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BLOW-UP FOR LOGARITHMIC VISCOELASTIC BOUSSINESQ EQUATIONS WITH DELAYED DISSIPATIVE TERM AND ACOUSTIC BOUNDARY CONDITIONS

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Abstract. The current work is devoted to the study of nonlinear viscoelastic wave equations owing a logarithmic source, nonlinear damping and a delay in the velocity with acoustic boundary conditions for which we provide a blow-up criterion. By the use of an energy method, we prove that the solution with negative initial energy blows up after finite time. Moreover, we examine the blow-up time upper and lower bounds.

1. Introduction

The prediction of waves for coastal regions has always been a big challenge for scientists since wave transformation harms constantly the property and humans near coastal regions. The non-linear transformation of water surface waves in coastal region with the effects of shoaling, diffraction, reflection and refraction are modeled by the use of Boussinesq equations which describe the shallow water waves in intermediate water depth and dissemination of strong

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long waves in surfing areas where breaking of the wave dominates [15]. These equations were formulated by Boussinesq in 1871 and 1872 [3] in order to provide a logical explanation to the existence of the solitary, long, shallow and water waves of permanent form, that was discovered by Scott Russell [16]. The the Boussinesq equations are a system of non linear partial differential equations that consider the influence of wave frequency on the behavior of waves. They are widely applied in numerical description of fairly long waves in the field of coastal engineering to model water waves in shallow seas and ports and weirs [2, 26].

Substantial mathematical interest has been given to Boussinesq equations which was investigated by several researchers from various counts, among them we mention without limitation Valdamir Valmarov [18, 21, 22, 23] and numerous papers such as [4, 9, 24, 25] have investigated the Boussinesq equation generalization. It is important to highlight that considerable amount of effort have been made to establish sufficient conditions for the nonexistence of global solutions to various associated boundary value problems [11, 19]. The effect of small nonlinearity and dispersion in Boussinesq equations is considered, but in many real situations, the damping effect is strong compared with the nonlinear and dispersive one see [8, 21, 22].

Several researcher have studied the blow-up in the Boussinesq equations. For instance, the author in [17] showed the collapse dynamics existence for the Boussinesq equation two basic forms in the case of the periodic boundary conditions. The sufficient criterion of the blow-up is found analytically. Whereas, in [27, 28], the author has studied respectively the existence and the blow-up of solutions to the initial boundary value problems of type Boussinesq equations. An other example ,to name but a few, is in [12] where the author investigated instability of the solitary waves for the generalized Boussinesq equation and established certain blow-up results for improved Boussinesq type equation.

In the current paper, we establish a blow-up result for solutions with vanishing initial energy of the following problem that involves a nonlinear viscoelastic equation of Boussinesq type owing a nonlinear damping term, logarithmic source and a delay in the velocity plus acoustic boundary conditions:

$$\begin{cases} u_{tt}+\Delta^2 u-\Delta u+\int_0^t g(t-s)\Delta u(s)ds+a_1u_t(t)|u_t(t)|^{m-2} & \\ +a_2u_t(t-r)|u_t(t-r)|^{m-2}=a_3u|u|^{q-2}\ln|u|, & \text{in } \Omega\times(0,T),\\ u(x,t)=0, & \text{on } \Gamma_0\times(0,T),\\ \frac{\partial u}{\partial\nu}+\int_0^t g(t-s)\frac{\partial u(s)}{\partial\nu}ds=h_1(x)\mathcal{Z}_t, & \text{on } \Gamma_1\times(0,T),\\ u_t+h_2(x)\mathcal{Z}_t+h_3(x)\mathcal{Z}=0, & \text{on } \Gamma_1\times(0,T),\\ u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x) \text{ in } \Omega, & \mathcal{Z}(x,0)=\mathcal{Z}_0 \text{ on } \Gamma_1,\\ u_t(x,t-r)=f_0(x,t-r), & \text{in } \Omega\times]0,r[. \end{cases} \tag{1.1}$$

With Ω is a bounded domain with a sufficiently smooth boundary in $\mathbb{R}^n$, $\partial\Omega = \Gamma_1 \cup \Gamma_0$, $\Gamma_1 \cap \Gamma_0 = \emptyset$, $|\Gamma_0| \neq 0$, $|\Gamma_1| \neq 0$, $r > 0$ is a time delay, $m \geq 2$, $q > 2$, $a_1 > 0$, $a_2 \in \mathbb{R}^*$, $a_3 > 0$ and h_1, h_2, $h_3 : \Gamma_1 \to (0, \infty)$ are essentially bounded functions satisfying

$$0 < h_0 \leq h_3, \quad \text{for all } \; x \in \Gamma_1,$$

where h_0 is a constant and the function $g : [0, \infty) \to [0, \infty)$ is nonincreasing differentiable function.

For that, we split the work into two parts: the second section that provides essential results and hypothesises and a third one where blow-up criteria is proved.

2. Preliminaries

Let $V = \{w \in H^2(\Omega) : \; w = 0 \, on \, \Gamma_0\}$ and $\langle .,. \rangle$ denote the scalar product in the space $L^2(\Omega)$. $\|.\|_\tau$ and $\|.\|_{\Gamma_1,\tau}$ represent the norms in the spaces $L^\tau(\Omega)$ and $L^\tau(\Gamma_1)$, respectively. Moreover, $\|.\|_Y$ denotes the norm of space Y. $C > 0$ denotes a generic constant. If there is no ambiguity, we omit the variables t and x. We recall the embedding $H^2_{\Gamma_0}(\Omega) \hookrightarrow L^\tau(\Omega)$, where $\tau \geq 2$ and

$$\begin{cases} \tau < \infty & \text{if } n = 1, \, 2, \, 3, \, 4; \\ \tau \leq \frac{2n}{n-4} & \text{if } n \geq 5. \end{cases} \tag{2.1}$$

Let λ_1 be the first eigenvalue of the spectral Dirichlet problem

$$\Delta^2 u = \lambda_1 u, \; \text{ in } \; \Omega, \qquad u = \frac{\partial u}{\partial \eta} = 0 \; \text{ in } \; \Gamma,$$

$$\|\nabla u\|_2 \leq \frac{1}{\sqrt{\lambda_1}} \|\Delta u\|_2. \tag{2.2}$$

As consequence $V \hookrightarrow L^\tau(\Omega)$, with $\|u\|_\tau \leq \frac{C_n}{\sqrt{\lambda_1}} \|\Delta u\|_2$.

Now, we introduce, as in the work of in Nicaise and Pignotti [14], the new variable

$$y(x, \rho, t) = u_t(x, t - r\rho), \; x \in \Omega, \quad \rho \in (0, 1), \quad t > 0.$$

Then, we obtain

$$r y_t(x, \rho, t) + y_\rho(x, \rho, t) = 0 \quad \text{in } \Omega \times (0, 1) \times (0, +\infty).$$

Then, the problem (1.1) is equivalent to

$$\begin{cases} u_{tt}+\Delta^2 u-\Delta u+\int_0^t g(t-s)\Delta u(s)ds+a_1 u_t(t)|u_t(t)|^{m-2} & \\ +a_2 y(x,1,t)|y(x,1,t)|^{m-2}=a_3 u|u|^{q-2}\ln|u|, & \text{in } \Omega\times(0,T), \\ r y_t(x,\rho,t)+y_\rho(x,\rho,t)=0, & \text{in } \Omega\times]0,1[\times]0,+\infty[, \\ u(x,t)=0, & \text{on } \partial\Omega\times[0,\infty[, \\ \frac{\partial^2 u}{\partial\nu^2}+\frac{\partial u}{\partial\nu}+\int_0^t g(t-s)\frac{\partial u(s)}{\partial\nu}ds=h_1(x)\mathcal{Z}_t, & \text{on } \Gamma_1\times(0,T), \\ u_t+h_2(x)\mathcal{Z}_t+h_3(x)\mathcal{Z}=0, & \text{on } \Gamma_1\times(0,T,) \\ u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x), & \text{in } \Omega, \\ \mathcal{Z}(x,0)=\mathcal{Z}_0, & \text{on } \Gamma_1, \\ y(x,\rho,0)=f_0(x,-\rho r)=y_0(x,\rho), & \text{in } \Omega\times]0,1[. \end{cases} \tag{2.3}$$

To state and prove our result, we set the following assumptions:

- **(A1)** The kernel g verifies $g_l := 1-\int_0^\infty g(s)ds > 0$.
- **(A2)** The coefficients a_1 and a_2 satisfy
$$a_2 \neq 0 \quad \text{and} \quad -a_1 < a_2 < a_1.$$
- **(A3)** $2 < m < q$.
- **(A4)** Let q satisfy
$$\begin{cases} 2<q<\infty & \text{if } n=1,\,2,\,3,\,4; \\ 2<q<min\left\{\frac{2(n-2)}{n-4},\,\frac{4+n}{n-4}\right\} & \text{if } n\geq 5. \end{cases}$$
- **(A5)** The kernel function g fulfills
$$\int_0^\infty g(s)ds < \frac{q-2}{q-2+\frac{1}{q}}. \tag{2.4}$$

The following theorem establishes the existence and uniqueness of a local solution of our problem, the proof can be determined by combining the proof given in [6, 13].

Theorem 2.1. *Assume that* **(A1)**, **(A2)** *and the exponent q satisfies*
$$\begin{cases} 2<q<\infty & \text{if } n=1,\,2,\,3,\,4; \\ 2<q<\frac{2(n-2)}{n-4} & \textit{if } n\geq 5. \end{cases} \tag{2.5}$$
Then, for every $(u_0,\,u_1,\,\mathcal{Z}_0,\,y_0)\in V\times L^2(\Omega)\times L^2(\Gamma)\times L^m(\Omega\times(0,1))$, *problem* (2.3) *admit a unique local solution* $(u,\,\mathcal{Z},\,y)$ *with* $u\in C(0,T;V)\cap C^1(0,T;\,L^2(\Omega))$, $\sqrt{h_1}\mathcal{Z}\in L^\infty(0,T;\,L^2(\Gamma_1))$ *and* $y\in L^\infty(0,T;\,L^m(\Omega\times(0,1)))$.

Our objective is to determine a blow-up criterion to problem (1.1).

3. Blow-up criteria

We define the energy associated to the solution of system (1.1) by

$$\begin{aligned} E(t) =& \frac{1}{2}\|\Delta u\|_2^2 + \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2}\int_0^t g(s)ds\|\nabla u\|_2^2 \\ &+ \frac{1}{2}(g \circ \nabla u) + \frac{a_3}{q^2}\|u\|_q^q - \frac{a_3}{q}\int_\Omega |u|^q ln|u|dx \\ &+ r\xi\int_0^1 \|y(x,\rho,t)\|_m^m d\rho + \frac{1}{2}\int_{\Gamma_1} h_3 h_1 \mathcal{Z}^2 d\Gamma, \end{aligned} \tag{3.1}$$

where

$$(g \circ \nabla u)(t) = \int_0^t g(t-s)\|\nabla u(t) - \nabla u(s)\|_2^2 ds$$

and ξ is a positive constant such that

$$\frac{|a_2|(m-1)}{m} < \xi < \frac{ma_1 - |a_2|}{m} \tag{3.2}$$

and

$$a_1 > |a_2|.$$

Lemma 3.1. *Under the conditions of Theorem* 2.1*, it holds*

$$\begin{aligned} \frac{d}{dt}E(t) \le& \frac{1}{2}\left(g' \circ \nabla u - g(t)\|\nabla u(t)\|_2^2\right) \\ &- \int_{\Gamma_1} h_2 h_1 \mathcal{Z}_t^2 d\Gamma - \gamma_1\left(\|u_t\|_m^m + \|y(x,1,t\|_m^m\right), \end{aligned} \tag{3.3}$$

where γ_1 is given later.

Proof. By multiplying the first equation in (2.3) by u_t, integrating over Ω, and using integration by parts, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega |u_t|^2 dx + \int_\Omega \Delta u \Delta u_t dx + \int_{\Gamma_1} u_t \frac{\partial \Delta u}{\partial \nu} + \int_\Omega \nabla u \nabla u_t - \int_{\Gamma_1} u_t \frac{\partial u}{\partial \nu} d\mu \\ &- \int_\Omega \int_0^t g(t-s)\nabla u(s)\nabla u_t ds dx + a_1\int_\Omega |u_t|^m dx \\ &+ a_2\int_\Omega y(x,1,t)|y(x,1,t)|^{m-2}u_t dx - a_3\int_\Omega u|u|^{q-2}\ln|u|u_t dx \\ &+ \int_{\Gamma_1}\int_0^t g(t-s)u_t(t)\frac{\partial u(s)}{\partial \nu} ds d\mu = 0. \end{aligned} \tag{3.4}$$

On the other hand,

$$\int_\Omega u|u|^{q-2}\ln|u|u_t dx = \frac{d}{dt}\left[\frac{1}{q}\int_\Omega |u|^q \ln|u|dx - \frac{1}{q^2}\|u\|_q^q\right], \tag{3.5}$$

$$
\begin{aligned}
&\int_\Omega \int_0^t g(t-s)\nabla u(s)\nabla u_t ds dx \\
&= -\frac{g(t)}{2}\|\nabla u\|_2^2+\frac{1}{2}(g'\circ\nabla u)(t)+\frac{1}{2}\frac{d}{dt}\left[\int_0^t g(s)ds\|\nabla u\|_2^2-(g\circ\nabla u)(t)\right].
\end{aligned} \tag{3.6}
$$

Then, substituting (3.5) and (3.6) into (3.4), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\Big[\|u_t\|_2^2+\|\Delta u\|_2^2+\|\nabla u\|_2^2+\int_0^t g(s)ds\|\nabla u\|_2^2+(g\circ\nabla u)(t) \\
&\quad -\frac{2a_3}{q}\int_\Omega |u|^q\ln|u|dx+\frac{2a_3}{q^2}\|u\|_q^q)\Big] \\
&= \int_{\Gamma_1} u_t\frac{\partial^2 u}{\partial^2\nu}d+\int_{\Gamma_1} u_t\frac{\partial u}{\partial \nu}-\frac{g(t)}{2}\|\nabla u\|_2^2+\frac{1}{2}(g'\circ\nabla u)(t)-a_1\int_\Omega |u_t|^m dx \\
&\quad -a_2\int_\Omega y(x,1,t)|y(x,1,t)|^{m-2}u_t dx-\int_{\Gamma_1}\int_0^t g(t-s)u_t(t)\frac{\partial u(s)}{\partial\nu}ds d\mu.
\end{aligned} \tag{3.7}
$$

We multiply the second equation in (2.3) by $\xi y(x,\rho,t)|y(x,\rho,t)|^{m-2}$, we integrate the result over $\Omega\times(0,1)$, we obtain

$$
\xi r\frac{\partial}{\partial t}\int_0^1 \|y(x,1,t)\|^m d\rho = -\xi\|y(x,1,t)\|_m^m+\xi\|u_t\|_m^m. \tag{3.8}
$$

We multiply the third equation in (2.3) by u_t, we integrate the result over Γ_1, we get

$$
\int_{\Gamma_1} u_t\frac{\partial^2 u}{\partial\nu^2}d\Gamma+\int_{\Gamma_1}\frac{\partial u}{\partial\nu}u_t d\Gamma+\int_{\Gamma_1}\int_0^t g(t-s)\frac{\partial u(s)}{\partial\nu}ds d\Gamma = \int_{\Gamma_1} h_1(x)\mathcal{Z}_t d\Gamma. \tag{3.9}
$$

We multiply the fourth equation in (2.3) by $h_1\mathcal{Z}_t$, we integrate the result over Γ_1, we get

$$
\int_{\Gamma_1} h_1\mathcal{Z}_t u_t = -\int_{\Gamma_1} h_1 h_2(x)\mathcal{Z}_t^2 d\Gamma-\frac{1}{2}\int_{\Gamma_1}\frac{\partial}{\partial t}h_1 h_3(x)\mathcal{Z}^2 d\Gamma. \tag{3.10}
$$

Next, substituting (3.10) into (3.9), we obtain

$$
\begin{aligned}
&\int_{\Gamma_1} u_t\frac{\partial^2 u}{\partial\nu^2}d\Gamma+\int_{\Gamma_1}\frac{\partial u}{\partial\nu}u_t d\Gamma+\int_{\Gamma_1}\int_0^t g(t-s)\frac{\partial u(s)}{\partial\nu}ds d\Gamma \\
&\qquad = -\int_{\Gamma_1} h_1 h_2(x)\mathcal{Z}_t^2 d\Gamma-\frac{1}{2}\int_{\Gamma_1}\frac{\partial}{\partial t}h_1 h_3(x)\mathcal{Z}^2 d\Gamma.
\end{aligned} \tag{3.11}
$$

Substituting (3.11) into (3.7), we get

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\Big[&\|u_t\|_2^2 - \|\Delta u\|_2^2 + \|\nabla u\|_2^2 - \int_0^t g(s)ds\|\nabla u\|_2^2 + (g\circ\nabla u)(t)\\ &- \frac{2a_3}{q}\int_\Omega |u|^q \ln|u|dx + \frac{2a_3}{q^2}\|u\|_q^q) + \int_{\Gamma_1} h_1h_3(x)\mathcal{Z}^2 d\Gamma\Big]\\ &= -\frac{g(t)}{2}\|\nabla u\|_2^2 + \frac{1}{2}(g'\circ\nabla u)(t) - a_1\int_\Omega |u_t|^m dx\\ &\quad - a_2\int_\Omega y(x,1,t)|y(x,1,t)|^{m-2}u_t dx - \int_{\Gamma_1} h_1h_2(x)\mathcal{Z}_t^2 d\Gamma.\end{aligned} \tag{3.12}$$

By combining (3.12) and (3.8), we obtain

$$\begin{aligned}\frac{\partial}{\partial t}E(t) = &-\frac{g(t)}{2}\|\nabla u\|_2^2 + \frac{1}{2}(g'\circ\nabla u)(t) - (a_1-\xi)\|u_t\|_m^m - \xi\|y(x,1,t)\|_m^m\\ &- a_2\int_\Omega y(x,1,t)|y(x,1,t)|^{m-2}u_t dx - \int_{\Gamma_1} h_1h_2(x)\mathcal{Z}_t^2 d\Gamma.\end{aligned} \tag{3.13}$$

By using Young's inequality, we get

$$-a_2\int_\Omega y(x,1,t)|y(x,1,t)|^{m-2}u_t dx \le |a_2|\frac{(m-1)}{m}\|y(x,1,t)\|_m^m + \frac{|a_2|}{m}\|u_t\|_m^m, \tag{3.14}$$

then we estimate $E'(t)$

$$\begin{aligned}E'(t) \le &-\frac{g(t)}{2}\|\nabla u\|_2^2 + \frac{1}{2}(g'\circ\nabla u)(t) - \gamma_1\left(\|u_t\|_m^m + \|y(x,1,t)\|_m^m\right)\\ &- \int_{\Gamma_1} h_1h_2(x)\mathcal{Z}_t^2 d\Gamma,\end{aligned} \tag{3.15}$$

where $\gamma_1 = \min\{a_1 - \xi - \frac{a_2}{m}, \frac{\xi m - |a_2|(m-1)}{m}\}$, we choose $a_1 - \xi - \frac{a_2}{m} > 0$ and $\frac{\xi m - |a_2|(m-1)}{m} > 0$

$$\frac{|a_2|(m-1)}{m} < \xi < \frac{ma_1 - a_2}{m}$$

and

$$a_1 > |a_2| > 0.$$

□

Lemma 3.2. *Assume* **(A1)** *and if* $E(0) < 0$, *then*

$$\int_\Omega |u|^q ln|u|dx \ge 0 \tag{3.16}$$

and

$$\|u_t\|_m^m + \|y(x,1,t)\|_m^m \leq \frac{I'(t)}{\gamma_1}, \tag{3.17}$$

where $I(t) = -E(t)$.

Proof. Since the decreasing energy, then

$$-\frac{a_3}{p}\int_\Omega |u|^p ln|u| dx < -E(t) < -E(0) < 0, \tag{3.18}$$

it implies that

$$\frac{a_3}{p}\int_\Omega |u|^p ln|u| dx > 0.$$

From (3.15) and g is a nonincreasing, we have

$$\gamma_1\left(\|u_t\|_m^m + \|y(x,1,t)\|_m^m\right) \leq -E'(t). \tag{3.19}$$

Then we obtain (3.17). □

Lemma 3.3. ([7]) *Let* $w \in H_0^2(\Omega)$ *and* τ *assure* $\tau \geq 2$ *and*

$$\begin{cases} 2 < \tau < \infty & \text{if } n = 1, 2, 3, 4; \\ 2 < \tau \leq \frac{2(n-2)}{n-4} & \textit{if } n \geq 5. \end{cases} \tag{3.20}$$

Then, there exist $c > 1$, *for all* $\frac{2}{\tau} \leq \frac{s}{\tau} \leq 1$, *it holds*

$$\|w\|_\tau^s \leq c(\|w\|_\tau^\tau + \|\nabla w\|_2^2).$$

Lemma 3.4. ([7]) *Let* $\tau \geq 2$. *For* $w \in H_0^2(\Omega)$ *with* (3.16), *it holds*

$$\|w\|_\tau^\tau \leq C\left(\|\Delta w\|_2^2 + \int_\Omega |w|^\tau ln|w| dx\right).$$

Lemma 3.5. ([7]) *Let* $w \in H_0^2(\Omega)$ *satisfy* (3.16) *and* τ *verify* $\tau \geq 2$ *and*

$$\begin{cases} 2 < \tau < \infty & \text{if } n = 1, 2, 3, 4; \\ 2 < \tau \leq \frac{2(n-2)}{n-4} & \text{if } n \geq 5. \end{cases}$$

Then, for all $\frac{2}{\tau} \leq \frac{s}{\tau} \leq 1$,

$$\left(\int_\Omega |w|^\tau ln|w| dx\right)^{\frac{s}{\tau}} \leq C\left(\|\Delta w\|_2^2 + \int_\Omega |w|^\tau ln|w| dx\right).$$

Lemma 3.6. ([7]) *Let $w \in H_0^2(\Omega)$ satisfy* (3.16) *and τ verify $\tau \geq 2$ and*

$$\begin{cases} 2 < \tau < \infty & \text{if } n = 1,\, 2,\, 3,\, 4; \\ 2 < \tau \leq \frac{2(n-2)}{n-4} & \text{if } n \geq 5. \end{cases}$$

Then there exists a positive constant C such that

$$\|w\|_\tau^\tau \leq C\left(\|\Delta w\|^{\frac{2\tau}{q}} + (\int_\Omega |w|^\tau ln|w|dx)^{\frac{\tau}{q}} \right).$$

Theorem 3.7. *Assume that* **(A1)**-**(A5)** *hold and if $E(0) < 0$, the solution to problems* (2.3) *blows up after a time*

$$T^* \leq \frac{(1-\beta)\,(F(0))^{\frac{-\beta}{1-\beta}}}{\beta c_2}$$

with c_2 nonnegative constant, β satisfies

$$\frac{2(q-m)}{q^2(m-1)} \leq \beta \leq min\left\{ \frac{q-m}{q(m-1)},\, \frac{q-2}{2q} \right\} \tag{3.21}$$

and

$$F(0) = I^{1-\beta}(0) + \varepsilon\langle u_0,\, u_1\rangle - \frac{\varepsilon}{2}\int_{\Gamma_1} h_1 h_2 \mathcal{Z}_\prime^{\,2} d\Gamma - \varepsilon\int_{\Gamma_1} h_1 \mathcal{Z}_\prime u_0 d\Gamma > 0.$$

Proof. Let set

$$F(t) = I^{1-\beta}(t) + \varepsilon\langle u,\, u_t\rangle - \frac{\varepsilon}{2}\int_{\Gamma_1} h_1 h_2 \mathcal{Z}^2 d\Gamma - \varepsilon\int_{\Gamma_1} h_1 \mathcal{Z} u d\Gamma, \tag{3.22}$$

where $\varepsilon > 0$. Differentiating with respect to t and using the first equation of (2.3)

$$\begin{aligned} F'(t) =& (1-\beta)I^{-\beta}(t)I'(t) + \varepsilon\|u_t\|_2^2 - \varepsilon\int_\Omega \Delta^2 u\, u dx + \varepsilon\int_\Omega \Delta u\, u dx \\ & - \varepsilon\int_\Omega\int_0^t g(t-s)\Delta u(s)u(t)dsdx - a_1\varepsilon\int_\Omega u_t(t)|u_t(t)|^{m-2}u(t)dx \\ & - a_2\varepsilon\int_\Omega y(x,1,t)|y(x,1,t)|^{m-2}u(t)dx + a_3\varepsilon\int_\Omega |u|^q \ln|u| \\ & - \varepsilon\int_{\Gamma_1} h_1 h_2 \mathcal{Z}\mathcal{Z}_t d\Gamma - \varepsilon\int_{\Gamma_1} h_1 \mathcal{Z}_t u d\Gamma - \varepsilon\int_{\Gamma_1} h_1 \mathcal{Z} u_t d\Gamma \end{aligned}$$

$$
\begin{aligned}
&= (1-\beta)I^{-\beta}(t)I'(t) + \varepsilon\|u_t\|_2^2 - \varepsilon\int_\Omega |\Delta u|^2 dx \\
&\quad + \varepsilon\int_{\Gamma_1} \frac{\partial^2 u}{\partial\nu^2} u d\Gamma - \varepsilon\int_\Omega |\nabla u|^2 dx \\
&\quad + \varepsilon\int_{\Gamma_1} \frac{\partial u}{\partial\nu} u d\Gamma - a_1\varepsilon\int_\Omega u_t(t)|u_t(t)|^{m-2}u(t)dx \\
&\quad - a_2\varepsilon\int_\Omega y(x,1,t)|y(x,1,t)|^{m-2}u(t)dx \\
&\quad + a_3\varepsilon\int_\Omega |u|^q \ln|u| + \varepsilon\int_0^t g(s)ds\|\nabla u\|^2 \\
&\quad + \varepsilon\int_0^t g(t-s)\langle \nabla u(t),\, \nabla u(s) - \nabla u(t)\rangle ds \\
&\quad - \varepsilon\int_{\Gamma_1} g(t-s)\frac{\partial u}{\partial\nu} u ds d\nu \\
&\quad - \varepsilon\int_{\Gamma_1} h_1 h_2 \mathcal{Z}\mathcal{Z}_t d\Gamma - \varepsilon\int_{\Gamma_1} h_1 \mathcal{Z}_t u d\Gamma - \varepsilon\int_{\Gamma_1} h_1 \mathcal{Z} u_t d\Gamma \\
&= (1-\beta)I^{-\beta}(t)I'(t) + \varepsilon\|u_t\|_2^2 - \varepsilon\|\Delta u\|_2^2 dx - \varepsilon(1 - \int_0^t g(s)ds)\|\nabla u\|_2^2 \\
&\quad - a_1\varepsilon\int_\Omega u_t(t)|u_t(t)|^{m-2}u(t)dx - a_2\varepsilon\int_\Omega y(x,1,t)|y(x,1,t)|^{m-2}u(t)dx \\
&\quad + a_3\varepsilon\int_\Omega |u|^q \ln|u| + \varepsilon\int_0^t g(t-s)\langle \nabla u(t),\, \nabla u(s) - \nabla u(t)\rangle ds \\
&\quad + \varepsilon\int_{\Gamma_1} \frac{\partial^2 u}{\partial\nu^2} u d\Gamma + \varepsilon\int_{\Gamma_1} \frac{\partial u}{\partial\nu} u d\Gamma - \varepsilon\int_{\Gamma_1} g(t-s)\frac{\partial u}{\partial\nu} u ds d\nu \\
&\quad - \varepsilon\int_{\Gamma_1} h_1 h_2 \mathcal{Z}\mathcal{Z}_t d\Gamma - \varepsilon\int_{\Gamma_1} h_1 \mathcal{Z}_t u d\Gamma - \varepsilon\int_{\Gamma_1} h_1 \mathcal{Z} u_t d\Gamma \\
&\geq (1-\beta)I^{-\beta}(t)I'(t) + \varepsilon\|u_t\|_2^2 - \varepsilon\|\Delta u\|_2^2 + \varepsilon\int_\Omega |u|^q \ln|u| dx \\
&\quad - \varepsilon\left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 - a_1\varepsilon\int_\Omega u_t(t)|u_t(t)|^{m-2}u(t)dx \\
&\quad - a_2\varepsilon\int_\Omega y(x,1,t)|y(x,1,t)|^{m-2}u(t)dx \\
&\quad + \varepsilon\int_0^t g(t-s)\langle \nabla u(t),\, \nabla u(s) - \nabla u(t)\rangle ds + \varepsilon\int_{\Gamma_1} h_1 h_3 \mathcal{Z}^2 d\Gamma.
\end{aligned}
\tag{3.23}
$$

By Young's inequality, we obtain

$$\begin{aligned}\varepsilon\int_0^t g(t-s)\langle\nabla u(t),\,\nabla u(s)-\nabla u(t)\rangle ds &\geq -\frac{\varepsilon}{4\eta}\|\nabla u\|^2-2\int_0^t g(s)ds\\ &\quad-\varepsilon\eta\int_0^t g(t-s)\|\nabla u(s)-\nabla u(t)\|_2^2 ds\\ &=-\frac{\varepsilon}{4\eta}\|\nabla u\|_2^2\int_0^t g(s)ds-\varepsilon\eta g\circ\nabla u.\end{aligned}$$

So, we have

$$\begin{aligned}F'(t)\geq{}&(1-\beta)I^{-\beta}(t)I'(t)+\varepsilon\|u_t\|_2^2-\varepsilon\|\Delta u\|_2^2+\varepsilon\int_\Omega|u|^q\ln|u|dx\\ &-\varepsilon\left(1-\int_0^t g(s)ds+\frac{1}{4\eta}\int_0^t g(s)ds\right)\|\nabla u\|_2^2\\ &-a_1\varepsilon\int_\Omega u_t(t)|u_t(t)|^{m-2}u(t)dx-a_2\varepsilon\int_\Omega y(x,1,t)|y(x,1,t)|^{m-2}u(t)dx\\ &+\varepsilon\int_{\Gamma_1}h_1h_3\mathcal{Z}^2d\Gamma-\varepsilon\eta g\circ\nabla u+\varepsilon\alpha I(t)-\varepsilon\alpha I(t),\end{aligned}$$

where $\alpha=p(1-\lambda)$ for some $\lambda\in(0,1)$. Therefore,

$$\begin{aligned}F'(t)\geq{}&(1-\beta)I^{-\beta}(t)I'(t)+\varepsilon(\frac{\alpha}{2}+1)\|u_t\|_2^2+\varepsilon(\frac{\alpha}{2}-1)\|\Delta u\|_2^2\\ &+\varepsilon\left(1-\frac{a_3\alpha}{q}\right)\int_\Omega|u|^q\ln|u|dx\\ &+\varepsilon\left(\frac{\alpha}{2}-\frac{\alpha}{2}\int_0^t g(s)ds-1+\int_0^t g(s)ds-\frac{1}{4\eta}\int_0^t g(s)ds\right)\|\nabla u\|_2^2\\ &+\varepsilon\left(\frac{\alpha}{2}-\eta\right)g\circ\nabla u-a_1\varepsilon\int_\Omega u_t(t)|u_t(t)|^{m-2}u(t)dx\\ &-a_2\varepsilon\int_\Omega y(x,1,t)|y(x,1,t)|^{m-2}u(t)dx\\ &+\varepsilon r\xi\alpha\int_0^1\|y(x,\rho,t)\|_m^m d\rho+\varepsilon\left(\frac{\alpha}{2}+1\right)\int_{\Gamma_1}h_1h_3\mathcal{Z}^2d\Gamma+\varepsilon\alpha I(t).\end{aligned}\tag{3.24}$$

From **(A2)** and using Young's inequalities with $\frac{m-1}{m}+\frac{1}{m}=1$, we get

$$\begin{aligned}&a_1\varepsilon\int_\Omega u_t(t)|u_t(t)|^{m-2}u(t)dx+a_2\varepsilon\int_\Omega y(x,1,t)|y(x,1,t)|^{m-2}u(t)dx\\ &\quad\leq\frac{a_1(m-1)}{m}\delta_1^{-\frac{m}{m-1}}\left(\|y(1,t)\|_m^m+\|u_t\|_m^m\right)+\frac{(a_1+|a_2|)\delta_1^m}{m}\|u\|_m^m.\end{aligned}\tag{3.25}$$

From (3.17) and taking $\delta_1 = \left(\theta I^{-\beta}(t)\right)^{-\frac{m-1}{m}}$ and $\theta > 0$, we obtain

$$\begin{aligned}
&\frac{a_1(m-1)}{m}\delta_1^{-\frac{m}{m-1}}\left(\|y(1,t)\|_m^m + \|u_t\|_m^m\right) + \frac{(a_1+|a_2|)\delta_1^m}{m}\|u\|_m^m \\
&\quad \le \frac{a_1(m-1)\theta}{m\gamma_1}\left(I(t)\right)^{-\beta} I'(t) + \frac{C}{m\theta^{m-1}}\left(I(t)\right)^{\beta(m-1)}\|u\|_m^m \\
&\quad \le \frac{a_1(m-1)\theta}{m\gamma_1}\left(I(t)\right)^{-\beta} I'(t) + \frac{C}{m\theta^{m-1}}\left(I(t)\right)^{\beta(m-1)}\|u\|_q^m.
\end{aligned} \tag{3.26}$$

Through the utility of **(A3)** and **(A4)**. On account of (3.18) and Lemma 3.4, we obtain

$$\left(I(t)\right)^{\beta(m-1)}\|u\|_q^m \le C\left(\|\Delta u\|_2^2 + \int_\Omega |u|^q ln|u|dx\right)^{\frac{m}{q}}\left(\int_\Omega |u|^q ln|u|dx\right)^{\beta(m-1)}. \tag{3.27}$$

By the elementary inequality $(a+b)^\Upsilon \le 2^\Upsilon(a^\Upsilon + b^\Upsilon)$, we get

$$\begin{aligned}
&\left(I(t)\right)^{\beta(m-1)}\|u\|_q^m \\
&\quad \le 2^{\frac{m}{q}} C\left(\|\Delta u\|_2^{\frac{2m}{q}} + \left(\int_\Omega |u|^q ln|u|dx\right)^{\frac{m}{q}}\right)\left(\int_\Omega |u|^q ln|u|dx\right)^{\beta(m-1)} \\
&\quad \le C\left[\|\Delta u\|_2^{\frac{2m}{q}}\left(\int_\Omega |u|^q ln|u|dx\right)^{\beta(m-1)} + \left(\int_\Omega |u|^q ln|u|dx\right)^{\beta(m-1)+\frac{m}{q}}\right].
\end{aligned} \tag{3.28}$$

Using Young's equality with $\frac{m}{q} + \frac{q-m}{q} = 1$, this estimation becomes

$$\begin{aligned}
&\left(I(t)\right)^{\beta(m-1)}\|u\|_q^m \\
&\quad \le C\left[\|\Delta u\|_2^2 + \left(\int_\Omega |u|^q ln|u|dx\right)^{\frac{\beta(m-1)q}{q-m}} + \left(\int_\Omega |u|^q ln|u|dx\right)^{\frac{\beta q(m-1)+m}{q}}\right].
\end{aligned} \tag{3.29}$$

By (3.21), we observe that $\frac{2}{q} \le \frac{\beta q(m-1)+m}{q} \le 1$ and $\frac{2}{q} \le \frac{\beta(m-1)q}{q-m} < 1$. Then, utilizing Lemma 3.5, we deduce

$$\left(\int_\Omega |u|^q ln|u|dx\right)^{\frac{\beta q(m-1)+m}{q}} \le C\left(\|\Delta u\|_2^2 + \int_\Omega |u|^q ln|u|dx\right) \tag{3.30}$$

and

$$\left(\int_\Omega |u|^q ln|u|dx\right)^{\frac{\beta(m-1)q}{q-m}} \le C\left(\|\Delta u\|_2^2 + \int_\Omega |u|^q ln|u|dx\right). \tag{3.31}$$

By inserting (3.30) and (3.31) to (3.29), we obtain

$$(I(t))^{\beta(m-1)}\|u\|_q^m \leq \tilde{C}\left(\|\Delta u\|_2^2 + \int_\Omega |u|^q ln|u|dx\right). \tag{3.32}$$

Combining (3.32) in (3.26), we obtain

$$\begin{aligned} &a_1\varepsilon\int_\Omega u_t(t)|u_t(t)|^{m-2}u(t)dx + a_2\varepsilon\int_\Omega y(x,1,t)|y(x,1,t)|^{m-2}u(t)dx \\ &\leq \frac{a_1(m-1)\theta}{m\gamma_1}(I(t))^{-\beta}I'(t) + \frac{C}{m\theta^{m-1}}\left(\|\Delta u\|_2^2 + \int_\Omega |u|^q ln|u|dx\right). \end{aligned} \tag{3.33}$$

By applying this to (3.24), we arrive at

$$\begin{aligned} F'(t) \geq &\left(1-\beta-\varepsilon\frac{a_1(m-1)\theta}{m\gamma_1}\right)I^{-\beta}(t)I'(t) + \varepsilon(\frac{\alpha}{2}+1)\|u_t\|_2^2 \\ &+\varepsilon\left(-\frac{C}{m\theta^{m-1}} + \frac{\alpha}{2} - 1\right)\|\Delta u\|_2^2 + \varepsilon\left(1 - \frac{a_3\alpha}{q} - \frac{c}{m\theta^{m-1}}\right)\int_\Omega |u|^q \ln|u|dx \\ &+\varepsilon\left(\frac{\alpha}{2} - 1 - \left(\frac{\alpha}{2} - 1 + \frac{1}{4\eta}\right)\int_0^t g(s)ds\right)\|\nabla u\|_2^2 + \varepsilon\left(\frac{\alpha}{2}-\eta\right)g\circ\nabla u \\ &+\varepsilon\frac{a_3}{p^2}\alpha\|u\|_m^m + \varepsilon\left(\frac{\alpha}{2}+1\right)\int_{\Gamma_1} h_1h_3\mathcal{Z}^2 d\Gamma + \varepsilon\alpha I(t). \end{aligned}$$

Let take $\eta = \frac{\alpha}{2}$ and use the relation

$$\begin{aligned} &\left(\frac{\alpha}{2}-1\right) - \left(\frac{\alpha}{2} - 1 + \frac{1}{4\eta}\right)\int_0^t g(s)ds \\ &= \left(\frac{\alpha}{2}-1\right)\left(1 - \int_0^t g(s)ds\right) - \frac{1}{2\alpha}\int_0^t g(s)ds \\ &\geq \left(\frac{\alpha}{2}-1\right)g_l - \frac{1}{2\alpha}(1-g_l) =: a_\alpha. \end{aligned}$$

Then from (3.24), we get

$$\begin{aligned} F'(t) \geq &\left(1-\beta-\varepsilon\frac{a_1(m-1)\theta}{m\gamma_1}\right)I^{-\beta}(t)I'(t) + \varepsilon(\frac{\alpha}{2}+1)\|u_t\|_2^2 \\ &+\varepsilon\left(1-\frac{a_3\alpha}{q} - \frac{c}{m\theta^{m-1}}\right)\int_\Omega |u|^q\ln|u|dx \\ &+\varepsilon a_\alpha\|\nabla u\|_2^2 + (\frac{\alpha}{2} - 1 - \frac{c}{m\theta^{m-1}}\|\Delta u\|_2^2 \\ &+\varepsilon\frac{a_3}{p^2}\alpha\|u\|_q^q + \varepsilon\left(\frac{\alpha}{2}+1\right)\int_{\Gamma_1} h_1h_3\mathcal{Z}^2 d\Gamma + \varepsilon\alpha I(t). \end{aligned} \tag{3.34}$$

So,

$$F'(t) \geq \tilde{C}\left(I(t) + \|u_t\|_2^2 + \|\nabla u\|_2^2 + \|u\|_q^q\right) > 0. \tag{3.35}$$

Consequently, we have $F(t) \geq F(0) > 0$ for all $t > 0$.

On the one hand, from (3.22), we know that

$$F(t) \leq I^{1-\beta}(t) + \varepsilon\langle u,\, u_t\rangle,$$

then

$$(F(t))^{\frac{1}{1-\beta}} \leq \left(I^{1-\beta}(t) + \varepsilon\langle u,\, u_t\rangle\right)^{\frac{1}{1-\beta}}. \tag{3.36}$$

We use Hölder's and Young's inequalities to get

$$\begin{aligned} \left|\int_\Omega u u_t dx\right|^{\frac{1}{1-\beta}} &\leq C\|u\|_q^{\frac{1}{1-\beta}}\|u_t\|_2^{\frac{1}{1-\beta}} \\ &\leq C\|u\|_q^{\frac{2}{1-2\beta}} + \|u_t\|_2^2 \end{aligned} \tag{3.37}$$

with $2 \leq \frac{2}{1-2\beta} \leq q$.

By using Lemma 3.3, we obtain

$$\|u\|_q^{\frac{2}{1-2\beta}} \leq c\left(\|u\|_q^q + \|\nabla u\|_2^2\right). \tag{3.38}$$

Substituting (3.38) into (3.37), we get

$$\left|\int_\Omega u u_t dx\right|^{\frac{1}{1-\beta}} \leq C\left(\|u\|_q^q + \|u_t\|_2^2 + \|\nabla u\|_2^2\right). \tag{3.39}$$

From (3.36) and (3.39), we have

$$(F(t))^{\frac{1}{1-\beta}} \leq \tilde{C}\left(I + \|u\|_q^q + \|u_t\|_2^2 + \|\nabla u\|_2^2\right). \tag{3.40}$$

Combining (3.40) and (3.35), we obtain

$$F'(t) \geq C_2\,(F(t))^{\frac{1}{1-\beta}}, \tag{3.41}$$

where C_2 is positive constant.

By integrating of (3.41) over $(0, t)$, we have

$$(F(t))^{\frac{\beta}{1-\beta}} \geq \frac{1}{(F(0))^{\frac{-\beta}{1-\beta}} - C_2\frac{\beta t}{1-\beta}},$$

which gives that the solution u blow up after finite time

$$T \leq T^* = \frac{(1-\beta)\,(F(0))^{\frac{-\beta}{1-\beta}}}{\beta C_2}.$$

□

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