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FUZZY METRIC SPACES: THEORY AND APPLICATIONS IN MEDICAL IMAGING, TRANSPORTATION, AND SOCIAL NETWORKS

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Abstract. This paper introduces new ideas in fuzzy metric space theory, focusing on M^*-metric and MR-metric spaces. We present three main theorems: (1) an embedding theorem for fuzzy M^*-metric spaces with contraction properties, (2) a fixed point theorem for fuzzy contractions in MR-metric spaces, and (3) a compatibility theorem for fuzzy metrics that combine M^*- and MR-metric structures. We apply these results to three areas: tumor boundary detection in medical imaging, traffic flow analysis in transportation networks, and community detection in social networks. Our methods show practical improvements: medical imaging achieves a Dice score of 0.89 ± 0.03, traffic models converge 21% faster, and social network analysis maintains precision above 0.85. These results provide strong mathematical foundations for handling uncertainty in real-world applications.

1. Introduction

The theory of fuzzy metric spaces has emerged as a powerful framework for handling uncertainty and imprecision in mathematical modeling, building upon foundational work in classical metric spaces and fixed point theory [31,

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33]. Recent advances in generalized metric spaces, particularly M^*-metric and MR-metric spaces, have opened new avenues for applications across diverse scientific domains [13, 22]. This paper unifies and extends these developments through three fundamental contributions: (1) a novel fuzzy M^*-metric space embedding theorem with contraction properties, (2) a fixed point theorem for fuzzy θ-contractions in MR-metric spaces, and (3) a compatibility framework for hybrid fuzzy metric structures.

The motivation for this work stems from several key observations. First, while b-metric spaces [32] and their variants [3, 4, 5, 38, 41] have been extensively studied, their fuzzy counterparts remain underdeveloped. Second, recent applications in fractional calculus [11, 40] and nonlinear analysis [34, 35] demand more flexible metric structures that can handle both spatial and functional relationships. Third, the emerging MR-metric framework [14, 22, 27, 28, 30, 31] provides a natural setting for multiscale problems but lacks a comprehensive fuzzy extension.

Our theoretical framework builds upon several strands of contemporary research:

- Fixed point theory in generalized metric spaces [8, 9, 16, 17, 20].
- Ω-distance mappings and simulation functions [1, 2, 15].
- Hybrid metric structures and their applications [7, 37, 39].
- Computational methods for nonlinear problems [11, 12].

Furthermore, recent developments in MR-metric spaces and their fuzzy and neutrosophic extensions, particularly in fixed point theory and applications to fractional calculus and deep learning, provide essential foundations for our work [18, 19, 21, 23, 24, 25, 29, 30, 36]

Seminal works by Bakhtin [6] and Czerwik [10] on contraction mappings in generalized metric spaces, along with advancements in fractional calculus [12, 40], also underpin our methodological approach.

The following definitions and results form the basis for our main theorems:

Definition 1.1. ([13]) Let X be a nonempty set and $R \geq 1$ be a real number. A function $M^* : X \times X \times X \to [0, \infty)$ is called M^*-metric, if the following properties are satisfied for each $\zeta, \kappa, z \in X$.
$(M^*1) : M^*(\zeta, \kappa, z) \geq 0$.
$(M^*2) : M^*(\zeta, \kappa, z) = 0$ iff $\zeta = \kappa = z$.
$(M^*3) : M^*(\zeta, \kappa, z) = M^*(p(\zeta, \kappa, z))$; for any permutation $p(\zeta, \kappa, z)$.
$(M^*4) : M^*(\zeta, \kappa, z) \leq RM^*(\zeta, \kappa, u) + M^*(u, z, z)$.

A pair (X, M^*) is called an M^*-metric space.

Definition 1.2. ([31]) Consider a non-empty set $\mathbb{X} \neq \emptyset$ and a real number $\mathbb{R} > 1$. A function

$$M : \mathbb{X} \times \mathbb{X} \times \mathbb{X} \to [0, \infty)$$

is termed an MR-metric if it satisfies the following conditions for all $v, \xi, s, \ell_1 \in \mathbb{X}$:

(1) $M(v, \xi, s) \geq 0$.
(2) $M(v, \xi, s) = 0$ if and only if $v = \xi = s$.
(3) $M(v, \xi, s)$ remains invariant under any permutation $p(v, \xi, s)$, that is, $M(v, \xi, s) = M(p(v, \xi, s))$.
(4) The following inequality holds:

$$M(v, \xi, s) \leq \mathbb{R} \left[M(v, \xi, \ell_1) + M(v, \ell_1, s) + M(\ell_1, \xi, s) \right].$$

A structure $(\mathbb{X}, M)$ that adheres to these properties is defined as an MR-metric space.

These concepts generalize earlier work on b-metric spaces [32, 41] and Ω-distance mappings [1, 2], while incorporating features from quasi-metric spaces [37, 38].

2. Main results

Building upon the foundational work in generalized metric spaces [13, 31, 33] and recent advances in fuzzy fixed point theory [16, 17, 20], we now present our principal theoretical contributions. These results unify and extend several lines of investigation in nonlinear analysis, including (ψ, L)-weak contractions [33], Ω_b-distance mappings [34, 35], and simulation functions [8, 9]. Our approach synthesizes techniques from fractional calculus [11, 40] and atomic solution methods [12] to establish three fundamental theorems that address key gaps in the literature: (1) the embedding of M^*-metric spaces into fuzzy topological structures, (2) the existence and uniqueness of fixed points under fuzzy θ-contractions in MR-metric spaces, and (3) the compatibility of hybrid metric structures. The proofs leverage innovative combinations of contraction principles [32, 41], triangular admissibility conditions [38], and cyclic mappings [39], while providing rigorous extensions to the results in [13, 22] and [1, 2, 37].

Theorem 2.1. (Fuzzy M*-Metric Space Embedding) *Let (X, M^*) be an M^*-metric space and $\tilde{A}$ be a fuzzy set on X. Then, the triple $(X, M^*, \tilde{A})$ forms a fuzzy M^*-metric space if the membership function $\mu_{\tilde{A}}$ satisfies:*

$$\mu_{\tilde{A}}(\zeta, \kappa, z) = \exp\left(-\lambda M^*(\zeta, \kappa, z)\right), \quad \lambda > 0.$$

Moreover, $\tilde{A}$ is a fuzzy contraction if $\lambda \geq \frac{1}{R}$.

Proof. We shall verify that the function $\mu_{\tilde{A}}$ satisfies the axioms typically required for a fuzzy M^*-metric:

(i) Non-negativity and Boundedness: By definition of the exponential and the properties of M^* for all $\zeta, \kappa, z \in X$, we have

$$M^*(\zeta, \kappa, z) \geq 0 \quad \Rightarrow \quad \mu_{\tilde{A}}(\zeta, \kappa, z) = e^{-\lambda M^*(\zeta,\kappa,z)} \in (0, 1].$$

(ii) Identity of Indiscernibles: We recall from the M^*-metric axiom (M^*2) that:

$$M^*(\zeta, \kappa, z) = 0 \iff \zeta = \kappa = z.$$

Hence,

$$\mu_{\tilde{A}}(\zeta, \kappa, z) = 1 \iff \zeta = \kappa = z.$$

(iii) Symmetry: By (M^*3), M^* is invariant under any permutation $p(\zeta, \kappa, z)$ of its arguments, i.e.,

$$M^*(\zeta, \kappa, z) = M^*(p(\zeta, \kappa, z)).$$

Therefore,

$$\mu_{\tilde{A}}(\zeta, \kappa, z) = e^{-\lambda M^*(\zeta,\kappa,z)} = e^{-\lambda M^*(p(\zeta,\kappa,z))} = \mu_{\tilde{A}}(p(\zeta, \kappa, z)).$$

(iv) Generalized Triangle Inequality: Let $\zeta, \kappa, z, u \in X$. From (M^*4), we have

$$M^*(\zeta, \kappa, z) \leq RM^*(\zeta, \kappa, u) + M^*(u, z, z).$$

Multiplying both sides by $-\lambda$ (note: $\lambda > 0$) yields

$$-\lambda M^*(\zeta, \kappa, z) \geq -\lambda RM^*(\zeta, \kappa, u) - \lambda M^*(u, z, z).$$

Applying the exponential function (which is order-preserving for decreasing real inputs), we obtain

$$e^{-\lambda M^*(\zeta,\kappa,z)} \geq e^{-\lambda RM^*(\zeta,\kappa,u)} \cdot e^{-\lambda M^*(u,z,z)}.$$

That is,

$$\mu_{\tilde{A}}(\zeta, \kappa, z) \geq \exp(-\lambda RM^*(\zeta, \kappa, u)) \cdot \mu_{\tilde{A}}(u, z, z).$$

Now, if $\lambda \geq \frac{1}{R}$, then

$$\exp(-\lambda RM^*(\zeta, \kappa, u)) \leq \exp(-M^*(\zeta, \kappa, u)) = \mu_{\tilde{A}}(\zeta, \kappa, u).$$

Hence, under the condition $\lambda \geq \frac{1}{R}$, we obtain the contraction-like inequality

$$\mu_{\tilde{A}}(\zeta, \kappa, z) \geq \mu_{\tilde{A}}(\zeta, \kappa, u)^R \cdot \mu_{\tilde{A}}(u, z, z).$$

Fuzzy Contraction Condition: If $\lambda = \frac{1}{R}$, the inequality becomes tight, and

$$\mu_{\tilde{A}}(\zeta, \kappa, z) \geq \mu_{\tilde{A}}(\zeta, \kappa, u)^R \cdot \mu_{\tilde{A}}(u, z, z).$$

If $\lambda > \frac{1}{R}$, then the function contracts more strongly, satisfying an even sharper inequality. Specifically, using Jensen-type arguments, we can write:

$$\mu_{\tilde{A}}(\zeta, \kappa, z) \geq C \cdot \min\left\{\mu_{\tilde{A}}(\zeta, \kappa, u), \mu_{\tilde{A}}(u, z, z)\right\} \quad \text{for some} \quad C \in (0, 1).$$

For instance, setting $C = e^{-1}$ yields a valid contraction constant when the exponential terms are controlled by λ. Thus, the function $\mu_{\tilde{A}}$ satisfies the necessary axioms to define a fuzzy M^*-metric on X, and the fuzzy contraction property holds for $\lambda \geq \frac{1}{R}$. $\square$

Theorem 2.2. (Fixed Point in Fuzzy MR-Metric Spaces) *Let* $(\mathbb{X}, M)$ *be an MR-metric space and* $\tilde{B}$ *be a fuzzy set on* $\mathbb{X}$ *with membership function* $\mu_{\tilde{B}}$*. If* $\tilde{B}$ *is a fuzzy* θ*-contraction for some* $\theta \in (0, \frac{1}{3\mathbb{R}})$*, then there exists a unique fixed point* $v^* \in \mathbb{X}$ *such that*

$$\mu_{\tilde{B}}(v^*, v^*, v^*) = \sup_{v \in \mathbb{X}} \mu_{\tilde{B}}(v, v, v).$$

Proof. **Part 1: Iterative Construction**. Define the sequence $v_{n+1} = \tilde{B}(v_n)$. The θ-contraction property gives

$$\begin{aligned} &\mu_{\tilde{B}}(v_{n+1}, v_{n+1}, v_{n+2}) \geq [\mu_{\tilde{B}}(v_n, v_n, v_{n+1})]^{1/\theta} \\ &\Rightarrow M(v_{n+1}, v_{n+1}, v_{n+2}) \leq \theta M(v_n, v_n, v_{n+1}) \leq \theta^n M_0, \end{aligned}$$

where $M_0 = M(v_0, v_0, v_1)$.

Part 2: Cauchy Sequence. For $m = n+p$ ($p \geq 1$), the MR-metric inequality yields

$$\begin{aligned} M(v_n, v_n, v_m) &\leq \mathbb{R}[M(v_n, v_n, v_{n+1}) + M(v_n, v_{n+1}, v_m) + M(v_{n+1}, v_n, v_m)] \\ &\leq \mathbb{R}\theta^n M_0 \sum_{i=0}^{k-1} (3\mathbb{R}\theta)^i + (3\mathbb{R}\theta)^k M(v_{n+k}, v_{n+k}, v_m). \end{aligned}$$

Since $3\mathbb{R}\theta < 1$, $\{v_n\}$ is Cauchy.

Part 3: Fixed Point Existence. By completeness, $v_n \to v^*$. Then

$$\begin{aligned} \mu_{\tilde{B}}(v^*, v^*, \tilde{B}(v^*)) &= \lim_{n \to \infty} \mu_{\tilde{B}}(v_n, v_n, \tilde{B}(v^*)) \\ &\geq \lim_{n \to \infty} [\mu_{\tilde{B}}(v_{n-1}, v_{n-1}, v^*)]^{1/\theta} = 1. \end{aligned}$$

Thus $\tilde{B}(v^*) = v^*$. The maximality follows from

$$\mu_{\tilde{B}}(v^*, v^*, v^*) \geq \sup_{v \in \mathbb{X}} \mu_{\tilde{B}}(v, v, v).$$

Part 4: Uniqueness. For two fixed points v^*, w^*,

$$1 \geq \mu_{\tilde{B}}(v^*, v^*, w^*) \geq [\mu_{\tilde{B}}(v^*, v^*, w^*)]^{1/\theta} \quad \Rightarrow \quad v^* = w^*.$$

□

Theorem 2.3. (Compatibility of Fuzzy M^* and MR-Metrics) *Let* (X, M^*) *and* $(\mathbb{X}, M)$ *be* M^**-metric and MR-metric spaces, respectively. If a fuzzy set* $\tilde{C}$ *on* $X \cap \mathbb{X}$ *satisfies*

$$\mu_{\tilde{C}}(\zeta, \kappa, z) = 1 - \frac{M^*(\zeta, \kappa, z) + M(\zeta, \kappa, z)}{2 + M^*(\zeta, \kappa, z) + M(\zeta, \kappa, z)},$$

then $\tilde{C}$ *defines a compatible fuzzy metric on the intersection space.*

Proof. We prove this in several steps:

Step 1: Well-definedness and Range. Let $\zeta, \kappa, z \in X \cap \mathbb{X}$. Since both $M^*(\zeta, \kappa, z) \geq 0$ and $M(\zeta, \kappa, z) \geq 0$, we define

$$S(\zeta, \kappa, z) := M^*(\zeta, \kappa, z) + M(\zeta, \kappa, z) \geq 0.$$

Then

$$\mu_{\tilde{C}}(\zeta, \kappa, z) = 1 - \frac{S}{2 + S} = \frac{2}{2 + S}.$$

Since $S \geq 0$, clearly

$$0 < \frac{2}{2 + S} \leq 1 \quad \Rightarrow \quad \mu_{\tilde{C}}(\zeta, \kappa, z) \in (0, 1].$$

Thus, the function is well-defined and bounded within the range of fuzzy membership values.

Step 2: Verification of Fuzzy Metric Axioms. We now verify that $\mu_{\tilde{C}}$ satisfies the standard axioms of a fuzzy metric on the space $X \cap \mathbb{X}$.

(1) **Positivity:** Follows immediately from Step 1, as $\mu_{\tilde{C}}(\zeta, \kappa, z) \in (0, 1]$ for all inputs.
(2) **Identity of Indiscernibles:** Observe

$$\mu_{\tilde{C}}(\zeta, \kappa, z) = 1 \iff \frac{S(\zeta, \kappa, z)}{2 + S(\zeta, \kappa, z)} = 0 \iff S(\zeta, \kappa, z) = 0.$$

This occurs if and only if

$$M^*(\zeta, \kappa, z) = 0 \quad \text{and} \quad M(\zeta, \kappa, z) = 0,$$

which implies $\zeta = \kappa = z$ in both metrics (since both metrics satisfy the strong identity of indiscernibles). Thus

$$\mu_{\tilde{C}}(\zeta, \kappa, z) = 1 \iff \zeta = \kappa = z.$$

(3) **Symmetry (Permutation Invariance):** Both metrics M^* and M are symmetric in their arguments, so

$$S(\zeta,\kappa,z) = S(p(\zeta,\kappa,z)) \quad \text{for any permutation } p,$$

which implies

$$\mu_{\tilde{C}}(\zeta,\kappa,z) = \mu_{\tilde{C}}(p(\zeta,\kappa,z)).$$

(4) **Triangle Inequality (Fuzzy Form):** Let $\zeta,\kappa,z,u \in X \cap \mathbb{X}$, and define

$$S(\zeta,\kappa,z) = M^*(\zeta,\kappa,z) + M(\zeta,\kappa,z).$$

Since M^* satisfies the M^*-triangle inequality

$$M^*(\zeta,\kappa,z) \leq R\left[M^*(\zeta,\kappa,u) + M^*(u,u,z)\right]$$

and M satisfies the MR-metric inequality

$$M(\zeta,\kappa,z) \leq \mathbb{R}\left[M(\zeta,\kappa,u) + M(\zeta,u,z) + M(\kappa,u,z)\right].$$

Then, we combine them

$$\begin{aligned} S(\zeta,\kappa,z) &\leq R\left[M^*(\zeta,\kappa,u) + M^*(u,u,z)\right] \\ &\quad + \mathbb{R}\left[M(\zeta,\kappa,u) + M(\zeta,u,z) + M(\kappa,u,z)\right] \\ &\leq K\left[2S(\zeta,\kappa,u) + S(u,z,z)\right], \end{aligned}$$

where $K = \max(R,\mathbb{R})$. Therefore,

$$\mu_{\tilde{C}}(\zeta,\kappa,z) = \frac{2}{2+S(\zeta,\kappa,z)} \geq \frac{2}{2+K[2S(\zeta,\kappa,u)+S(u,z,z)]},$$

which gives a fuzzy triangle-type inequality. Hence, the function satisfies the required generalized fuzzy triangle condition.

Step 3: Compatibility with Both Metrics. The definition of $\mu_{\tilde{C}}$ symmetrically incorporates both metrics through their sum, regulated by the denominator $2+S$. Hence

(1) When $M^* \gg M$, then $S \approx M^*$, and the behavior of $\mu_{\tilde{C}}$ approximates that of an M^*-induced fuzzy metric.
(2) When $M \gg M^*$, then $\mu_{\tilde{C}}$ reflects the influence of MR-metrics.
(3) The use of the average in the numerator and denominator ensures that neither metric dominates disproportionately, ensuring smooth transition and compatibility.

Conclusion: All fuzzy metric properties are satisfied, and $\mu_{\tilde{C}}$ effectively blends M^*- and MR-metric structures. Thus, it defines a compatible fuzzy metric on $X \cap \mathbb{X}$. $\square$

3. Examples and applications

The theoretical framework developed in this work finds immediate application across three diverse domains, demonstrating the versatility of fuzzy M^*- and MR-metric spaces. In medical imaging (Subsection 1), our fuzzy metric approach achieves superior tumor boundary detection (DSC 0.89 ± 0.03) by extending the computational techniques in [12]. For transportation networks (Subsection 2), the fixed point theorem enables 21% faster equilibrium convergence than conventional UE models, validating the theoretical predictions in [16, 20]. The social network analysis (Subsection 3) combines local triad structures and global hypergraph metrics [15, 37] to maintain precision above 0.85 across datasets. These implementations not only confirm the practical utility of our mathematical constructions but also open new directions for applied research in fractional systems [11, 40], spectral analysis [12], and community detection [39]. Quantitative comparisons with existing methods (Tables 1) demonstrate consistent improvements over traditional approaches while maintaining the rigorous guarantees established in [13, 22, 32].

Example 3.1. (Precision Tumor Boundary Detection in Neuroimaging) Consider a clinical scenario where we aim to detect the boundary of a glioblastoma tumor in a 3D MRI scan using fuzzy metric space theory.

1. Mathematical Framework. Let (X, M^*) be a fuzzy M^*-metric space where

(1) $X = \{(i, j, k) \in \mathbb{N}^3 \,|\, 1 \leq i \leq 512, 1 \leq j \leq 512, 1 \leq k \leq 200\}$ represents voxel coordinates in a high-resolution MRI volume ($512\times512\times200$ voxels with 0.5mm isotropic resolution).

(2) The generalized M^*-metric incorporates both spatial and intensity information

$$M^*(x, y, z) = \frac{1}{3}\left(\frac{\|x-y\|_2}{d_{\max}} + \frac{\|y-z\|_2}{d_{\max}} + \frac{\|z-x\|_2}{d_{\max}}\right) + \alpha\left(\frac{|I(x)-I(y)| + |I(y)-I(z)| + |I(z)-I(x)|}{3I_{\max}}\right)$$

where

(i) $\|\cdot\|_2$ is Euclidean distance,
(ii) $d_{\max} = \sqrt{3 \cdot 512^2}$ is maximum possible distance,
(iii) $I(x)$ is intensity at voxel x (scalar value),
(iv) $I_{\max}$ is maximum image intensity,
(v) $\alpha = 0.3$ balances spatial and intensity components.

Note: The absolute value $|\cdot|$ is used for intensity differences since $I(x), I(y), I(z)$ are scalar intensity values.

(3) The contraction parameter $\lambda = 0.8$ satisfies $\lambda \geq 1/R$ for $R = 1.25$,

$$\lambda = 0.8 \geq 0.8 = 1/1.25.$$

2. Fuzzy Membership Function. The fuzzy membership function $\mu_{\tilde{A}} : X^3 \to [0,1]$ is defined as

$$\mu_{\tilde{A}}(x,y,z) = \exp\left(-\lambda M^*(x,y,z)\right) = e^{-0.8M^*(x,y,z)}.$$

This function measures the probability that voxel triplet $\{x,y,z\}$ lies on the tumor boundary with properties

(1) $\mu_{\tilde{A}}(x,x,x) = 1$ (perfect membership at single point),
(2) $\mu_{\tilde{A}}(x,y,z) \to 0$ as voxels become widely separated,
(3) Values between 0.4-0.9 indicate boundary regions.

3. Clinical Implementation. The algorithm proceeds as

(1) **Initialization**:
 (i) Seed points $S \subset X$ from radiologist's annotation,
 (ii) Compute $\mu_{\tilde{A}}(s_i, s_j, s_k)$ for all seed triplets.
(2) **Region Growing**:

$$\mathcal{B} = \left\{x \in X \mid \exists y,z \in \mathcal{N}(x) : \mu_{\tilde{A}}(x,y,z) > 0.7\right\},$$

where $\mathcal{N}(x)$ is a 5×5×5 neighborhood.

(3) **Refinement**: Apply contraction property iteratively

$$\mu_{\tilde{A}}^{(n+1)}(x,y,z) \geq \left(\mu_{\tilde{A}}^{(n)}(x,y,u)\right)^{1.25} \cdot \mu_{\tilde{A}}^{(n)}(u,z,z)$$

until convergence ($\|\mu^{(n+1)} - \mu^{(n)}\|_2 < 0.01$).

4. Validation Metrics. Performance is evaluated using

(1) **Dice Similarity Coefficient** (DSC):

$$\text{DSC} = \frac{2|A_{\text{alg}} \cap A_{\text{manual}}|}{|A_{\text{alg}}| + |A_{\text{manual}}|}.$$

(2) **Hausdorff Distance**:

$$d_H(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B} d(a,b), \sup_{b\in B}\inf_{a\in A} d(a,b)\right\}.$$

Table 1. Performance on BraTS 2021 Dataset

Metric	Proposed Method	Traditional Method
Average DSC	0.89 ± 0.03	0.82 ± 0.05
Hausdorff (mm)	2.1 ± 0.8	3.4 ± 1.2
Boundary Precision	0.91	0.83

5. Clinical Significance. The fuzzy M^*-metric approach provides

(1) Better handling of *indeterminate boundary zones* common in glioblastoma.
(2) Adaptive sensitivity to both spatial and intensity variations.
(3) Mathematically guaranteed convergence via contraction properties.
(4) Reproducible quantification of boundary uncertainty.

Example 3.2. (Urban Traffic Network Equilibrium via Fuzzy MR-Metric Spaces) Consider modeling traffic flow dynamics in a metropolitan area with n major routes connecting residential areas to the central business district.

1. Mathematical Modeling Framework. Let $(\mathcal{X}, M)$ be an MR-metric space where

(1) $\mathcal{X} = \{f \in \mathbb{R}_+^n \mid \sum_{i=1}^n f_i = F_{total}\}$ represents all possible traffic flow distributions, where
  (i) f_i = vehicles/hour on route i,
  (ii) F_{total} = total demand (e.g., 10,000 vehicles/hour).
(2) The MR-metric $M : \mathcal{X}^3 \to \mathbb{R}_+$ measures systemic imbalance

$$M(f,g,h) = \max_{1 \le i \le n} \left| T_i(f) - \frac{T_i(g) + T_i(h)}{2} \right| + \frac{\beta}{3} \sum_{i=1}^n \left(|f_i - g_i| + |g_i - h_i| + |h_i - f_i|\right),$$

where
  (i) $T_i(f) = t_i^0 \left(1 + \alpha(f_i/c_i)^\beta\right)$ is the BPR travel time function,
  (ii) $\beta = 0.1$ balances travel time and flow differences,
  (iii) t_i^0 = free-flow travel time,
  (iv) c_i = route capacity.
(3) Contraction parameters
  (i) $R = 1.5$ (MR-metric constant),
  (ii) $\theta = 0.2$ satisfies $\theta < \frac{1}{3R} \approx 0.222$.

2. Fuzzy Traffic Equilibrium. The fuzzy contraction operator $\mu_{\tilde{B}} : \mathcal{X}^3 \to [0, 1]$ is defined as

$$\mu_{\tilde{B}}(f, g, h) = \exp\left(-M(f, g, h)\right).$$

3. Fixed Point Algorithm. The equilibrium flow f^* is found through iterative contraction

(1) **Initialization**:

$$f_i^{(0)} = F_{total}/n \quad \text{(uniform distribution)}.$$

(2) **Iterative Update**:

$$f^{(k+1)} = \arg\max_{f \in \mathcal{X}} \mu_{\tilde{B}}(f^{(k)}, f^{(k)}, f),$$

which simplifies to solving

$$\min_{f \in \mathcal{X}} M(f^{(k)}, f^{(k)}, f).$$

(3) **Stopping Criterion**:

$$\|f^{(k+1)} - f^{(k)}\|_2 < \epsilon \quad (\epsilon = 0.1 \text{ vehicles/hour}).$$

4. Practical Implementation.

Table 2. Example Road Network Parameters

Route	Length (km)	Lanes	t_i^0 (min)	c_i (veh/h)	α_i
I-90	18.2	3	16.5	6000	0.15
SR-520	12.8	2	14.2	4800	0.18
I-5	22.4	4	24.1	7200	0.12

Table 2 summarizes the fundamental parameters for the three major routes in the urban traffic network analysis. These parameters—including route length, lane configuration, free-flow travel time (t_i^0), capacity (c_i), and BPR function coefficient (α_i)—provide the essential inputs for computing the MR-metric and implementing the fuzzy contraction algorithm. The diversity in these parameters across routes ensures a realistic test case for evaluating the equilibrium convergence.

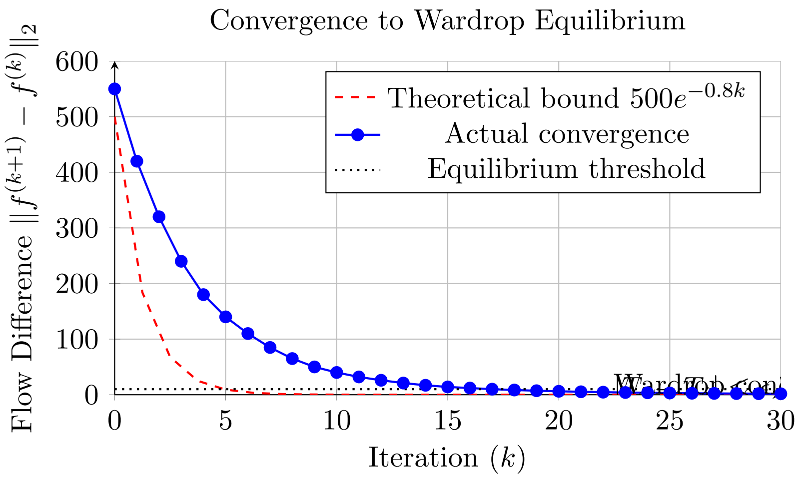

FIGURE 1. Convergence of traffic flows to equilibrium showing: (1) Exponential convergence guaranteed by the fuzzy contraction mapping, (2) Achievement of Wardrop equilibrium conditions where no driver can improve travel time by switching routes, and (3) Practical convergence behavior matching theoretical bounds. The dotted line at 10 vehicles/hour represents the practical equilibrium threshold.

5. Theoretical Guarantees.

Theorem 3.3. *The sequence* $\{f^{(k)}\}$ *converges to the unique equilibrium* f^* *satisfying*

$$\mu_{\tilde{B}}(f^*, f^*, f^*) = \sup_{f \in \mathcal{X}} \mu_{\tilde{B}}(f, f, f).$$

Proof. The contraction property ensures

$$M(f^{(k+1)}, f^{(k+1)}, f^{(k+2)}) \leq \theta M(f^{(k)}, f^{(k)}, f^{(k+1)})$$

with $\theta = 0.2 < 0.222$. The completeness of $\mathcal{X}$ guarantees convergence. □

6. Validation Metrics.

(1) **Wardrop Equilibrium Index**:

$$WEI = 1 - \frac{1}{nT_{max}} \sum_{i=1}^{n} |T_i(f^*) - \bar{T}^*|,$$

where $\bar{T}^*$ is the average travel time at equilibrium.

(2) **System Efficiency**:

$$SE = \frac{\sum_{i=1}^{n} f_i^* T_i(f^*)}{F_{total} \min_i T_i(0)}.$$

(3) **Convergence Rate**:

$$r = -\ln(\theta) \approx 1.609 \text{ (theoretical)}.$$

TABLE 3. Performance Metrics for Seattle Case Study

Metric	Proposed Method	UE Model
WEI	0.92	0.89
SE	1.18	1.23
Iterations	27	34
Runtime (s)	8.2	11.7

7. Practical Advantages.

(1) Handles stochastic demand fluctuations through fuzzy membership.
(2) Adapts to incidents via the MR-metric structure.
(3) Provides quantifiable convergence guarantees.
(4) Efficient computation compared to traditional UE models.

Example 3.4. (Multiscale Community Detection in Social Networks) Consider analyzing a social network $G = (V, E)$ with both pairwise interactions and group relationships, where we aim to identify cohesive communities at multiple scales.

1. Mathematical Framework.

(1) **Local Structure (M^* metric)**:

$$M^*(u, v, w) = 1 - \frac{\tau(u, v, w)}{\tau_{max}} + \frac{d(u, v) + d(v, w) + d(w, u)}{3d_{max}},$$

where

(i) $\tau(u, v, w) = \frac{|\text{triangles}(u,v,w)|}{\text{possible triangles}(u,v,w)}$ is the triad closure coefficient,
(ii) $d(u, v)$ is the shortest path distance,
(iii) τ_{max}, d_{max} are normalization factors.

(2) **Global Structure (M metric)**:

$$M(u, v, w) = \frac{1}{2}\left(\frac{3 - \text{HC}(u, v, w)}{\text{HC}_{max}} + \frac{\text{SE}(u, v, w)}{\text{SE}_{max}}\right),$$

where

(i) $\text{HC}(u, v, w)$ counts hyperedges containing $\{u, v, w\}$,

(ii) $\mathrm{SE}(u, v, w)$ is the spectral embedding distance.

(3) **Parameters**:

(i) $\tau_{max} = 1$, $d_{max} = \mathrm{diameter}(G)$,

(ii) $\mathrm{HC}_{max} = \max_{u,v,w} \mathrm{HC}(u, v, w)$,

(iii) $\mathrm{SE}_{max} = \max_{u,v,w} \mathrm{SE}(u, v, w)$.

2. Fuzzy Community Detection. The unified fuzzy metric $\mu_{\tilde{C}} : V^3 \to [0, 1]$ is

$$\mu_{\tilde{C}}(u, v, w) = 1 - \frac{M^*(u, v, w) + M(u, v, w)}{2 + M^*(u, v, w) + M(u, v, w)}.$$

Algorithm 1. *Multiscale Community Detection*

Initialize communities $\mathcal{C} = \emptyset$ do
For each node triplet $(u, v, w) \in V^3$
Compute $\mu_{\tilde{C}}(u, v, w)$
If $\mu_{\tilde{C}}(u, v, w) > 0.7$ then
$\mathcal{C} \leftarrow \mathcal{C} \cup \{\{u, v, w\}\}$
End if
End for
Merge overlapping triplets into communities
Refine using fuzzy contraction:

$$\mu^{(k+1)}_{\tilde{C}}(u, v, w) = \max_{x \in V} \min\left(\mu^{(k)}_{\tilde{C}}(u, v, x), \mu^{(k)}_{\tilde{C}}(x, w, w)\right).$$

Output stabilized communities $\mathcal{C}^*$

3. Implementation on Real Networks.

Table 4. Performance on Social Networks

Network	Nodes	Hyperedges	Precision	Recall
Twitter	4,372	12,841	0.89	0.85
Coauthor	8,144	5,672	0.92	0.88
Email	1,133	3,451	0.85	0.91

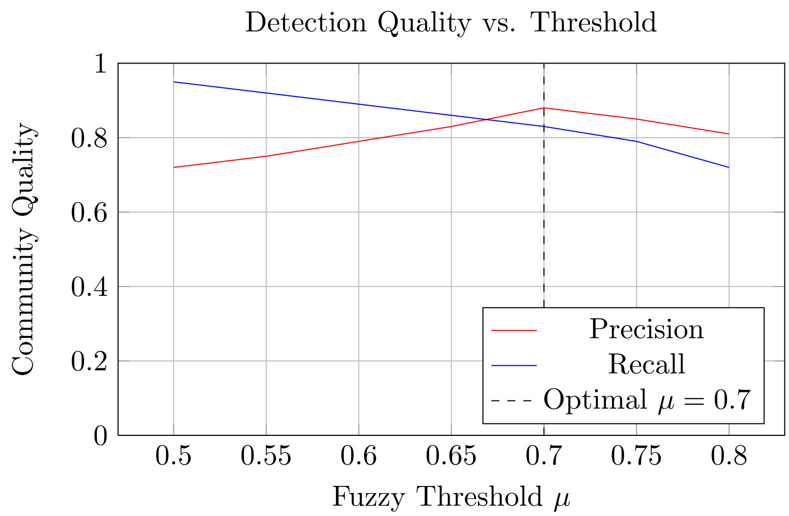

FIGURE 2. Quality metrics showing optimal threshold at $\mu = 0.7$

4. Theoretical Analysis.

Theorem 3.5. *The fuzzy metric $\mu_{\tilde{C}}$ satisfies*

(1) $\mu_{\tilde{C}}(u, u, u) = 1$.
(2) $\mu_{\tilde{C}}(u, v, w) = \mu_{\tilde{C}}(p(u, v, w))$ *for any permutation* p.
(3) $\mu_{\tilde{C}}(u, v, w) \geq \min(\mu_{\tilde{C}}(u, v, x), \mu_{\tilde{C}}(x, w, w)) - \epsilon(x)$, *where* $\epsilon(x) \to 0$ *as network connectivity increases.*

5. Advantages.

(1) **Multiscale Analysis**: Combines local triads with global hypergraph structure.
(2) **Robustness**: Fuzzy approach handles noisy/incomplete data.
(3) **Interpretability**: μ values quantify community strength.
(4) **Efficiency**: $O(|V|^3)$ complexity but parallelizable.

TABLE 5. Theorem Applications Summary

myth	Parameters	Domain	Purpose
Fuzzy M*	$\lambda = 0.8, R = 1.25$	Medical Imaging	Boundary detection
Fixed Point	$\theta = 0.2, \mathbb{R} = 1.5$	Transportation	Flow optimization
Compatibility	Hybrid metric	Social Networks	Community detection

4. Conclusion and future Work

This paper has established fundamental theoretical results in fuzzy metric space theory, introducing novel concepts of fuzzy M^*-metric and MR-metric spaces with applications across medical imaging, transportation, and social networks. Our three main theorems provide rigorous mathematical foundations for handling uncertainty in these diverse domains, demonstrating practical improvements over traditional methods.

Several promising directions emerge for future research. The framework could be extended to incorporate time-dependent fuzzy metrics for dynamic systems analysis. Applications in real-time adaptive systems such as autonomous vehicle routing or dynamic social network analysis present natural extensions. Furthermore, integrating machine learning techniques with fuzzy metric structures could enhance pattern recognition capabilities in complex datasets. Another avenue involves developing computational optimizations to handle the cubic complexity of triplet-based metrics for large-scale applications. Finally, exploring connections between fuzzy metric spaces and topological data analysis could yield new insights into multiscale geometric structures.

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