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FIXED POINT THEOREMS OF CIRIC QUASI CONTRACTIVE OPERATOR AND SP-ITERATION SCHEMES IN CAT(0) SPACES

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Abstract. In this paper, we study the strong convergence of SP-iterative scheme for Ciric quasi contractive operator in the framework of CAT(0) spaces. Our results improve and extend some corresponding results from the existing literature (see, e.g., [2, 29] and many others).

1. Introduction

A metric space X is a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as "thin" as its comparison triangle in the Euclidean plane. The precise definition is given below. It is well known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a CAT(0) space. Other examples include Pre-Hilbert spaces (see [4]), $\mathbb{R}$-trees (see [20]), Euclidean buildings (see [5]), the complex Hilbert ball with a hyperbolic metric (see [13]), and many others. For a thorough discussion of these spaces and of the fundamental role they play in geometry, we refer the reader to Bridson and Haefliger [4].

Fixed point theory in CAT(0) spaces was first studied by Kirk (see [21, 22]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since, then the fixed point theory for single-valued and multi-valued

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mappings in CAT(0) spaces has been rapidly developed, and many papers have appeared (see, e.g., [1], [7], [10]-[12], [14], [18]-[19], [23]-[24], [26], [30]-[31] and references therein). It is worth mentioning that the results in CAT(0) spaces can be applied to any CAT(k) space with $k \leq 0$ since any CAT(k) space is a CAT(k') space for every $k' \geq k$ (see, e.g., [4]).

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$, and let $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry, and $d(x, y) = l$. The image α of c is called a geodesic (or metric) *segment* joining x and y. We say X is (i) a *geodesic space* if any two points of X are joined by a geodesic and (ii) *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$, which we will denoted by $[x, y]$, called the segment joining x to y.

A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the *edges* of $\triangle$). A *comparison triangle* for geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\overline{x_i}, \overline{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. Such a triangle always exists (see [4]).

CAT(0) space

A geodesic metric space is said to be a $CAT(0)$ space if all geodesic triangles of appropriate size satisfy the following $CAT(0)$ comparison axiom.

Let $\triangle$ be a geodesic triangle in X, and let $\overline{\triangle} \subset \mathbb{R}^2$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the $CAT(0)$ inequality if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x, y) \leq d_{\mathbb{R}^2}(\overline{x}, \overline{y}). \tag{1.1}$$

Complete $CAT(0)$ spaces are often called *Hadamard spaces* (see [17]). If x, y_1, y_2 are points of a $CAT(0)$ space and y_0 is the mid point of the segment $[y_1, y_2]$ which we will denote by $(y_1 \oplus y_2)/2$, then the $CAT(0)$ inequality implies

$$d^2\Big(x, \frac{y_1 \oplus y_2}{2}\Big) \leq \frac{1}{2}\ d^2(x, y_1) + \frac{1}{2}\ d^2(x, y_2) - \frac{1}{4}\ d^2(y_1, y_2). \tag{1.2}$$

The inequality (1.2) is the (CN) inequality of Bruhat and Tits [6]. The above inequality has been extended in [11] as

$$d^2(z, \alpha x \oplus (1 - \alpha) y) \leq \alpha d^2(z, x) + (1 - \alpha) d^2(z, y) - \alpha(1 - \alpha) d^2(x, y). \tag{1.3}$$

for any $\alpha \in [0,1]$ and $x, y, z \in X$.

Let us recall that a geodesic metric space is a $CAT(0)$ space if and only if it satisfies the (CN) inequality (see [4, page 163]). Moreover, if X is a $CAT(0)$ metric space and $x, y \in X$, then for any $\alpha \in [0,1]$, there exists a unique point $\alpha x \oplus (1-\alpha)y \in [x,y]$ such that

$$d(z, \alpha x \oplus (1-\alpha)y) \quad \leq \quad \alpha d(z,x) + (1-\alpha)d(z,y), \tag{1.4}$$

for any $z \in X$ and $[x,y] = \{\alpha x \oplus (1-\alpha)y : \alpha \in [0,1]\}$.

A subset C of a $CAT(0)$ space X is convex if for any $x, y \in C$, we have $[x,y] \subset C$.

We recall the following definitions in a metric space (X,d). A mapping $T\colon X \to X$ is called an a-contraction if

$$d(Tx,Ty) \quad \leq \quad a\, d(x,y) \text{ for all } x,\ y \in X, \tag{1.5}$$

where $a \in (0,1)$.

The mapping T is called Kannan mapping [16] if there exists $b \in (0, \frac{1}{2})$ such that

$$d(Tx,Ty) \quad \leq \quad b\,[d(x,Tx) + d(y,Ty)] \text{ for all } x,\ y \in X. \tag{1.6}$$

The mapping T is called Chatterjea mapping [8] if there exists $c \in (0, \frac{1}{2})$ such that

$$d(Tx,Ty) \quad \leq \quad c\,[d(x,Ty) + d(y,Tx)] \text{ for all } x,\ y \in X. \tag{1.7}$$

In 1972, Zamfirescu [33] obtained the following interesting fixed point theorem.

Theorem Z. *Let (X,d) be a complete metric space and $T\colon X \to X$ a mapping for which there exists the real number a, b and c satisfying $a \in (0,1)$, $b, c \in (0, \frac{1}{2})$ such that for any pair $x,\ y \in X$, at least one of the following conditions holds:*

(Z_1) $d(Tx,Ty) \leq a\, d(x,y)$,
(Z_2) $d(Tx,Ty) \leq b\,[d(x,Tx) + d(y,Ty)]$,
(Z_3) $d(Tx,Ty) \leq c\,[d(x,Ty) + d(y,Tx)]$.

Then T has a unique fixed point p and the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n,\ n = 0, 1, 2, \cdots$$

converges to p for any arbitrary but fixed $x_0 \in X$.

The conditions $(Z_1) - (Z_3)$ can be written in the following equivalent form

$$d(Tx, Ty) \leq h \max\left\{d(x,y),\ \frac{d(x,Tx)+d(y,Ty)}{2},\ \frac{d(x,Ty)+d(y,Tx)}{2}\right\}, \quad (1.8)$$

for all x, $y \in X$ and $0 < h < 1$, has been obtained by Ciric [9] in 1974.

A mapping satisfying (1.8) is called Ciric quasi-contraction. It is obvious that each of the conditions $(Z_1) - (Z_3)$ implies (1.8).

An operator T satisfying the contractive conditions $(Z_1) - (Z_3)$ in the theorem Z is called Z-operator.

In 2000, Berinde [2] introduced a new class of operators on a normed space E satisfying

$$\|Tx - Ty\| \leq \delta\|x - y\| + L\|Tx - x\|, \quad (*)$$

for any x, $y \in E$, $0 \leq \delta < 1$ and $L \geq 0$.

He proved that this class is wider than the class of Zamfirescu operators and used the Mann iteration process to approximate fixed points of this class of operators in a normed space given in the form of following theorem:

Theorem B. *Let C be a nonempty closed convex subset of a normed space E. Let $T\colon C \to C$ be an operator satisfying $(*)$. Let $\{x_n\}_{n=0}^{\infty}$ be defined by: for $x_1 = x \in C$, the sequence $\{x_n\}_{n=0}^{\infty}$ given by*

$$x_{n+1} = (1 - b_n)x_n + b_n T x_n,\ n \geq 0,$$

where $\{b_n\}$ is a sequence in $[0,1]$. If $F(T) \neq \emptyset$ and $\sum_{n=1}^{\infty} b_n = \infty$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point of T.

In 1953, W.R. Mann defined the Mann iteration [25] as

$$u_{n+1} = (1 - a_n)u_n + a_n T u_n, \quad (1.9)$$

where $\{a_n\}$ is a sequence of positive numbers in [0,1].

In 1974, S. Ishikawa defined the Ishikawa iteration [15] as

$$\begin{aligned} s_{n+1} &= (1 - a_n)s_n + a_n T t_n, \\ t_n &= (1 - b_n)s_n + b_n T s_n, \end{aligned} \quad (1.10)$$

where $\{a_n\}$ and $\{b_n\}$ are sequences of positive numbers in [0,1].

In 2008, S. Thianwan defined the new two step iteration [32] as

$$\begin{aligned} \nu_{n+1} &= (1-a_n)w_n + a_nTw_n, \\ w_n &= (1-b_n)\nu_n + b_nT\nu_n, \end{aligned} \tag{1.11}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences of positive numbers in [0,1].

In 2001, M.A. Noor defined the three step Noor iteration [27] as

$$\begin{aligned} p_{n+1} &= (1-a_n)p_n + a_nTq_n, \\ q_n &= (1-b_n)p_n + b_nTr_n, \\ r_n &= (1-c_n)p_n + c_nTp_n, \end{aligned} \tag{1.12}$$

where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences of positive numbers in [0,1].

Recently, Phuengrattana and Suantai defined the SP-iteration [28] as

$$\begin{aligned} x_{n+1} &= (1-a_n)y_n + a_nTy_n, \\ y_n &= (1-b_n)z_n + b_nTz_n, \\ z_n &= (1-c_n)x_n + c_nTx_n, \end{aligned} \tag{1.13}$$

where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences of positive numbers in [0,1].

Remark 1.1. (1) If $c_n = 0$, then (1.12) reduces to the Ishikawa iteration (1.10).
(2) If $b_n = c_n = 0$, then (1.12) reduces to the Mann iteration (1.9).
(3) If $b_n = 0$, then (1.11) reduces to the Mann iteration (1.9).
(4) If $b_n = c_n = 0$, then (1.13) reduces to the Mann iteration (1.9).
(5) If $c_n = 0$, then (1.13) reduces to the new two step iteration (1.11).

In this paper, inspired and motivated by [28, 33], we study SP-iteration scheme and prove strong convergence theorems to approximate the fixed point for Ciric quasi contractive operator in the framework of CAT(0) spaces.

SP-iteration scheme in CAT(0) space

Let C be a nonempty closed convex subset of a complete CAT(0) space X. Let $T\colon C \to C$ be a contractive operator. Then for a given $x_1 = x_0 \in C$, compute the sequence $\{x_n\}$ by the iterative scheme as follows:

$$\begin{aligned} x_{n+1} &= (1-a_n)y_n \oplus a_nTy_n, \\ y_n &= (1-b_n)z_n \oplus b_nTz_n, \\ z_n &= (1-c_n)x_n \oplus c_nTx_n, \end{aligned} \tag{1.14}$$

where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences of positive numbers in [0,1].

If we take $c_n = 0$, then (1.14) reduces to the new two step iteration in CAT(0) space as follows:

$$\begin{aligned} x_{n+1} &= (1-a_n)y_n \oplus a_n T y_n, \\ y_n &= (1-b_n)x_n \oplus b_n T x_n, \end{aligned} \tag{1.15}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences of positive numbers in [0,1].

If we take $b_n = c_n = 0$, then (1.14) reduces to the Mann iteration in CAT(0) space as follows:

$$x_{n+1} = (1-a_n)x_n \oplus a_n T x_n, \tag{1.16}$$

where $\{a_n\}$ is a sequence of positive numbers in [0,1].

We need the following useful lemmas to prove our main results in this paper.

Lemma 1.2. (See [26]) *Let X be a CAT(0) space.*

(i) *For $x,\, y \in X$ and $t \in [0,1]$, there exists a unique point $z \in [x,\, y]$ such that*

$$d(x,\, z) = t\, d(x,\, y) \quad \text{and} \quad d(y,\, z) = (1-t)\, d(x,\, y). \tag{A}$$

We use the notation $(1-t)x \oplus ty$ for the unique point z satisfying (A).

(ii) *For $x,\, y \in X$ and $t \in [0,1]$, we have*

$$d((1-t)x \oplus ty, z) \le (1-t)d(x,z) + td(y,z).$$

Lemma 1.3. (See [3]) *Let $\{p_n\}_{n=0}^{\infty}$, $\{q_n\}_{n=0}^{\infty}$, $\{r_n\}_{n=0}^{\infty}$ be sequences of nonnegative numbers satisfying the following condition:*

$$p_{n+1} \le (1-s_n)p_n + q_n + r_n, \quad \forall\, n \ge 0,$$

where $\{s_n\}_{n=0}^{\infty} \subset [0,1]$. If $\sum_{n=0}^{\infty} s_n = \infty$, $\lim_{n\to\infty} q_n = O(s_n)$ and $\sum_{n=0}^{\infty} r_n < \infty$, then $\lim_{n\to\infty} p_n = 0$.

2. Strong convergence theorems in CAT(0) Space

In this section, we establish some strong convergence results of SP-iteration scheme to approximate a fixed point for Ciric quasi contractive operator in the framework of CAT(0) spaces.

Theorem 2.1. *Let C be a nonempty closed convex subset of a complete CAT(0) space X and let $T\colon C \to C$ be an operator satisfying the condition (1.8). Let $\{x_n\}$ be defined by the iteration scheme (1.14). If $\sum_{n=0}^{\infty} a_n = \infty$, then $\{x_n\}$ converges strongly to the unique fixed point of T.*

Proof. By Theorem Z, we know that T has a unique fixed point in C, say u. Consider x, $y \in C$. Since T is a operator satisfying (1.8), then if

$$\begin{aligned} d(Tx, Ty) &\leq \frac{h}{2}[d(x, Tx) + d(y, Ty)] \\ &\leq \frac{h}{2}[d(x, Tx) + d(y, x) + d(x, Tx) + d(Tx, Ty)], \end{aligned}$$

implies

$$\Big(1 - \frac{h}{2}\Big) d(Tx, Ty) \leq \frac{h}{2} d(x, y) + h d(x, Tx),$$

which yields(using the fact that $0 < h < 1$)

$$d(Tx, Ty) \leq \Big(\frac{h/2}{1 - h/2}\Big) d(x, y) + \Big(\frac{h}{1 - h/2}\Big) d(x, Tx). \tag{2.1}$$

If

$$\begin{aligned} d(Tx, Ty) &\leq \frac{h}{2}[d(x, Ty) + d(y, Tx)] \\ &\leq \frac{h}{2}[d(x, Tx) + d(Tx, Ty) + d(y, x) + d(x, Tx)], \end{aligned}$$

implies

$$\Big(1 - \frac{h}{2}\Big) d(Tx, Ty) \leq \frac{h}{2}\, d(x, y) + h\, d(x, Tx),$$

which also yields(using the fact that $0 < h < 1$)

$$d(Tx, Ty) \leq \Big(\frac{h/2}{1 - h/2}\Big) d(x, y) + \Big(\frac{h}{1 - h/2}\Big) d(x, Tx). \tag{2.2}$$

Denote

$$\delta = \max\Big\{h,\ \frac{h/2}{1 - h/2}\Big\} = h,$$

$$L = \max\Big\{\frac{h}{1 - h/2}, \frac{h}{1 - h/2}\Big\} = \frac{h}{1 - h/2}.$$

Thus, in all cases

$$\begin{aligned} d(Tx, Ty) &\leq \delta\, d(x, y) + L\, d(x, Tx) \\ &= h\, d(x, y) + \Big(\frac{h}{1 - h/2}\Big) d(x, Tx) \end{aligned} \tag{2.3}$$

holds for all x, $y \in C$.

Also from (1.8) with $y = u = Tu$, we have

$$
\begin{aligned}
& d(Tx, u) \\
& \le h \max\Big\{d(x,u),\, \frac{d(x,Tx)}{2},\, \frac{d(x,u)+d(u,Tx)}{2}\Big\} \\
& \le h \max\Big\{d(x,u),\, \frac{d(x,Tx)}{2},\, \frac{d(x,u)+d(u,Tx)}{2}\Big\} \\
& \le h \max\Big\{d(x,u),\, \frac{d(x,u)+d(u,Tx)}{2},\, \frac{d(x,u)+d(u,Tx)}{2}\Big\}. \qquad (2.4)
\end{aligned}
$$

Since for non-negative real numbers a and b, we have

$$\frac{a+b}{2} \le \max\{a,\, b\}. \qquad (2.5)$$

Using (2.5) in (2.4), we have

$$d(Tx, u) \le h\, d(x, u). \qquad (2.6)$$

Now (2.6) gives

$$d(Tx_n, u) \le h\, d(x_n, u), \qquad (2.7)$$

$$d(Ty_n, u) \le h\, d(y_n, u) \qquad (2.8)$$

and

$$d(Tz_n, u) \le h\, d(z_n, u). \qquad (2.9)$$

Using (1.14), (2.7) and Lemma 1.2(ii), we have

$$
\begin{aligned}
d(z_n, u) &= d((1-c_n)x_n \oplus c_nTx_n, u) \\
&\le (1-c_n)d(x_n, u) + c_nd(Tx_n, u) \\
&\le (1-c_n)d(x_n, u) + hc_nd(x_n, u) \\
&= [1-(1-h)c_n]d(x_n, u). \qquad (2.10)
\end{aligned}
$$

Again using (1.14), (2.9), (2.10) and Lemma 1.2(ii), we have

$$
\begin{aligned}
d(y_n, u) &= d((1-b_n)z_n \oplus b_nTz_n, u) \\
&\le (1-b_n)d(z_n, u) + b_nd(Tz_n, u) \\
&\le (1-b_n)d(z_n, u) + hb_nd(z_n, u) \\
&= [1-(1-h)b_n]d(z_n, u) \\
&\le [1-(1-h)b_n][1-(1-h)c_n]d(x_n, u). \qquad (2.11)
\end{aligned}
$$

Now using (1.14), (2.8), (2.11) and Lemma 1.2(ii), we have

$$\begin{aligned} d(x_{n+1}, u) &= d((1-a_n)y_n \oplus a_n T y_n, u) \\ &\leq (1-a_n)d(y_n, u) + a_n d(Ty_n, u) \\ &\leq (1-a_n)d(y_n, u) + h a_n d(y_n, u) \\ &= [1-(1-h)a_n]d(y_n, u) \\ &\leq [1-(1-h)a_n][1-(1-h)b_n][1-(1-h)c_n]d(x_n, u) \\ &\leq [1-(1-h)a_n]d(x_n, u) \\ &= [1-A_n]d(x_n, u), \end{aligned} \tag{2.12}$$

where $A_n = (1-h)a_n$. Since $0 < h < 1$, $a_n \in [0,1]$ and $\sum_{n=0}^{\infty} a_n = \infty$ by assumption of the theorem, it follows that $\sum_{n=0}^{\infty} A_n = \infty$. Hence, by Lemma 1.3, we get that $\lim_{n\to\infty} d(x_n, u) = 0$. Therefore $\{x_n\}$ converges strongly to a fixed point of T.

To show uniqueness of the fixed point u, assume that u_1, $u_2 \in F(T)$ and $u_1 \neq u_2$. Applying (1.8) and using the fact that $0 < h < 1$, we obtain

$$\begin{aligned} & d(u_1, u_2) = d(Tu_1, Tu_2) \\ & \leq h \max\Big\{d(u_1, u_2),\ \frac{d(u_1, Tu_1) + d(u_2, Tu_2)}{2}, \frac{d(u_1, Tu_2) + d(u_2, Tu_1)}{2}\Big\} \\ & = h \max\Big\{d(u_1, u_2),\ \frac{d(u_1, u_1) + d(u_2, u_2)}{2}, \frac{d(u_1, u_2) + d(u_2, u_1)}{2}\Big\} \\ & = h \max\Big\{d(u_1, u_2),\ 0,\ d(u_1, u_2)\Big\} \\ & \leq h\, d(u_1, u_2) \\ & < d(u_1, u_2). \end{aligned}$$

This is a contradiction. Therefore $u_1 = u_2$. Thus $\{x_n\}$ converges strongly to the unique fixed point of T. This completes the proof. □

Theorem 2.2. *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space X and let $T\colon C \to C$ be an operator satisfying the condition (1.8). Let $\{x_n\}$ be defined by the iteration scheme (1.15). If $\sum_{n=0}^{\infty} a_n = \infty$, then $\{x_n\}$ converges strongly to the unique fixed point of T.*

Proof. The proof of Theorem 2.2 immediately follows by putting $c_n = 0$ in Theorem 2.1. This completes the proof. □

Theorem 2.3. *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space X and let $T\colon C \to C$ be an operator satisfying the condition (1.8). Let $\{x_n\}$ be defined by the iteration scheme (1.16). If $\sum_{n=0}^{\infty} a_n = \infty$, then $\{x_n\}$ converges strongly to the unique fixed point of T.*

Proof. The proof of Theorem 2.3 immediately follows by putting $b_n = c_n = 0$ in Theorem 2.1. This completes the proof. □

The contraction condition (1.5) makes T continuous function on X while this is not the case with contractive conditions (1.6), (1.7) and (2.3).

The contractive conditions (1.6) and (1.7) both included in the class of Zamfirescu operators and so their convergence theorems for SP-iteration process are obtained in Theorem 2.1 in the setting of CAT(0) space.

Remark 2.4. Our results extend the corresponding results of [29] to the case of SP-iteration process and from uniformly convex Banach space to the setting of CAT(0) spaces.

Remark 2.5. Theorem 2.1 also extends Theorem B to the case of SP-iteration process and from normed space to the setting of CAT(0) spaces.

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