,s,x(s))\,ds$$

has a unique fixed point $x^* \in X$, which is the unique solution to the integral equation.

Proof. We verify that T satisfies the condition of Theorem 2.2. For any $x, y \in X$, $x \neq y$, we have

$$\begin{aligned}
d(Tx,Ty) &= \sup_{t\in[0,1]} \left| \int_0^1 (K(t,s,x(s)) - K(t,s,y(s)))\,ds \right| \\
&\leq \sup_{t\in[0,1]} \int_0^1 |K(t,s,x(s)) - K(t,s,y(s))|\,ds \\
&\leq \int_0^1 \left(c|x(s)-y(s)|^{\lambda} + a|x(s)-y(s)|^{1-\lambda} \right) ds \\
&\leq c \left(\sup_{s\in[0,1]} |x(s)-y(s)| \right)^{\lambda} + a \left(\sup_{s\in[0,1]} |x(s)-y(s)| \right)^{1-\lambda} \\
&= c\, d(x,y)^{\lambda} + a\, d(x,y)^{1-\lambda}.
\end{aligned}$$

Thus, T satisfies

$$d(Tx,Ty) \leq c\, d(x,y)^{\lambda} + a\, d(x,y)^{1-\lambda}.$$

By Theorem 2.2, T has a unique fixed point x^*, which solves the integral equation. □

5. Conclusion

In this paper, we introduced and studied two new classes of contractive mappings in complete metric spaces of diameter at most one: the *mixed power-type contraction* and the *generalized mixed Kannan-type contraction*. We established fixed point theorems ensuring the existence and uniqueness of fixed points under these conditions. Our main results extend and generalize classical fixed point theorems by incorporating mixed-type terms and power-type expressions, controlled by parameters a, c, and λ under suitable constraints.

We demonstrated that strict inequalities in the contraction conditions are essential for guaranteeing uniqueness, as highlighted through carefully constructed counterexamples. Furthermore, we provided illustrative examples in both scalar and functional settings to show the broad applicability of our results. Special cases of our main theorems were also explored in the form of corollaries, which recover known results such as power-type and Kannan-type contractions. To validate the practical utility of our results, we presented an application to a nonlinear integral equation.

Future work may include extending these results to more general spaces such as partial metric spaces, fuzzy metric spaces, or spaces endowed with $\mathcal{T}$-distance, also exploring results on fixed point theory in MR-metric spaces as developed in [25, 26, 29, 30].

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