Nonlinear Functional Analysis and Applications
Vol. 19, No. 1 (2014), pp. 131-143

http://nfaa.kyungnam.ac.kr/jour-nfaa.htm

NORMAL STRUCTURE AND POLYGONS IN BANACH SPACES

Satit Saejung[1] **and Ji Gao**[2]

[1]Department of Mathematics, Faculty of Science
Khon Kaen University, Khon Kaen 40002, Thailand
e-mail: saejung@kku.ac.th

[2]Department of Mathematics, Community College of Philadelphia,
Philadelphia, PA 19130-3991, USA
e-mail: jgao@ccp.edu

Abstract. Let X be a Banach space with the unit sphere $S(X)$. In this paper, we study the inscribed isosceles triangles, parallelograms and circumscribed hexagons in $S(X)$. The relationships of these polygons with normal structure, uniformly non-square and other geometric properties are obtained. Some existing results about fixed points and normal structure are improved.

1. Introduction

Let X be a normed linear space. Let $B(X) = \{x \in X : \|x\| \leq 1\}$ and $S(X) = \{x \in X : \|x\| = 1\}$ be the unit ball, and the unit sphere of X, respectively. Let X^* be the dual space of X.

Brodskiĭ and Mil'man [2] introduced the following geometric concepts in 1948:

Definition 1.1. A bounded and convex subset K of a Banach space X is said to have normal structure if every convex subset H of K that contains more than one point contains a point $x_0 \in H$, such that $\sup\{\|x_0 - y\| : y \in H\} < d(H)$, where $d(H) = \sup\{\|x - y\| : x, y \in H\}$ denotes the diameter

[0]Received September 14, 2013. Revised January 9, 2014.
[0]2010 Mathematics Subject Classification: 46B20, 46C05, 52A07.
[0]Keywords: Fixed points, inscribed polygons, normal structure, ultraproduct, uniformly nonsquare.
[0]Correspointing author: Ji Gao.

of H. A Banach space X is said to have normal structure if every bounded and convex subset of X has normal structure. A Banach space X is said to have weak normal structure if for each weakly compact convex set K in X has normal structure. X is said to have uniform normal structure if there exists $0 < c < 1$ such that for any subset bounded closed convex subset K of X that contains more than one point, there exists $x_0 \in K$ such that $\sup\{\|x_0 - y\| : y \in K\} \leq c \cdot d(K)$.

For a reflexive Banach space, the normal structure and weak normal structure coincide.

In 1965, Kirk [10] proved that if a Banach space X has weak normal structure then it has weak fixed point property, that is, every nonexpansive mapping from a weakly compact and convex subset of X into itself has a fixed point.

In this paper, we study the inscribed isosceles triangle and parallelogram and circumscribed hexagon in $S(X)$. The relationships of these polygons with normal structure, uniformly non-square and other geometric properties are obtained. Finally we consider uniform normal structure in section 3. Some existing results about fixed points and normal structure are improved.

2. Inscribed and circumscribed Polygons and Normal structure

Definition 2.1. ([5]) Let X be a Banach space. A hexagon H in X is called a normal hexagon if the length of each side is 1 and each pair of two opposite sides are parallel.

Remark 2.2. The concept of normal hexagon is different from the concept of regular hexagon in Euclidean spaces. We may consider the normal hexagon as an image of a regular hexagon under a bounded linear mapping from an Euclidean space to a Banach space.

Lemma 2.3. ([5], [8]) *Let X be a Banach space without weak normal structure. Then for any $0 < \delta < 1$, there are x_1, x_2, and x_3 in $S(X)$ satisfying*

(i) $x_2 - x_3 = x_1$;
(ii) $\|\frac{x_1+x_2}{2}\| > 1 - \delta$;
(iii) $\|\frac{x_3+(-x_1)}{2}\| > 1 - \delta$.

The geometric meaning of Lemma 2.3 is that if a Banach space X fails to have weak normal structure then there is an inscribed normal hexagons with four sides arbitrarily closed to the unit sphere $S(X)$.

Lemma 2.4. ([7]) *If $x, y \in B(X)$ and $0 < \epsilon < 1$ are such that $\frac{\|x+y\|}{2} > 1 - \epsilon$, then for all $0 \le c \le 1$ and $z = cx + (1-c)y \in [x, y]$, the line segment connecting x and y, it follows that $\|z\| > 1 - 2\epsilon$.*

A curve in a Banach space X is a continuous mapping $x : [a, b] \to X$ and in this case it is denoted by $C := \{x(t) : a \le t \le b\}$. A curve is called simple if it does not have multiple points. A curve is called closed if $x(a) = x(b)$. A closed curve is called symmetric about the origin if $x \in C$, then also $-x \in C$.

The concept of the length of a curve in Banach spaces resembles the same concept in Euclidean spaces. For a curve $C = \{x(t) : t \in [a, b]\}$ and a partition $P := \{t_0, t_1, t_2, \dots, t_n\} \subset [a, b]$ where

$$a = t_0 < t_1 < t_2 < ... < t_i < ... < t_n = b,$$

let

$$l(C, P) = \sum_{i=1}^{n} \|x(t_i) - x(t_{i-1})\|.$$

The length $l(C)$ of a curve C is defined as the least upper bound of $l(C, P)$ for all partitions P of $[a, b]$, that is,

$$l(C) = \sup\{l(C, P) : P \text{ is a partition of } [a, b]\}.$$

If $l(C)$ is finite, then the curve C is called rectifiable.

For a curve $C = \{x(t) : t \in [a, b]\}$ and $a \le t \le b$, let $l_a^t(C)$ denote the length of the curve $\{x(s) : s \in [a, t]\}$. If C is rectifiable, then $l_a^t(C)$ is a continuous function of t.

Theorem 2.5. ([3], [12]) *Let X_2 be a two-dimensional Banach space and K_1, K_2 be closed convex subsets of X_2 with nonvoid interiors. If $K_1 \subseteq K_2$, then $l(\partial(K_1)) \le l(\partial(K_2))$, where $l(\partial(K_i))$ denote the lengths of the circumferences of $K_i, i = 1, 2$.*

Definition 2.6. ([3], [12]) Let $C = \{x(t) : t \in [a, b]\}$ be a rectifiable curve in a normed space X. Suppose that for $0 \le s \le l(C)$ the element $y(s)$ represents the point $x(t) \in C$ for which $l_a^t(C) = s$, Then the curve $\{y(s) : 0 \le s \le l(C)\}$ is called the standard representation of C.

For a normed linear space X, it is clear that $S(X_2)$ is a simple closed curve which is symmetric about the origin and unique up to orientation. For $x \in S(X_2)$, let κ be the one of arcs of $S(X_2)$ from x to $-x$, and let $g : [0, L] \to \kappa$ be the standard representation in terms of arc length, where L is the length of κ.

Theorem 2.7. ([7]) *The functions* ϕ*, and* ψ*:* $[0, L] \to [0, 2]$ *defined by* $\phi = \|g(s) - x\|$*, and* $\psi = \|g(s) + x\|$ *are continuously increasing, and decreasing functions respectively.*

Theorem 2.8. ([12]) *Let* X_2 *be a two-dimensional Banach space. The following statements are true:*

(1) $6 \leq l(S(X_2)) \leq 8$*;*
(2) $l(S(X_2)) = 8$ *if and only if* $S(X_2)$ *is a parallelogram;*
(3) $l(S(X_2)) = 6$ *if and only if* $S(X_2)$ *is an affinely regular hexagon.*

Let $\delta_X(\epsilon) = \inf\{1 - \frac{\|x+y\|}{2} : x, y \in S(X), \|x - y\| \geq \epsilon\}$ where $0 \leq \epsilon \leq 2$ be the modulus of convexity of X. (For example, see [4].) Compare to the concept of modulus of convexity, the modulus of flatness is defined as follows:

Definition 2.9. The $\alpha_X(\epsilon) = \sup\{1 - \frac{\|x+y\|}{2} : x, y \in S(X), \|x - y\| \leq \epsilon\}$ where $0 \leq \epsilon \leq 2$ is called the modulus of flatness of X.

Proposition 2.10. *For any Banach space* X*, the following are true:*

(1) $0 \leq \alpha_X(\epsilon) \leq \frac{\epsilon}{2}$*;*
(2) *If* X *is a Hilbert space, then* $\alpha_X(\epsilon) = \delta_X(\epsilon) = 1 - \frac{\sqrt{4-\epsilon^2}}{2}$.

For any polygon Y inscribed in a two dimensional space, let $p(Y)$ denote the perimeter of Y. We now define the following concept.

Definition 2.11. Let X be a normed space. Define

$$l_X(\triangle) := \sup\left\{p(\triangle) : \begin{array}{c} \triangle \text{ is an isosceles inscribed triangle} \\ \text{in a two dimensional subspace of } X \end{array}\right\}.$$

It is clear that if X is a Banach space, then

(1) $l_X(\triangle) \leq 6$;
(2) $l_X(\triangle) = 3\sqrt{3}$ whenever X is a Hilbert space.

Theorem 2.12. *If a Banach space* X *fails to have weak normal structure, then*

$$l_X(\triangle) + 2\alpha_X(1) \geq 6.$$

Proof. Suppose X fails to have weak normal structure and $\epsilon > 0$. First let x_1, x_2, and x_3 in $S(X)$ satisfy the three conditions in Lemma 2.3. It follows then that

$$\|x_1 - x_3\| \geq 2 - 4\epsilon,$$

$$\|x_1 + x_2\| \geq 2 - 4\epsilon,$$

and

$$\|x_3 - (-x_2)\| = 2\frac{\|x_3 + x_2\|}{2} = 2 - 2(1 - \frac{\|x_3 + x_2\|}{2}) \geq 2 - 2\alpha_X(1).$$

Secondly, let X_2 be the two-dimensional subspace of X containing x_1, x_2, x_3. Let $x_1' \in S(X_2)$ be such that $\|x_1' - x_3\| = \|x_1' + x_2\|$, then from Theorem 2.7, we have

$$\|x_1' - x_3\| = \|x_1' + x_2\| \geq 2 - 4\epsilon.$$

Finally, let $\triangle$ be the isosceles triangle whose vertices are x_1', x_2 and x_3. So,

$$l_X(\triangle) \geq \|x_1' - x_3\| + \|x_1' + x_2\| + \|x_3 - (-x_2)\| \geq 6 - 2\alpha_X(1) - 8\epsilon.$$

Since ϵ can be arbitrary small, we have

$$l_X(\triangle) + 2\alpha_X(1) \geq 6.$$

□

Definition 2.13. ([9]) A normed linear space X is uniformly nonsquare if there exists a $\delta > 0$ such that for any $x, y \in S(X)$, either $\|x + y\| \leq 2(1 - \delta)$ or $\|x - y\| \leq 2(1 - \delta)$.

Theorem 2.14. *If a Banach space X is not uniformly nonsquare then $l_X(\triangle) = 6$.*

Proof. Since X is not uniformly nonsquare, for any $\delta > 0$, there are $x, y \in S(X)$ such that $\|x + y\| \geq 2(2 - \delta)$ and $\|x - y\| \geq 2(2 - \delta)$. We consider the triangle $\triangle$ with vertices x, y and $-x$. Using the same technique as the preceding theorem, we have $l_X(\triangle) \geq 6 - 4\delta$. Since δ can be arbitrary small, we have $l_X(\triangle) = 6$. □

Corollary 2.15. *A Banach space X with $l_X(\triangle) < 6$ is uniformly nonsquare, therefore X is super-reflexive and then reflexive.*

Corollary 2.16. *A Banach space X with $l_X(\triangle) + 2\alpha_X(1) < 6$ has normal structure.*

Definition 2.17. Let X be a normed space. Define

$$l_X(\diamond) := \sup \left\{ p(\diamond) : \begin{array}{c} \diamond \text{ is a parallelogram in a two} \\ \text{dimensional subspace of } X \end{array} \right\}.$$

It is clear that for any Banach space X, then

(1) $l_X(\diamond) \leq 8$;
(2) $l_X(\diamond) = 4\sqrt{2}$ provided that X is a Hilbert space.

Let $\rho_X(\tau) = \frac{1}{2}\sup\{\|x+y\| + \|x-y\| - 1 : x \in S(X), \|y\| = \tau\}$ be the modulus of smoothness of X, where $\tau \geq 0$ (for example, see [4]). In 1982, Turett [13] proved the following.

Theorem 2.18. *For a Banach space X, if $\lim_{\tau\to 0}\frac{\rho_X(\tau)}{\tau} < 1/2$ then X has uniform normal structure.*

In 2006, Gao [6] proved the following

Theorem 2.19. *For a Banach space X, if $\rho_X(1) < \frac{\sqrt{5}-1}{2}$ then X has uniform normal structure.*

In 2000, Baronti, et al. [1] introduced the parameter

$$A_2(X) = \frac{1}{2}\sup\{\|x+y\| + \|x-y\| : x, y \in S(X)\}.$$

It is easy to see that $l_X(\diamond) = 4A_2(X) = 4(\rho_X(1)+1)$.

Theorem 2.20. *If a Banach space X fails to have weak normal structure, then for any $\epsilon > 0$, there exists a two dimensional subspace X_2 of X such that*

(i) *there exists an inscribed parallelogram $\diamond$ in $S(X_2)$ such that*

$$p(\diamond) \geq l(S(X_2)) - \epsilon;$$

(ii) *there exists a circumscribed hexagon $\overline{H}$ of $S(X_2)$ such that*

$$p(\overline{H}) \leq l(S(X_2)) + \epsilon.$$

Proof. Suppose X does not have weak normal structure. For $\epsilon > 0$, let x_1, x_2 and x_3 in $S(X)$ satisfying the three conditions in Lemma 2.3. Let X_2 be the two dimensional space spanned by x_1, x_2 and x_3. (see Figure 1 below).

Since $\frac{\|x_1+x_2\|}{2} > 1-\epsilon$, it follows that $\|z\| > 1-2\epsilon$ for any $z \in [x_1, x_2]$. So, the line segment

$$[\frac{x_1}{1-2\epsilon}, \frac{x_2}{1-2\epsilon}] \subseteq X_2 \backslash B_0(X_2),$$

where $B_0(X_2) = \{x : \|x\| < 1\}$ is a open unit ball of X.

Let $x = -x_1 + 2(x_3 + x_1) = x_1 + 2(x_2 - x_1) \in X_2$. Then

$$\|x + x_1\| = \|x - x_1\| = 2.$$

Let $z_2 \in [x, x_2]$ and $z_3 \in [x, x_3]$ be such that $z_2 - z_3 = \alpha x_1$ for some $0 \leq \alpha \leq 1$, $[z_2, z_3] \subseteq X_2\backslash B_0(X_2)$, and $[z_2, z_3] \bigcap S(X_2) \neq \emptyset$. Then

$$\|z_2 - z_3\| + \|z_3 - x_3\| = 1.$$

Let $z_1 \in [z_2, z_3] \bigcap S(X_2)$, we have

$$\|z_2 - z_1\| + \|z_1 - z_3\| + \|z_3 + x_1\| = 2. \tag{2.1}$$

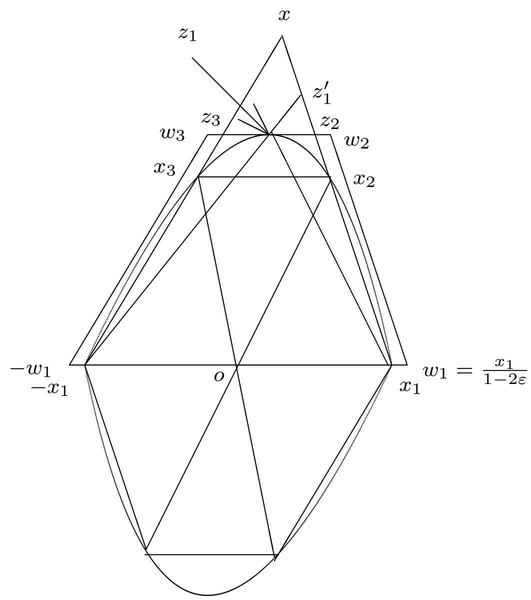

FIGURE 1. For the proof of Theorem 2.20.

Let $z_1' \in [x, x_2]$ such that $z_1 \in [-x_1, z_1']$. Then it follows from Lemma 2.4 that

$$2 - 4\epsilon \leq \|z_1' + x_1\| \leq 2. \tag{2.2}$$

Since $\frac{\|z_1'-z_1\|}{\|z_1'+x_1\|} = \frac{\|z_1-z_2\|}{2\|x_1\|}$, we have

$$\|z_1' - z_1\| \leq \|z_1 - z_2\| \tag{2.3}$$

From (2.1), (2.2) and (2.3), we have

$$\begin{aligned} \|z_1 - (-x_1)\| &= \|z_1 + x_1\| \\ &= \|z_1' + x_1\| - \|z_1' - z_1\| \\ &\geq 2 - \|z_1' - z_1\| - 4\epsilon \\ &\geq 2 - \|z_1 - z_2\| - 4\epsilon \\ &= \|z_1 - z_3\| + \|z_3 + x_1\| - 4\epsilon, \end{aligned}$$

that is,

$$\|z_1 - (-x_1)\| \geq \|z_1 - z_3\| + \|z_3 + x_1\| - 4\epsilon. \tag{2.4}$$

Similarly, we have

$$\|z_1 - x_1\| \geq \|z_1 - z_2\| + \|z_2 - x_1\| - 4\epsilon. \tag{2.5}$$

From (2.4) and (2.5), we have

$$\begin{aligned}
&\|z_1 - (-x_1)\| + \|z_1 - x_1\| \\
&\geq \|z_1 - z_2\| + \|z_2 - x_1\| + \|z_1 - z_3\| + \|z_3 + x_1\| - 8\epsilon \\
&= \|z_2 - x_1\| + \|z_2 - z_3\| + \|z_3 + x_1\| - 8\epsilon,
\end{aligned}$$

that is,

$$\|z_1 - (-x_1)\| + \|z_1 - x_1\| \geq \|z_2 - x_1\| + \|z_2 - z_3\| + \|z_3 + x_1\| - 8\epsilon. \quad (2.6)$$

Let $w_1 = x_1 + \frac{2\epsilon}{1-2\epsilon}x_1$, $w_2 = z_2 + \frac{2\epsilon}{1-2\epsilon}x_1$, and $w_3 = z_3 - \frac{2\epsilon}{1-2\epsilon}x_1$. We consider the hexagon $\overline{H}$ with vertices w_1, w_2, w_3, $-w_1$, $-w_2$, and $-w_3$. Then

$$B(X) \subseteq \overline{co}(\overline{H}),$$

where $\overline{co}(\overline{H})$ denotes the closed convex hull of $\overline{H}$, $\|w_2 - w_1\| = \|z_2 - x_1\|$, $\|w_3 - w_2\| = \|z_3 - z_2\| + \frac{4\epsilon}{1-2\epsilon}$, and $\|w_3 + w_1\| = \|z_3 + x_1\|$. From (2.6), we have

$$\begin{aligned}
l(\overline{H}) &= 2(\|w_2 - w_1\| + \|w_3 - w_2\| + \|w_3 + w_1\|) \\
&= 2(\|z_2 - x_1\| + \|z_3 - z_2\| + \|z_3 + x_1\| + \frac{8\epsilon}{1-2\epsilon}) \\
&\leq 2(\|z_1 - x_1\| + \|z_1 + x_1\| + \frac{8\epsilon}{1-2\epsilon} + 8\epsilon).
\end{aligned}$$

We next consider the parallelogram $\diamond$ with vertices $z_1, x_1, -z_1$, and $-x_1$. Then $l(\overline{H}) - 64\epsilon \leq l(\diamond)$. Since $\overline{co}(\diamond) \subseteq \overline{co}(S(X_2)) \subseteq \overline{co}(\overline{H})$, it follows from Theorem 2.5 that $l(\diamond) \leq l(S(X_2)) \leq l(\overline{H})$. Now we have

$$l(\overline{H}) \leq l(S(X_2)) + 64\epsilon$$

and

$$l(\diamond) \geq l(S(X_2)) - 64\epsilon,$$

as desired. □

Definition 2.21. Let X be a normed space. Define

$$d(X) := \inf \left\{ \begin{array}{l} \max\left\{1 - \|\frac{x_1+x_2}{2}\|, 1 - \|\frac{x_2+x_3}{2}\|, 1 - \|\frac{x_3+(-x_1)}{2}\|\right\} : \\ x_1,\ x_2,\ x_3,\ -x_1,\ -x_2, \text{ and } -x_3 \text{ are counterclockwise} \\ \text{vertices of an inscribed normal hexagon in a two} \\ \text{dimensional subspace of } X \end{array} \right\}.$$

It is easy to see that $0 \leq d(X) \leq \frac{1}{2}$.

Theorem 2.22. *If a Banach space X fails to have weak normal structure, then for any $\epsilon > 0$, there exists a two-dimensional subspace $X_2 \subseteq X$ and an inscribed parallelogram $\diamond$ in $S(X_2)$ such that*

$$l_X(\diamond) \geq (4 + \frac{2}{1-d(X)})(1-\epsilon).$$

Proof. Suppose X does not have weak normal structure. For $\varepsilon > 0$, let x_1, x_2 and x_3 in $S(X)$ satisfying the three conditions in Lemma 2.3. Let X_2 be the two dimensional space spanned by x_1, x_2 and x_3. (See the Figure 2 below).

Let $x = -x_1 + 2(x_3 + x_1) = x_1 + 2(x_2 - x_1) \in X_2$ be as in the preceding theorem, and let $x'' = [0, x] \bigcap S(X_2)$ and $x'' = \beta \frac{x_2+x_3}{2} = \beta \frac{2x_2 - x_1}{2}$, we have

$$\beta \geq \frac{1}{1-d(X)},$$

and

$$\begin{aligned} x'' - x_1 &= \beta \frac{x_2 + x_3}{2} - x_1 \\ &= \beta \frac{2x_2 - x_1}{2} - x_1 \\ &= \beta x_2 - (\frac{\beta}{2} + 1)x_1 \\ &= (\frac{\beta}{2} + 1)(\frac{2\beta}{\beta+2}x_2 - x_1). \end{aligned}$$

Since $\frac{2\beta}{\beta+2} \leq 1$, we have

$$\|\frac{2\beta}{\beta+2}x_2 - x_1\| \geq 1 - 2\epsilon.$$

Therefore, we have

$$\|x'' - x_1\| \geq (\frac{\beta}{2} + 1)(1 - 2\epsilon).$$

Similarly, we have

$$\|x'' + x_1\| \geq (\frac{\beta}{2} + 1)(1 - 2\epsilon).$$

We consider the inscribed parallelogram with vertices x'', x_1, $-x''$ and $-x_1$. Then

$$\begin{aligned} l(\diamond) &\geq 4(\frac{\beta}{2} + 1)(1 - 2\epsilon) \\ &= (4 + 2\beta)(1 - 2\epsilon) \\ &\geq (4 + \frac{2}{1-d(X)})(1 - 2\epsilon). \end{aligned}$$

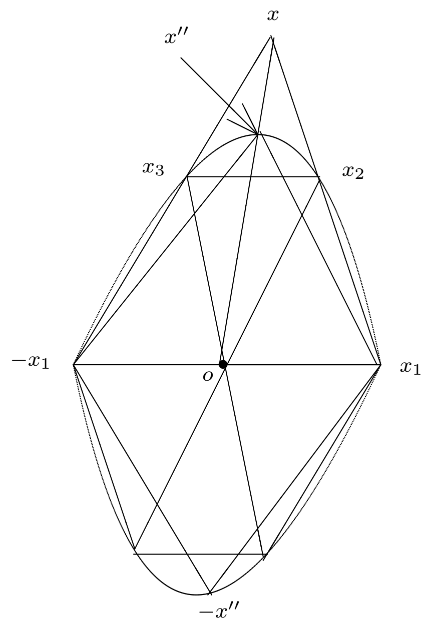

FIGURE 2. For the proof of Theorem 2.22

□

As a consequence of the preceding theorem, we obtain the following result.

Theorem 2.23. *For a Banach space X, if $l_X(\diamond) < 4 + \frac{2}{1-d(X)}$, then X has weak normal structure.*

Theorem 2.24. *If Banach space X is not uniform nonsquare, then $l_X(\diamond) = 8$.*

Proof. We consider the parallelogram $\diamond$ with vertices $x, y, -x$ and $-y$ in the definition of uniformly nonsquare, we have $l_X(\diamond) \geq 8 - 8\delta$. Since δ can be arbitrary small, we have $l_X(\diamond) = 8$. □

Corollary 2.25. *For a Banach space X, if $l_X(\diamond) < 8$, then X is uniformly nonsquare, therefore X is super-reflexive and then reflexive.*

Corollary 2.26. *For a Banach space X, if $l_X(\diamond) < 4 + \frac{2}{1-d(X)}$, or $A_2(X) < 1 + \frac{1}{2(1-d(X))}$, or $\rho_X(1) < \frac{1}{2(1-d(X))}$, then X has normal structure.*

3. Uniform Normal Structure

Let $\mathcal{F}$ be a filter on an index set I, and let $\{x_i\}_{i\in I}$ be a subset in a Hausdorff topological space X, $\{x_i\}_{i\in I}$ is said to converge to x with respect to $\mathcal{F}$, denote by $\lim_{\mathcal{F}} x_i = x$, if for each neighborhood V of x, $\{i \in I : x_i \in V\} \in \mathcal{F}$. A filter $\mathcal{U}$ on I is called an ultrafilter if it is maximal with respect to the ordering of the set inclusion. An ultrafilter is called trivial if it is of the form $\{A : A \subseteq I, i_0 \in A\}$ for some $i_0 \in I$. We will use the fact that if $\mathcal{U}$ is an ultrafilter, then

(i) for any $A \subseteq I$, either $A \in \mathcal{U}$ or $I \setminus A \in \mathcal{U}$;
(ii) if $\{x_i\}_{i\in I}$ has a cluster point x, then $\lim_{\mathcal{U}} x_i$ exists and equals to x.

Let $\{X_i\}_{i\in I}$ be a family of Banach spaces and let $l_\infty(I, X_i)$ denote the subspace of the product space equipped with the norm $\|(x_i)\| = \sup_{i\in I} \|x_i\| < \infty$.

Definition 3.1. ([11]) Let $\mathcal{U}$ be an ultrafilter on I and let

$$N_{\mathcal{U}} = \{(x_i) \in l_\infty(I, X_i) : \lim_{\mathcal{U}} \|x_i\| = 0\}.$$

The ultraproduct of $\{X_i\}_{i\in I}$ is the quotient space $l_\infty(I, X_i)/N_{\mathcal{U}}$ equipped with the quotient norm.

We will use $(x_i)_{\mathcal{U}}$ to denote the element of the ultraproduct. It follows from the assertion (ii) above, and the definition of quotient norm that

$$\|(x_i)_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|x_i\| \tag{3.1}$$

In the following we will restrict our index set I to be $\mathbb{N}$, the set of natural numbers, and let $X_i = X$ for all $i \in \mathbb{N}$ for some Banach space X. For an ultrafilter $\mathcal{U}$ on $\mathbb{N}$, we use $X_{\mathcal{U}}$ to denote the ultraproduct.

Theorem 3.2. *For any Banach space X, any $\epsilon > 0$, and for any nontrivial ultrafilter U on $\mathbb{N}$, $\alpha_{X_{\mathcal{U}}}(\epsilon) = \alpha_X(\epsilon)$, $l_{X_{\mathcal{U}}}(\triangle) = l_X(\triangle)$, $d(X_{\mathcal{U}}) = d(X)$, and $l_{X_{\mathcal{U}}}(\diamond) = l_X(\diamond)$.*

Proof. We only prove $l_{X_{\mathcal{U}}}(\triangle) = l_X(\triangle)$, and the proofs of the rest are the same. Since X can be isometrically embedded onto $X_{\mathcal{U}}$, we have $l_X(\triangle) \le l_{X_{\mathcal{U}}}(\triangle)$.

To prove the reverse inequality, for any $\eta > 0$ we choose $(x_i)_{\mathcal{U}} \in S(X_{\mathcal{U}})$, $(y_i)_{\mathcal{U}} \in S(X_{\mathcal{U}})$, and $(z_i)_{\mathcal{U}} \in S(X_{\mathcal{U}})$ in a two dimensional subspace of $X_{\mathcal{U}}$ such that

$$\|(x_i)_{\mathcal{U}} - (y_i)_{\mathcal{U}}\| + \|(y_i)_{\mathcal{U}} - (z_i)_{\mathcal{U}}\| + \|(z_i)_{\mathcal{U}} - (x_i)_{\mathcal{U}}\| > l_{X_{\mathcal{U}}}(\triangle) - \eta.$$

Then x_i, y_i and z_i are in some two dimensional subspace $(X_i)_2$, for each i. From the remarks (i) and (ii), the expression (3.1) and the paragraphs above, the following sets:

$$\begin{aligned}
C &= \{i \in \mathbb{N} : 1-\eta \le \|x_i\| \le 1-\eta\},\\
D &= \{i \in \mathbb{N} : 1-\eta \le \|y_i\| \le 1-\eta\},\\
E &= \{i \in \mathbb{N} : 1-\eta \le \|z_i\| \le 1-\eta\}, \text{ and}\\
F &= \{i \in \mathbb{N} : \|x_i - y_i\| + \|y_i - z_i\| + \|z_i - x_i\| > l_{X_\mathcal{U}}(\triangle) - \eta\}.
\end{aligned}$$

are all in $\mathcal{U}$. So the intersection $C \cap D \cap E \cap F$ is in $\mathcal{U}$ too, and is hence not empty.

Let $i \in C \cap D \cap E \cap F$ and $(X_i)_2$ be a two dimensional subspace of X spanned by x_i y_i and z_i, we have $l_X(\triangle) > l_{X_\mathcal{U}}(\triangle) - \eta$. Since η can be arbitrarily small, we conclude that $l_X(\triangle) \ge l_{X_U}(\triangle)$. □

Theorem 3.3. *If X is a Banach space with $l_X(\triangle) + 2\alpha_X(1) < 6$, then X has uniform normal structure.*

Proof. The idea of the proof is same as that of Theorem 4.4 in [7]. Suppose $l_X(\triangle) + 2\alpha_X(1) < 6$, and X does not have uniform normal structure, we find a sequence $\{C_n\}$ of bounded closed convex subsets of X such that for each n,

$$0 \in C_n, \quad d(C_n) = 1, \text{ and}$$

$$\operatorname{rad}(C_n) = \inf\{\sup\{\|x-y\| : y \in C_n\} : x \in C_n\} > 1 - \frac{1}{n}.$$

Let $\mathcal{U}$ be any nontrivial ultrafilter on $\mathbb{N}$, and let

$$C = \{(x_n)_\mathcal{U} : x_n \in C_n, n \in \mathbb{N}\}.$$

Then C is a nonempty bounded closed convex subset of $X_\mathcal{U}$. It follows from the properties of C_n above that $d(C) = \operatorname{rad}(C) = 1$, so $X_\mathcal{U}$ does not have normal structure. On the other hand, from Theorem 3.2, we have

$$l_{X_\mathcal{U}}(\triangle) + 2\alpha_{X_\mathcal{U}}(1) = l_X(\triangle) + 2\alpha_X(1) < 6.$$

This contradicts to Corollary 2.16, and hence X must have uniform normal structure. □

Theorem 3.4. *For a Banach space X, if $l_X(\diamond) < 4 + \frac{2}{1-d(X)}$ or $A_2(X) < 1 + \frac{1}{2(1-d(X))}$ or $\rho_X(1) < \frac{1}{2(1-d(X))}$, then X has uniform normal structure.*

Remark 3.5. Theorem 3.4 improves Theorem 2.19 for Banach spaces X with $d(X) \ge \frac{\sqrt{5}-2}{\sqrt{5}-1}$.

References

[1] M. Baronti, E. Casini and P.L. Papini, *Triangles Inscribed in a Semicircle, in Monkowski Planes and in Normed Spaces*, J. Math. Anal. Appl, **252(1)** (2000), 124-146.

[2] M.S. Brodskiĭ and D.P. Mil'man, *On the center of a convex set*(Russian), Doklady Akad. Nauk SSSR (N.S.), **59** (1948), 837–840.

[3] H. Buseman, The Geometry of Geodesics, Academic Press, New York(1955).

[4] J. Diestel, The Geometry of Banach Spaces - Selected Topics Lecture Notes in Math., Vol. 485, Spring - Verlag, Berlin and New York(1975).

[5] J. Gao, *Normal Structure hexagon and more general Banach Spaces with Uniform Normal Structure*, J. Math. (Wuhan), **20(3)** (2000), 241-248.

[6] J. Gao, *On Modulus of Convexity and Smooethness in Banach Spaces*, Adv. Algebra Anal., **1(1)** (2006), 39-50.

[7] J. Gao and K.S. Lau, *On Two Classes of Banach Spaces with Normal Structure*, Studia Mathematica, **99(1)** 1991, 41-56.

[8] J. Gao and S. Saejung, *A Constant Related to Fixed points and Normal Structure in Banach Spaces*, Nonlinear Functional Analysis and Applications, **16(1)** (2011), 17-28.

[9] R.C. James, *Uniformly Nonsquare Banach Spaces*, Annals of Math., **80** (1964), 542-550.

[10] W.A. Kirk, *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Monthly, **72** (1965), 1004–1006.

[11] B. Sims, "Ultra"-techniques in Banach space theory. Queen's Papers in Pure and Applied Mathematics, 60. Queen's University, Kingston, ON, 1982.

[12] J.J. Schăffer, Geometry of Spheres in Normed Spaces, Marcel Dekker, New York (1976).

[13] B. Turett, A dual view of a theorem of Baillon. Nonlinear analysis and applications (St. Johns, Nfld., 1981), pp. 279–286, Lecture Notes in Pure and Appl. Math., 80, Dekker, New York, 1982.