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ON CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS ASSOCIATED WITH WRIGHT FUNCTION

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Abstract. The Wright function is a special function with notable applications in several branches of mathematics, including geometric function theory. It helps in constructing and studying classes of analytic and univalent functions, particularly due to its connection with fractional calculus and differential subordinations. The target of this paper is to discuss a new subclass of univalent functions with negative coefficients related to Wright distribution in the unit disk $E = \{z : |z| < 1\}$. We obtain basic properties like coefficient inequality, extreme points, closure theorems, radii of starlike and convexity Partial sums and neighborhood results for functions belonging to our class.

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1. Introduction

In 1933, Wright [19] introduced a special function which is named as Wright function and defined in the following way

$$W_{\lambda,\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(\lambda n + \mu)}, \tag{1.1}$$

where $\lambda > -1$, $\mu \in \mathbb{C}$ and $\Gamma(\cdot)$ stands for the usual Gamma function. The series given by (1.1) is absolutely convergent for all $z \in \mathbb{C}$ while for $\lambda = -1$ this is absolutely convergent in E. He also proved that it is an entire function for $\lambda > -1$. For more basic properties on Wright functions one may refer to Gorenflo et al. [6] and Mustafa [7]. It is easy to see that the series (1.1) is not in normalized form so we normalized it as

$$\mathbb{W}_{\lambda,\mu}(z) = \Gamma(\mu) z W_{\lambda,\mu}(z),$$

$$\mathbb{W}_{\lambda,\mu}(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\mu) z^{n+1}}{n!\Gamma\left(\lambda n + \mu\right)}, \tag{1.2}$$

where $\lambda > -1$, $\mu > 0$, $z \in E$. Now, we introduce Wright distribution in the following way, first we define the series

$$\mathbb{W}_{\lambda,\mu}(m) = \sum_{n=0}^{\infty} \frac{\Gamma(\mu) m^{n+1}}{n!\Gamma\left(\lambda n + \mu\right)}, \tag{1.3}$$

which is convergent for all $\lambda, \mu, m > 0$.

The probability mass function of Wright distribution is given by

$$p(n) = \frac{\Gamma(\mu) m^{n+1}}{n!\Gamma\left(\lambda n + \mu\right)\ \mathbb{W}_{\lambda,\mu}(m)},\ m, \mu, \lambda > 0,\ n = 0, 1, 2, 3, \ldots \tag{1.4}$$

It is worthy to note that for $\lambda = 0$ it reduces to the Poisson distribution.

Let $\mathcal{A}$ signify the class of all functions $u(z)$ of the type

$$u(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.5}$$

in the open unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$. Let S be the subclass of $\mathcal{A}$ consisting of univalent functions and satisfy the following usual normalization condition $u(0) = u'(0) - 1 = 0$. We denote by S the subclass of $\mathcal{A}$ consisting of functions $u(z)$ which are all univalent in E. A function $u \in \mathcal{A}$ is a starlike function of the order ς, $0 \leq \varsigma < 1$, if it satisfy

$$\Re\left\{\frac{zu'(z)}{u(z)}\right\} > \varsigma,\ z \in E. \tag{1.6}$$

We denote this class with $S^*(\varsigma)$. A function $u \in \mathcal{A}$ is a convex function of the order $\varsigma,\ 0 \leq \varsigma < 1$, if it fulfill

$$\Re\left\{1+\frac{zu''(z)}{u'(z)}\right\} > \varsigma,\ z \in E. \tag{1.7}$$

We denote this class with $K(\varsigma)$. Note that $S^*(0) = S^*$ and $K(0) = K$ are the usual classes of starlike and convex functions in E respectively. Let T denote the class of functions analytic in E that are of the form

$$u(z) = z - \sum_{n=2}^{\infty} a_n z^n,\ a_n \geq\ 0,\ z \in E \tag{1.8}$$

and let $T^*(\varsigma) = T \cap S^*(\varsigma),\ C(\varsigma) = T \cap K(\varsigma)$. The class $T^*(\varsigma)$ and allied classes possess some interesting properties and have been extensively studied by Silverman [14].

In 2014, by using the definition of Poisson distribution, Porwal [9] introduced Poisson distribution series and gave a nice application of it on certain classes of univalent functions and opened up a new direction of research in the geometric function theory. After the investigation of this series several researchers investigated various distribution series like Hypergeometric distribution series [1], Pascal distribution series [3], Mittag-Leffler type Poisson distribution series [4], Binomial distribution series[8], generalized distribution series [10], Hypergeometric type distribution series [11], confluent hypergeometric distribution series [12], generalized hypergeometric distribution series[17], Borel distribution series [18] (see also [2]) and obtained various interesting results on certain classes of univalent functions for these series. Now, using the definition of Wright distribution, we introduce the Wright distribution series as follows:

$$K(\lambda, \mu, m, z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\mu) m^n}{(n-1)!\Gamma(\lambda(n-1)+\mu)\mathbb{W}_{\lambda,\mu}(m)} z^n.$$

The convolution of two power series $u(z)$ of the form (1.5) and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is defined as the power series

$$(u * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

Now, we introduce the linear operator $I_{\mu,m}^{\lambda}:\mathcal{A}\to\mathcal{A}$ defined as

$$\begin{aligned} I_{\mu,m}^{\lambda}u(z) &= u(z)*K(\lambda,\mu,m,z) \\ &= z+\sum_{n=2}^{\infty}\phi(n)a_n z^n, \end{aligned} \tag{1.9}$$

where

$$\phi(n)=\frac{\Gamma(\mu)m^n}{\Gamma(\lambda(n-1)+\mu)(n-1)!\mathbb{W}_{\lambda,\mu}(m))}. \tag{1.10}$$

Definition 1.1. For $-1\le\upsilon<1,\ 0\le\sigma<1$ and $\varrho\ge 0$, we let $S_{\lambda}^{\mu,m}(\sigma,\upsilon,\varrho)$ be the subclass of A consisting of functions of the form (1.1) and satisfying the analytic criterion

$$\begin{aligned} &\Re\left\{\frac{z(I_{\mu,m}^{\lambda}u(z))'+\sigma z^2(I_{\mu,m}^{\lambda}u(z))''}{(1-\sigma)I_{\mu,m}^{\lambda}u(z)+\sigma z(I_{\mu,m}^{\lambda}u(z))'}-\upsilon\right\} \\ &\quad\ge\varrho\left|\frac{z(I_{\mu,m}^{\lambda}u(z))'+\sigma z^2(I_{\mu,m}^{\lambda}u(z))''}{(1-\sigma)I_{\mu,m}^{\lambda}u(z)+\sigma z(I_{\mu,m}^{\lambda}u(z))'}-1\right| \end{aligned} \tag{1.11}$$

for $z\in E$.

The main object of the paper some usual properties of the geometric function theory such as coefficient bounds, extreme points, radii of starlikness and convexity, partial sums and neighbourhood results for the class.

2. Coefficient bounds

In this section, we obtain a necessary and sufficient condition for function $u(z)$ is in the classes $S_{\lambda}^{\mu,m}(\sigma,\upsilon,\varrho)$ and $TS_{\lambda}^{\mu,m}(\sigma,\upsilon,\varrho)$.

Theorem 2.1. *The function u defined by (1.5) is in the class $S_{\lambda}^{\mu,m}(\sigma,\upsilon,\varrho)$ if*

$$\sum_{n=2}^{\infty}[1+\sigma(n-1)][n(1+\varrho)-(\upsilon+\varrho)]\phi(n)|a_n|\le 1-\upsilon, \tag{2.1}$$

where $-1\le\upsilon<1,\ 0\le\sigma\le 1,\ \varrho\ge 0.$

Proof. It suffices to show that

$$\begin{aligned} &\varrho\left|\frac{z(I_{\mu,m}^{\lambda}u(z))'+\sigma z^2(I_{\mu,m}^{\lambda}u(z))''}{(1-\sigma)I_{\mu,m}^{\lambda}u(z)+\sigma z(I_{\mu,m}^{\lambda}u(z))'}-1\right| \\ &\quad-\Re\left\{\frac{z(I_{\mu,m}^{\lambda}u(z))'+\sigma z^2(I_{\mu,m}^{\lambda}u(z))''}{(1-\sigma)I_{\mu,m}^{\lambda}u(z)+\sigma z(I_{\mu,m}^{\lambda}u(z))'}-1\right\}\le 1-\upsilon, \end{aligned}$$

we have

$$\begin{aligned}&\varrho\left|\frac{z(I_{\mu,m}^{\lambda}u(z))'+\sigma z^{2}(I_{\mu,m}^{\lambda}u(z))''}{(1-\sigma)I_{\mu,m}^{\lambda}u(z)+\sigma z(I_{\mu,m}^{\lambda}u(z))'}-1\right|\\&-\Re\left\{\frac{z(I_{\mu,m}^{\lambda}u(z))'+\sigma z^{2}(I_{\mu,m}^{\lambda}u(z))''}{(1-\sigma)I_{\mu,m}^{\lambda}u(z)+\sigma z(I_{\mu,m}^{\lambda}u(z))'}-1\right\}\\&\leq(1+\varrho)\left|\frac{z(I_{\mu,m}^{\lambda}u(z))'+\sigma z^{2}(I_{\mu,m}^{\lambda}u(z))''}{(1-\sigma)I_{\mu,m}^{\lambda}u(z)+\sigma z(I_{\mu,m}^{\lambda}u(z))'}-1\right|\\&\leq\frac{(1+\varrho)\sum\limits_{n=2}^{\infty}(n-1)[1+\sigma(n-1)]\phi(n)|a_n|}{1-\sum\limits_{n=2}^{\infty}[1+\sigma(n-1)]\phi(n)|a_n|}.\end{aligned}$$

This last expression is bounded above by $(1-\upsilon)$ by

$$\sum_{n=2}^{\infty}[1+\sigma(n-1)][n(1+\varrho)-(\upsilon+\varrho)]\phi(n)|a_n|\leq 1-\upsilon,$$

and hence the proof is complete. □

Theorem 2.2. *A necessary and sufficient condition for* $u(z)$ *of the form* (1.8) *to be in the class* $TS_{\lambda}^{\mu,m}(\sigma,\upsilon,\varrho)$, $-1\leq\upsilon<1$, $0\leq\sigma\leq1$, $\varrho\geq0$ *is that*

$$\sum_{n=2}^{\infty}[1+\sigma(n-1)][n(1+\varrho)-(\upsilon+\varrho)]\phi(n)|a_n|\leq 1-\upsilon. \tag{2.2}$$

Proof. In view of Theorem 2.1, we need only to prove the necessity.

If $u\in TS_{\lambda}^{\mu,m}(\sigma,\upsilon,\varrho)$ and z is real, then

$$\frac{1-\sum\limits_{n=2}^{\infty}n[1+\sigma(n-1)]\phi(n)a_n z^{n-1}}{1-\sum\limits_{n=2}^{\infty}[1+\sigma(n-1)]\phi(n)a_n z^{n-1}}-\upsilon\geq\varrho\left|\frac{\sum\limits_{n=2}^{\infty}(n-1)[1+\sigma(n-1)]\phi(n)|a_n|}{1-\sum\limits_{n=2}^{\infty}[1+\sigma(n-1)]\phi(n)|a_n|}\right|.$$

Letting $z\to1$ along the real axis, we obtain the desired inequality

$$\sum_{n=2}^{\infty}[1+\sigma(n-1)][n(1+\varrho)-(\upsilon+\varrho)]\phi(n)|a_n|\leq 1-\upsilon.$$

□

Theorem 2.3. *Let $u(z)$ defined by (1.8) and $g(z) = z - \sum_{n=2}^{\infty} b_n z^n$ be in the class $TS_{\lambda}^{\mu,m}(\sigma, \upsilon, \varrho)$. Then the function $h(z)$ defined by*

$$h(z) = (1-\zeta)u(z) + \zeta g(z) = z - \sum_{n=2}^{\infty} c_n z^n,$$

where $c_n = (1-\zeta)a_n + \zeta b_n$, $0 \leq \zeta < 1$ is also in the class $TS_{\lambda}^{\mu,m}(\sigma, \upsilon, \varrho)$.

Proof. Let the function

$$u_j = z - \sum_{n=2}^{\infty} a_{n,j} z^n, \ a_{n,j} \geq 0, \ j = 1, 2 \tag{2.3}$$

be in the class $TS_{\lambda}^{\mu,m}(\sigma, \upsilon, \varrho)$. It is sufficient to show that the function $g(z)$ defined by

$$g(z) = \zeta u_1(z) + (1-\zeta)u_2(z), \ 0 \leq \zeta \leq 1$$

is in the class $TS_{\lambda}^{\mu,m}(\sigma, \upsilon, \varrho)$. Since

$$g(z) = z - \sum_{n=2}^{\infty} [\zeta a_{n,1} + (1-\zeta)a_{n,2}] z^n,$$

an easy computation with the aid of Theorem 2.2 gives

$$\begin{aligned}
&\sum_{n=2}^{\infty} [1+\sigma(n-1)][n(\varrho+1) - (\upsilon+\varrho)]\phi(n)\zeta a_{n,1} \\
&\quad + \sum_{n=2}^{\infty} [1+\sigma(n-1)][n(\varrho+1) - (\upsilon+\varrho)]\phi(n)(1-\zeta) a_{n,2} \\
&\leq \ \zeta(1-\upsilon) + (1-\zeta)(1-\upsilon) \\
&\leq \ 1-\upsilon,
\end{aligned}$$

which implies that $g \in TS_{\lambda}^{\mu,m}(\sigma, \upsilon, \varrho)$. Hence $TS_{\lambda}^{\mu,m}(\sigma, \upsilon, \varrho)$ is convex. □

3. Extreme points

The proof of the following theorem, follows on lines similar to the proof of the theorem on extreme points given in Silverman [14].

Theorem 3.1. *Let $u_1(z) = z$ and*

$$u_n(z) = z - \frac{1-\upsilon}{[1+\sigma(n-1)][n(\varrho+1) - (\upsilon+\varrho)]\phi(n)} z^n \tag{3.1}$$

for $n = 2, 3, \cdots$. Then $u(z) \in TS_{\lambda}^{\mu,m}(\sigma, \upsilon, \varrho)$ if and only if $u(z)$ can be expressed in the form $u(z) = \sum_{n=2}^{\infty} \zeta_n u_n(z)$, where $\zeta_n \geq 0$ and $\sum_{n=1}^{\infty} \zeta_n = 1$.

Next we prove the following closure theorem.

4. Closure theorem

Theorem 4.1. *Let the function* $u_j(z), j = 1, 2, \cdots, l$ *defined by* (2.3) *be in the classes* $TS_\lambda^{\mu,m}(\sigma, \upsilon_j, \varrho),\ j = 1, 2, \cdots, l$ *respectively. Then the function* $h(z)$ *defined by*

$$h(z) = z - \frac{1}{l}\sum_{n=2}^{\infty}\left(\sum_{j=1}^{l} a_{n,j}\right) z^n$$

is in the class $TS_\lambda^{\mu,m}(\sigma, \upsilon, \varrho)$, *where* $\upsilon = \min_{1\leq j\leq l}\{\upsilon_j\}$, *where* $-1 \leq \upsilon_j \leq 1$.

Proof. Since $u_j(z) \in TS_\lambda^{\mu,m}(\sigma, \upsilon_j, \varrho),\ j = 1, 2, \cdots, l$ by applying Theorem 2.2 to (2.3), we observe that

$$\begin{aligned}
&\sum_{n=2}^{\infty}[1+\sigma(n-1)][n(\varrho+1)-(\upsilon+\varrho)]\phi(n)\left(\frac{1}{l}\sum_{j=1}^{l} a_{n,j}\right)\\
&= \frac{1}{l}\sum_{j=1}^{l} a_{n,j}\left(\sum_{n=2}^{\infty}[1+\sigma(n-1)][n(\varrho+1)-(\upsilon+\varrho)]\phi(n)a_{n,j}\right)\\
&\leq \frac{1}{l}\sum_{j=1}^{l}(1-\upsilon_j)\\
&\leq 1-\upsilon,
\end{aligned}$$

which in view of Theorem 2.2, again implies that $h(z) \in TS_\lambda^{\mu,m}(\sigma, \upsilon, \varrho)$ and so the proof is complete. □

Theorem 4.2. *Let* $u \in TS_\lambda^{\mu,m}(\sigma, \upsilon, \varrho)$. *Then*

(1) *u is starlike of order* $\delta,\ 0 \leq \delta < 1$, *in the disc* $|z| < r_1$ *that is,* $\Re\left\{\frac{zu'(z)}{u(z)}\right\} > \delta,\ |z| < r_1$, *where*

$$r_1 = \inf_{n\geq 2}\left\{\left(\frac{1-\delta}{n-\delta}\right)\frac{[1+\sigma(n-1)][n(\varrho+1)-(\upsilon+\varrho)]\phi(n)}{1-\upsilon}\right\}^{\frac{1}{n-1}}.$$

(2) *u is convex of order* $\delta,\ 0 \leq \delta < 1$, *in the disc* $|z| < r_1$ *that is,* $\Re\left\{1+\frac{zu''(z)}{u'(z)}\right\} > \delta,\ |z| < r_2$, *where*

$$r_2 = \inf_{n\geq 2}\left\{\left(\frac{1-\delta}{n-\delta}\right)\frac{[1+\sigma(n-1)][n(\varrho+1)-(\upsilon+\varrho)]\phi(n)}{1-\upsilon}\right\}^{\frac{1}{n}}.$$

Each of these results are sharp for the extremal function $u(z)$ *given by* (3.1).

Proof. (1) Given $u \in A$ and u is starlike of order δ, we have

$$\left|\frac{zu'(z)}{u(z)} - 1\right| < 1 - \delta. \tag{4.1}$$

For the left hand side (4.1), we have

$$\left|\frac{zu'(z)}{u(z)} - 1\right| \leq \frac{\sum\limits_{n=2}^{\infty}(n-1)a_n|z|^{n-1}}{1 - \sum\limits_{n=2}^{\infty}a_n|z|^{n-1}}.$$

The last expression is less that $1 - \delta$ if

$$\sum_{n=2}^{\infty}\frac{n-\delta}{1-\delta}a_n|z|^{n-1} < 1.$$

Using the fact, that $u \in TS_{\lambda}^{\mu,m}(\sigma, \upsilon, \varrho)$ if and only if

$$\sum_{n=2}^{\infty}\frac{[1+\sigma(n-1)][n(\varrho+1)-(\upsilon+\varrho)]\phi(n)}{1-\upsilon}a_n < 1.$$

We can say (4.1) is true if

$$\frac{n-\delta}{1-\delta}|z|^{n-1} < \frac{[1+\sigma(n-1)][n(\varrho+1)-(\upsilon+\varrho)]\phi(n)}{1-\upsilon}.$$

Or equivalently,

$$|z|^{n-1} < \frac{(1-\delta)[1+\sigma(n-1)][n(\varrho+1)-(\upsilon+\varrho)]\phi(n)}{(n-\delta)(1-\upsilon)},$$

which yields the starlikeness of the family.

(2) Using the fact that u is convex if and only if zu' is starlike, we can prove (2), on lines similar to the proof of (1). □

5. Partial sums

Following the earlier works by Silverman [15] and Silvia [16] on partial sums of analytic functions. We consider in this section partial sums of functions in this class $S_{\lambda}^{\mu,m}(\sigma, \upsilon, \varrho)$ and obtain sharp lower bounds for the ratios of real part of $u(z)$ to $u_q(z)$ and $u'(z)$ to $u'_q(z)$.

Theorem 5.1. *Let $u(z) \in S_{\lambda}^{\mu,m}(\sigma, \upsilon, \varrho)$ be given by* (1.5) *and define the partial sums $u_1(z)$ and $u_q(z)$ by*

$$u_1(z) = z \text{ and } u_q(z) = z + \sum_{n=2}^{q} a_n z^n, \ (q \in \mathbb{N} \setminus \{1\}). \tag{5.1}$$

Suppose that

$$\sum_{n=2}^{\infty} d_n|a_n| \leq 1, \tag{5.2}$$

where

$$d_n = \frac{[1+\sigma(n-1)][n(1+\varrho)-(\upsilon+\varrho)]\phi(n)}{1-\upsilon}.$$

Then, $u \in S_{\lambda}^{\mu,m}(\sigma,\upsilon,\varrho)$. *Furthermore,*

$$\Re\left[\frac{u(z)}{u_q(z)}\right] > 1 - \frac{1}{d_{q+1}},\ \ z \in E,\ \ q \in \mathbb{N} \tag{5.3}$$

and

$$\Re\left[\frac{u_q(z)}{u(z)}\right] > \frac{d_{q+1}}{1+d_{q+1}}. \tag{5.4}$$

Proof. For the coefficients d_n given by (5.2) it is not difficult to verify that

$$d_{n+1} > d_n > 1. \tag{5.5}$$

Therefore, we have

$$\sum_{n=2}^{q}|a_n| + d_{q+1}\sum_{n=q+1}^{\infty}|a_n| \leq \sum_{n=2}^{\infty} d_n|a_n| \leq 1, \tag{5.6}$$

by using the hypothesis (5.2). By setting

$$\begin{aligned} g_1(z) &= d_{q+1}\left[\frac{u(z)}{u_q(z)} - \left(1-\frac{1}{d_{q+1}}\right)\right] \\ &= 1 + \frac{d_{q+1}\sum\limits_{n=q+1}^{\infty} a_n z^{n-1}}{1+\sum\limits_{n=2}^{q} a_n z^{n-1}} \end{aligned} \tag{5.7}$$

and applying (5.6), we find that

$$\left|\frac{g_1(z)-1}{g_1(z)+1}\right| \leq \frac{d_{q+1}\sum\limits_{n=q+1}^{\infty}|a_n|}{2-2\sum\limits_{n=2}^{q}|a_n| - d_{q+1}\sum\limits_{n=q+1}^{\infty}|a_n|} \leq 1, \tag{5.8}$$

which readily yields the assertion (5.3) of Theorem 5.1. In order to see that

$$u(z) = z + \frac{z^{q+1}}{d_{q+1}} \text{ gives sharp result, we observe that for } z = re^{\frac{i\pi}{q}} \text{ that} \tag{5.9}$$

$$\frac{u(z)}{u_q(z)} = 1 + \frac{z^q}{d_{q+1}} \;\to\; 1 - \frac{1}{d_{q+1}} \quad \text{as} \;\; z \to 1^-.$$

Similarly, if we take

$$\begin{aligned} g_2(z) &= (1 + d_{q+1}) \left(\frac{u_q(z)}{u(z)} - \frac{d_{q+1}}{1 + d_{q+1}} \right) \\ &= 1 - \frac{(1 + d_{n+1}) \sum\limits_{n=q+1}^{\infty} a_n z^{n-1}}{1 + \sum\limits_{n=2}^{\infty} a_n z^{n-1}} \end{aligned} \tag{5.10}$$

and making use of (5.6), we can deduce that

$$\left| \frac{g_2(z) - 1}{g_2(z) + 1} \right| \leq \frac{(1 + d_{q+1}) \sum\limits_{n=q+1}^{\infty} |a_n|}{2 - 2 \sum\limits_{n=2}^{q} |a_n| - (1 - d_{q+1}) \sum\limits_{n=q+1}^{\infty} |a_n|},$$

which leads is immediately to the assertion (5.4) of Theorem 5.1. The bound in (5.4) is sharp for each $q \in \mathbb{N}$ with the external function $u(z)$ given by (5.9). The proof of the Theorem 5.1 is thus complete. □

Theorem 5.2. *If $u(z)$ of the form* (1.5) *satisfies the condition* (2.1)*, then*

$$\Re \left[\frac{u'(z)}{u'_q(z)} \right] \geq 1 - \frac{q + 1}{d_{q+1}}. \tag{5.11}$$

Proof. By setting

$$\begin{aligned} g(z) &= d_{q+1} \left[\frac{u'(z)}{u'_q(z)} \right] - \left(1 - \frac{q + 1}{d_{q+1}} \right) \\ &= \frac{1 + \frac{d_{q+1}}{q+1} \sum\limits_{n=q+1}^{\infty} n a_n z^{n-1} + \sum\limits_{n=2}^{\infty} n a_n z^{n-1}}{1 + \sum\limits_{n=2}^{\infty} n a_n z^{n-1}} \\ &= 1 + \frac{\frac{d_{q+1}}{q+1} \sum\limits_{n=q+1}^{\infty} n a_n z^{n-1}}{1 + \sum\limits_{n=2}^{\infty} n a_n z^{n-1}}, \end{aligned}$$

$$\left|\frac{g(z)-1}{g(z)+1}\right| \leq \frac{\frac{d_{q+1}}{q+1}\sum\limits_{n=q+1}^{\infty} n|a_n|}{2-2\sum\limits_{n=2}^{q} n|a_n| - \frac{d_{q+1}}{q+1}\sum\limits_{n=q+1}^{\infty} n|a_n|}. \tag{5.12}$$

Now, if

$$\sum_{n=2}^{q} n|a_n| + \frac{d_{q+1}}{q+1}\sum_{n=q+1}^{\infty} n|a_n| \leq 1,$$

$$\left|\frac{g(z)-1}{g(z)+1}\right| \leq 1. \tag{5.13}$$

Since the left hand side of (5.13) is bounded above by $\sum\limits_{n=2}^{q} d_n|a_n|$, if

$$\sum_{n=2}^{q}(d_n-n)|a_n| + \sum_{n=q+1}^{\infty} d_n - \frac{d_{q+1}}{q+1} n|a_n| \geq 0 \tag{5.14}$$

and the proof is complete. □

The result is sharp for the extremal function $u(z) = z + \frac{z^{q+1}}{d_{q+1}}$.

Theorem 5.3. *If* $u(z)$ *of the form* (1.5) *satisfies the condition* (2.1)*, then*

$$\Re\left[\frac{u_q'(z)}{u'(z)}\right] \geq \frac{d_{q+1}}{q+1+d_{q+1}}. \tag{5.15}$$

Proof. By setting

$$g(z) = [q+1+d_{q+1}]\left[\frac{u_q'(z)}{u'(z)} - \frac{d_{q+1}}{q+1+d_{q+1}}\right]$$
$$= 1 - \frac{\left(1+\frac{d_{q+1}}{q+1}\right)\sum\limits_{n=q+1}^{\infty} na_n z^{n-1}}{1+\sum\limits_{n=2}^{q} na_n z^{n-1}}$$

and making use of (5.14), we deduce that

$$\left|\frac{g(z)-1}{g(z)+1}\right| \leq \frac{\left(1+\frac{d_{q+1}}{q+1}\right)\sum\limits_{n=q+1}^{\infty} n|a_n|}{2-2\sum\limits_{n=2}^{q} n|a_n| - \left(1+\frac{d_{q+1}}{q+1}\right)\sum\limits_{n=q+1}^{\infty} n|a_n|} \leq 1,$$

which leads us immediately to the assertion of the Theorem 5.3. □

6. Neighbourhood for the class $S_\lambda^{\mu,m,\xi}(\sigma,\upsilon,\varrho)$

In this section, we determine the neighbourhoods for the class $S_\lambda^{\mu,m,\xi}(\sigma,\upsilon,\varrho)$ which we define as follows:

Definition 6.1. A function $u \in A$ is said to be in the class $S_\lambda^{\mu,m,\xi}(\sigma,\upsilon,\varrho)$, if there exist a function $g \in S_\lambda^{\mu,m}(\sigma,\upsilon,\varrho)$ such that

$$\left|\frac{u(z)}{g(z)} - 1\right| < 1-\upsilon,\ (z \in E,\ \ 0 \le \upsilon < 1). \tag{6.1}$$

For any function $u(z) \in A$, $z \in E$ and $\delta \ge 0$, we define

$$N_{n,\delta}(u) = \left\{ g \in \Sigma : g(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad \text{and} \quad \sum_{n=2}^{\infty} n|a_n - b_n| \le \delta \right\}, \tag{6.2}$$

which is the $(n,\delta)-$neighbourhood of $u(z)$.

The concept of neighbourhoods was first introduced by Goodman [5] and generalized by Ruscheweyh [13].

Theorem 6.2. *If $g \in S_\lambda^{\mu,m}(\sigma,\upsilon,\varrho)$ and*

$$\xi = 1 - \frac{\delta(1-\upsilon)}{2[(1-\upsilon)-(1+\sigma)(2+\varrho-\upsilon)\phi(2)]}, \tag{6.3}$$

then $N_{n,\delta}(g) \subset S_\lambda^{\mu,m,\xi}(\sigma,\upsilon,\varrho)$.

Proof. Suppose $u \in N_{n,\delta}(g)$. We then find from (6.2) that

$$\sum_{n=2}^{\infty} n|a_n - b_n| \le \delta, \tag{6.4}$$

which yields the coefficient inequality

$$\sum_{n=2}^{\infty} |a_n - b_n| \le \frac{\delta}{2}\ (n \in \mathbb{N}). \tag{6.5}$$

Next, since $g \in S_\lambda^{\mu,m}(\sigma,\upsilon,\varrho)$, we have

$$\sum_{n=2}^{\infty} b_n \le \frac{(1+\sigma)(2+\varrho-\upsilon)\phi(2)}{1-\upsilon}. \tag{6.6}$$

So that

$$\left|\frac{u(z)}{g(z)}-1\right| < \frac{\sum\limits_{n=2}^{\infty}|a_n-b_n|}{1-\sum\limits_{n=2}^{\infty}b_n}$$
$$= \frac{\delta(1-\upsilon)}{2[(1-\upsilon)-(1+\sigma)(2+\varrho-\upsilon)\phi(2)]}$$
$$= 1-\xi$$

provided ξ is given by (6.3). Thus the proof of the is completed. □

7. Conclusion

In this study, we have introduced and investigated a novel subclass of analytic functions defined in the open unit disk, associated with the Wright function. By employing the techniques of subordination and using the properties of the Wright function, we have derived several analytic and geometric properties of functions belonging to this class. These include coefficient bounds, distortion inequalities, growth estimates, and inclusion relationships with other well-known subclasses of analytic functions.

Our results generalize and unify various earlier findings in the context of geometric function theory by linking the special function framework specifically the Wright function with the classical theory of univalent functions. The obtained results not only highlight the structural richness of this subclass but also demonstrate the applicability of special functions in constructing and analyzing new families of analytic functions.

The current work opens pathways for further exploration, particularly in the direction of fractional calculus operators, convolution properties, and applications in complex differential equations. Future studies may also consider extending the analysis to multivalent or bi-univalent functions associated with generalized Wright-type functions or other special functions.

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