Nonlinear Functional Analysis and Applications
Vol. 19, No. 1 (2014), pp. 127-130

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A CRITICAL REMARK ON SUBSEQUENTIAL CONTINUITY

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Abstract. Recent results due to various authors (Chauhan *et al.*; Journal of Applied Mathematics (2013), Beg and Chauhan; Novi Sad J. Math. (2013), Pant *et al.*; Advances in Fuzzy Systems (2013), Chauhan *et al.*; Vietnam Journal of Mathematics (2013), Rouzkard *et al.*; Bull. Belg. Math. Soc. Simon Stevin (2012), Imdad *et al.*; Appl. Math. Lett. (2011), Gopal and Imdad; Ann Univ Ferrara (2011) etc.) have been proved under the presumption that subsequential continuity is the generalization of continuity or reciprocal continuity. In this short note we communicate some important remarks about the concept of subsequential continuity and show that subsequential continuity is independent of the notion of continuity or reciprocal continuity.

1. Introduction and preliminaries

In 2009, Bouhadjera and Godet-Thobie [5] coined the term of subsequential continuity and claimed it as a generaliztion of reciprocal continuity or continuity for a pair of given mappings. Following Bouhadjera and Godet-Thobie [5], various authors, e.g., Pant *et al.* ([1], p.3), Singh *et al.* ([2], p.1304), Imdad and Gopal ([3] p.308), Fayyaz Rouzkard *et al.* ([4], p.314), Beg and Chauhan ([6], p.135), Khan and Chauhan ([7], p.71), Imdad *et al.* ([8], p. 1166), Sumitra *et al.* ([11], p.335), Manro *et al.* ([12], p.2703), Chauhan and

[0]Received September 2, 2013. Revised September 15, 2013.
[0]2010 Mathematics Subject Classification: 54H25, 47H10.
[0]Keywords: Common fixed point, subsequentially continuous mappings, metric space.

Kumar ([13], p.228), Chauhan *et al.* ([14], p.3) and Chauhan and Kim ([15], p.181) have also asserted that if f and g are both continuous or if f and g are reciprocally continuous then they are subsequentially continuous. This is, however, not true. The two concepts are independent of each other.

2. Main results

First, we recall some relevant definitions.

Definition 2.1. A pair of self-maps (f, g) of a metric space (X, d) is said to be

(i) reciprocally continuous [9] if $\lim_n fgx_n = ft$ and $\lim_n gfx_n = gt$, whenever $\{x_n\}$ is a sequence in X such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X.

(ii) subsequentially continuous [5] if there exists a sequence $\{x_n\}$ in X such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X and satisfies $\lim_n fgx_n = ft$ and $\lim_n gfx_n = gt$.

(iii) conditional reciprocal continuous (CRC) [5] iff whenever the set of sequences $\{x_n\}$ satisfying $\lim_n fx_n = \lim_n gx_n$ is nonempty, there exists a sequence $\{y_n\}$ satisfying $\lim_n fy_n = \lim_n gy_n = t(say)$ for some t in X such that $\lim_n fgy_n = ft$ and $\lim_n gfy_n = gt$.

If f and g are both continuous then they are obviously reciprocally continuous but the converse is not true (see Example 2.2 below). Also, if f and g are reciprocally continuous then they are obviously conditionally reciprocally continuous but as shown in Example 2.4 below, the converse is not true (see also [10]).

Example 2.2. Let $X = [0, 1]$ and d be the usual metric on X. Define $f, g : X \to X$ by $fx = [x]$, the greatest integer function and $gx = sgn(x)$, the signum function. Then f and g are reciprocally continuous mappings but not continuous mappings.

The two concepts recirpocal continuity and subsequential continuity are independent as is obvious from Examples 2.3 and 2.4 given below.

Example 2.3. Let $X = [0, \infty)$ and d be the usual metric on X. Define $f, g : X \to X$ by $fx = x$ for all x and $gx = x + a$ for all x and $a > 0$. Then f and g are continuous or reciprocally continuous mappings but not subsequentially continuous mappings.

Example 2.4. Let $X = [2, 20]$ and d be the usual metric on X. Define $f, g : X \to X$ as follows

$$\begin{cases} fx = 2\,, & \text{if } x = 2 \text{ or } x > 5, \\ fx = 6\,, & \text{if } 2 < x \leq 5, \end{cases}$$

$$\begin{cases} g2 = 2\,, & \\ gx = 12\,, & \text{if } 2 < x \leq 5, \\ gx = \frac{x+1}{3}\,, & \text{if } x > 5. \end{cases}$$

Then f and g are subsequentially continuous mappings. But f and g are neither continuous nor reciprocally continuous. To see this let us consider the constant sequence $\{x_n = 2\}$ then $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 2, \lim_{n\to\infty} fgx_n = 2 = f2$ and $\lim_{n\to\infty} gfx_n = 2 = g2$. If we consider the sequence $y_n = 5 + \frac{1}{n}$: $n \geq 0$, then $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 2$, $\lim_{n\to\infty} fgx_n = 6 \neq f2$ and $\lim_{n\to\infty} gfx_n = 2 = g2$. Thus f and g are neither continuous nor reciprocally continuous.

Examples 2.2, 2.3 and 2.4 clearly show that reciprocal continuity and subsequential continuity are independent concepts (see also [10]).

Remark 2.5. It may be observed that the notion of subsequential continuity imposes a strong condition on the mappings f and g by requiring the existence of a sequence $\{x_n\}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$. Such a precondition is not required when we define f and g to be reciprocally continuous. By assuming the existence of a sequence $\{x_n\}$ the notion of subsequential continuity circumvents the most crucial part of common fixed point theorems consisting of constructive procedures yielding a Cauchy sequence. Undercompleteness of the metric space the Cauchy sequence converges to a limit point. It may also be noted that constructive procedures are important parts of common fixed point theorems and strong assumptions like subsequential continuity do not and should not obviate the need for such constructive procedures. More recently Pant and Bisht [10] introduced the notion of conditional reciprocal continuity which does not require such a precondition and yet is a proper generalization of reciprocal continuity or subsequential continuity.

Remark 2.6. All the results proved by various authors ([1]-[8], [11]-[13], [15], [16]) can be improved and generalized by using the notion of conditional reciprocal continuity in place of subsequential continuity.

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