



## SEQUENTIAL LANGEVIN EQUATIONS INVOLVING A GENERALIZED TYPE OF CAPUTO DERIVATIVES

Houari Fettouch<sup>1</sup>, Zoubir Dahmani<sup>2</sup>, Hamid Beddani<sup>3</sup>,  
Rabha Wael Ibrahim<sup>4</sup> and Fahd Jarad<sup>5</sup>

<sup>1</sup>Laboratory of Pure and Applied Mathematics, Abdelhamid Bni Badis University,  
27000, Algeria  
e-mail: fettouch72@yahoo.fr

<sup>2</sup>Laboratory of LAMDA RO, University of Bldia A, 09000 Blida, Algeria  
e-mail: zzdahmani@yahoo.fr

<sup>3</sup>Ecole Supérieure de Génie Electrique et Energetique d'Oran, 31000 Oran, Algeria  
e-mail: beddanihamid@gmail.com

<sup>4</sup>Information and Communication Technology Research Group, Scientific Research Center,  
Al-Ayen University, Thi-Qar, Nasiriyah, 64001, Iraq  
e-mail: rabhaibrahim@yahoo.com

<sup>5</sup>Department of Mathematics, Çankaya University, Etimesgut 06530, Ankara, Türkiye;  
Department of Engineering Mathematics and Artificial Intelligence,  
Azerbaijan Technical University, Hüseyin Cavid Av., Baku 1073, Azerbaijan  
e-mail: fahd@cankaya.edu.tr

**Abstract.** This paper delves into a class of sequential Langevin equations characterized by three  $\varphi$ -Caputo fractional derivatives and a  $\varphi$ -RiemannLiouville integral. By harnessing the power of Schauders and Banachs fixed-point theorems, we reveal the existence, uniqueness, and well-posedness of solutions to these equations. To further strengthen our theoretical framework, we present a numerical example that illustrates our findings.

### 1. INTRODUCTION

Fractional calculus is an important area of study that has recently seen applications across various scientific and engineering fields. For more information, one can refer to the works cited in [1, 10, 11, 13, 14, 17, 21, 30, 31, 32, 34]. Research has utilized the Riemann–Liouville, Hadamard, and Katugampola

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<sup>0</sup>Corresponding author: Fahd Jarad(fahd@cankaya.edu.tr).

derivatives. However, further investigation is needed into differential systems involving derivatives related to another function; in this context, the derivatives mentioned earlier are simply particular cases. The concept of these generalized derivatives is elaborated in [26], and several analytical results concerning these operators are presented in [2, 3, 6, 23]. Additionally, we refer to the work in [18], where the authors addressed the following problem:

$$\begin{cases} {}^c\mathcal{D}_{0+}^\beta ({}^c\mathcal{D}_{0+}^\alpha + \mu) u(t) = g(t, u(t)), t \in (0, 1), & 0 < \alpha, \beta \leq 1, 1 < \beta \leq 2, \\ u(0) + u(1) = 0, {}^c\mathcal{D}_{0+}^\alpha u(0) + {}^c\mathcal{D}_{0+}^\alpha u(1) = 0, & {}^c\mathcal{D}_{0+}^{2\alpha} u(0) + {}^c\mathcal{D}_{0+}^{2\alpha} u(1) = 0. \end{cases}$$

Kosari et al.[29] studied the existence and uniqueness of solutions by analyzing the equation

$$\begin{cases} {}^cD_{0+}^\beta ({}^c\mathcal{D}_{0+}^\alpha + \mu) u(t) = g(t, u(t), u'(t)), & t \in (0, 1), \\ u(0) = u(1) = u'(0) = u'(1) = 0, & 0 < \alpha, \beta \leq 1, 2 < \beta \leq 3, \end{cases}$$

where  ${}^c\mathcal{D}^\alpha$  is the Caputo fractional derivative.

The research conducted by Seemab et al.[35] focuses on the Caputo Langevin model of  $\varphi$ -type:

$$\begin{cases} {}^c\mathcal{D}_{a+,t}^{\beta;\varphi} ({}^c\mathcal{D}_{a+,t}^{\alpha;\varphi} + \mu) [u] = g(t, u(t), {}^c\mathcal{D}_{a+,t}^{\delta;\varphi}[u]), & t \in (a, T), \\ u(a) = 0, u(\eta) = 0, u(T) = \lambda(\mathcal{I}_{a+, \zeta}^{\delta;\varphi})[u], & \mu, \lambda > 0. \end{cases}$$

Recently, Beddani and Dahmani[9] undertook a thorough investigation of the following sequential Langevin equation:

$$\begin{cases} {}^c\mathcal{D}_{a+}^{\alpha_1;\varphi} ({}^c\mathcal{D}_{a+}^{\alpha_2;\varphi} ({}^c\mathcal{D}_{a+}^{\alpha_3;\varphi} + \mu)) u(t) = g(t, u(t), {}^c\mathcal{D}_{a+}^{\alpha_4;\varphi} u(t)), & t \in J = (a, b), \\ u(a) = ({}^c\mathcal{D}_{a+}^{\alpha_4;\varphi}) u(b) = 0, u(b) = \rho \sum_{i=1}^n u(\zeta_i), \\ \mu, \rho > 0, 0 \leq a < \zeta_i < b < \infty, \varphi(b) - \varphi(a) = M > 0, \end{cases}$$

under the conditions that  ${}^c\mathcal{D}_{a+}^{\alpha_i;\varphi}, i = \overline{1, 4}$  are the  $\varphi$ -Caputo fractional derivatives of orders  $\alpha_i$  with  $0 < \alpha_i < 1$  and  $\alpha_4 < \alpha_3$ .

In recent years, fractional Langevin equations have become an effective mathematical tool to model memory-dependent stochastic dynamics and viscoelastic systems [12, 15, 16, 25, 33]. Their sequential form, involving multiple fractional orders, captures multi-scale relaxation phenomena more accurately than single-order models. The motivation for considering a sequential form of the Langevin equation lies in its ability to represent physical processes, where each derivative order reflects a different level of memory or diffusion intensity.

Moreover, the choice of the  $\varphi$ -Caputo derivative rather than the classical Caputo or Riemann Liouville operators, which have numerous applications [4, 5, 8, 19, 20, 24, 28, 37, 38, 39, 40], is motivated by its flexibility: it generalizes the time-scaling function through  $\varphi(t)$ , allowing the problem to adapt to

nonuniform temporal structures or nonlinear deformation in time. This property enables a better description of complex processes, such as anomalous diffusion, relaxation, or damping phenomena in heterogeneous media [7, 22, 27, 36].

The problem we study in this paper is therefore significant. It extends traditional fractional Langevin models to a more general framework governed by three sequential  $\varphi$ -Caputo derivatives combined with a  $\varphi$ -Riemann–Liouville integral. This formulation provides richer dynamics and more realistic modeling capabilities for systems with memory effects.

In addition, the use of fixed point theory, particularly Schauder and Banach theorems, provides an efficient analytical framework to prove the existence and uniqueness of solutions for such nonlinear fractional systems. This method is advantageous because it ensures rigorous results while being simple to implement in functional analysis settings. The Schauder theorem guarantees the existence of at least one solution under compactness assumptions, and the Banach contraction principle allows us to establish uniqueness under appropriate Lipschitz conditions.

In this study, we analyze the existence and uniqueness of solutions for a new  $\varphi$ -Caputo problem given by

$$\begin{aligned} {}^c\mathcal{D}_{a^+}^{\alpha_1;\varphi} ({}^c\mathcal{D}_{a^+}^{\alpha_2;\varphi} ({}^c\mathcal{D}_{a^+}^{\alpha_3;\varphi} + \mu)) u(t) &= d_1 g(t, u(t), {}^c\mathcal{D}_{a^+}^{\alpha_4;\varphi} u(t)) \\ &\quad + d_2 f(t, u(t), \mathcal{I}_{a^+}^{\alpha_5;\varphi} u(t)), \quad (1.1) \\ t \in J = (a, b), \quad u(a) &= {}^c\mathcal{D}_{a^+}^{\alpha_4;\varphi} u(b) = 0, \quad \mathcal{I}_{a^+}^{\alpha_5;\varphi} u(b) = \sum_{i=1}^n \rho_i u(\zeta_i), \end{aligned}$$

$$\mu, \rho_i > 0, \quad 0 \leq a < \zeta_i < b < \infty, \quad \varphi(b) - \varphi(a) = M > 0, \quad d_1 > 0, \quad d_2 > 0.$$

We consider  ${}^c\mathcal{D}_{a^+}^{\alpha_i;\varphi}, i = \overline{1, 4}$  as the  $\varphi$ -Caputo fractional derivatives of orders  $\alpha_i$  with  $\alpha_4 < \alpha_3$ , and  $\mathcal{I}_{a^+}^{\alpha_5;\varphi}$  the left-sided  $\varphi$ -Riemann–Liouville fractional integral of order  $\alpha_5$ , where  $0 < \alpha_i < 1, i = \overline{1, 5}$ , and  $\mu, \rho_i \in \mathbb{R}_+^*$ . Other conditions will be imposed later.

## 2. CAPUTO DERIVATIVES OF A FUNCTION REGARDING ANOTHER FUNCTION

We consider  $\varphi : J \rightarrow \mathbb{R}$  as an increasing function and  $\varphi'(t) \neq 0$  for all  $t \in J$ .

**Definition 2.1.** ([6]) For  $\alpha > 0$ , the left-sided  $\varphi$ -Riemann Liouville fractional integral of order  $\alpha$  for an integrable function  $u : J \rightarrow \mathbb{R}$  regarding  $\varphi : J \rightarrow \mathbb{R}$  is provided by:

$$\mathcal{I}_{a^+}^{\alpha;\varphi} u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \varphi'(s) (\varphi(t) - \varphi(s))^{\alpha-1} u(s) ds. \quad (2.1)$$

Based on the information provided earlier, the fractional derivatives with regards to  $\varphi$  is defined as follow:

**Definition 2.2.** ([6]) The left-sided  $\varphi$ -Riemann Liouville fractional derivative of a function  $u$  of order  $\alpha$  is:

$$\begin{aligned} \mathcal{D}_{a^+}^{\alpha,\varphi} u(t) &= \left( \frac{1}{\varphi'(t)} \frac{d}{dt} \right)^n \mathcal{I}_{a^+}^{n-\alpha,\varphi} u(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \left( \frac{1}{\varphi'(t)} \frac{d}{dt} \right)^n \int_a^t \varphi'(s) (\varphi(t) - \varphi(s))^{n-\alpha-1} u(s) ds. \end{aligned}$$

**Definition 2.3.** ([6]) The left-sided  $\varphi$ -Caputo fractional derivative of  $u$  is given by

$${}^c\mathcal{D}_{a^+}^{\alpha,\varphi} u(t) = I_{a^+}^{n-\alpha,\varphi} \left( \frac{1}{\varphi'(t)} \frac{d}{dt} \right)^n u(t),$$

where

$$u_{\varphi}^{[n]}(t) = \left( \frac{1}{\varphi'(t)} \frac{d}{dt} \right)^n u(t).$$

Therefore, we arrive at

$${}^c\mathcal{D}_{a^+}^{\alpha,\varphi} u(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_a^t \varphi'(s) (\varphi(t) - \varphi(s))^{n-\alpha-1} u_{\varphi}^{[n]}(s) ds, & \text{if } \alpha \notin \mathbb{N}, \\ u_{\varphi}^{[n]}(t), & \text{if } \alpha \in \mathbb{N}. \end{cases} \quad (2.2)$$

We would also like to highlight the following supporting results.

**Lemma 2.4.** ([6]) Suppose  $\alpha, \beta > 0, u \in L^1(J)$ . Then

$$\mathcal{I}_{a^+}^{\alpha,\varphi} \mathcal{I}_{a^+}^{\beta,\varphi} u(t) = \mathcal{I}_{a^+}^{\alpha+\beta,\varphi} u(t), \quad \text{a.e. } t \in J.$$

**Lemma 2.5.** ([6]) For  $u \in C([a, b])$ , we have

$${}^c\mathcal{D}_{a^+}^{\alpha,\varphi} \mathcal{I}_{a^+}^{\alpha,\varphi} u(t) = u(t), \quad t \in [a, b].$$

If  $u \in C^n(J)$ ,  $n-1 < \alpha < n$ , then

$$\mathcal{I}_{a^+}^{\alpha,\varphi} ({}^c\mathcal{D}_{a^+}^{\alpha,\varphi} u(t)) = u(t) - \sum_{k=0}^{n-1} \frac{u_{\varphi}^{[k]}(a)}{k!} [\varphi(t) - \varphi(a)]^k.$$

Taking  $0 < \alpha < 1$ , so one has

$$\mathcal{I}_{a^+}^{\alpha,\varphi} ({}^c\mathcal{D}_{a^+}^{\alpha,\varphi} u(t)) = u(t) - u(a).$$

**Lemma 2.6.** ([6, 26]) *Suppose that  $t > a$ ,  $\alpha \geq 0$ ,  $\beta > 0$ . Then*

- (1)  $\mathcal{I}_{a^+}^{\alpha, \varphi} [\varphi(t) - \varphi(a)]^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} [\varphi(t) - \varphi(a)]^{\beta+\alpha-1}$ .
- (2)  ${}^c\mathcal{D}_{a^+}^{\alpha, \varphi} [\varphi(t) - \varphi(a)]^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} [\varphi(t) - \varphi(a)]^{\beta-\alpha-1}$ .
- (3)  ${}^c\mathcal{D}_{a^+}^{\alpha, \varphi} [\varphi(t) - \varphi(a)]^k = 0$ , for all  $k \in \{0, \dots, n-1\}$ ,  $n \in \mathbb{N}$ .

**Lemma 2.7.** ([26, 31]) *Let  $\alpha > 0$ ,  $n \in \mathbb{N}$  such that  $n-1 < q \leq n$ . Then*

- (1)  ${}^c\mathcal{D}_{a^+}^{q, \varphi} \mathcal{I}_{a^+}^{\alpha, \varphi} u(t) = {}^c\mathcal{D}_{a^+}^{q-\alpha, \varphi} u(t)$ ; if  $q > \alpha$ .
- (2)  ${}^c\mathcal{D}_{a^+}^{q, \varphi} \mathcal{I}_{a^+}^{\alpha, \varphi} u(t) u(t) = \mathcal{I}_{a^+}^{\alpha-q, \varphi} u(t)$ ; if  $\alpha > q$ .

**Lemma 2.8.** ([35]) *Suppose that  $u \in C^n[a, b]$  and  $0 < q < 1$ . Then we have*

$$|\mathcal{I}_{a^+}^{q, \varphi} u(t_2) - \mathcal{I}_{a^+}^{q, \varphi} u(t_1)| \leq \frac{2 \|u\|}{\Gamma(q+1)} (\varphi(t_2) - \varphi(t_1))^q.$$

We need also the following lemma:

**Lemma 2.9.** *For any  $g \in L^1(J, \mathbb{R}, \mathbb{R})$ , the solution of*

$$\begin{cases} {}^c\mathcal{D}_{a^+}^{\alpha_1, \varphi} ({}^c\mathcal{D}_{a^+}^{\alpha_2, \varphi} ({}^c\mathcal{D}_{a^+}^{\alpha_3, \varphi} + \mu)) u(t) = h(t) \\ t \in J = (a, b), u(a) = {}^c\mathcal{D}_{a^+}^{\alpha_4, \varphi} u(b) = 0, \mathcal{I}_{a^+}^{\alpha_4, \varphi} u(b) = \sum_{i=1}^n \rho_i u(\zeta_i), \\ \mu, \rho_i > 0, 0 \leq a < \zeta_i < b < \infty, \text{ and } \varphi(b) - \varphi(a) = M > 0 \end{cases} \quad (2.3)$$

is given by

$$\begin{aligned} u(t) = & \mathcal{I}_{a^+}^{\alpha_1 + \alpha_2 + \alpha_3, \varphi} (h(t)) - \mu \mathcal{I}_{a^+}^{\alpha_3, \varphi} u(t) \\ & + \left( \frac{M^{\alpha_2 - \alpha_5} (\varphi(t) - \varphi(a))^{\alpha_3}}{\Delta \Gamma(\alpha_3 + 1) \Gamma(\alpha_2 + \alpha_3 - \alpha_4 + 1)} \right. \\ & \left. - \frac{M^{-\alpha_5} (\varphi(t) - \varphi(a))^{\alpha_2 + \alpha_3}}{\Delta \Gamma(\alpha_2 + \alpha_3 + 1) \Gamma(\alpha_3 - \alpha_4 + 1)} \right) \mathcal{I}_{a^+}^{\alpha_5, \varphi} u(b) \end{aligned}$$

$$\begin{aligned}
& + \frac{M^{-\alpha_5}(\varphi(t) - \varphi(a))^{\alpha_2 + \alpha_3}}{\Delta\Gamma(\alpha_2 + \alpha_3 + 1)\Gamma(\alpha_3 - \alpha_4 + 1)} \\
& \times \left( \mathcal{I}_{a^+}^{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5, \varphi}(h(b)) - \mu \mathcal{I}_{a^+}^{\alpha_3 + \alpha_5, \varphi}u(b) \right) \\
& - \frac{M^{\alpha_2 - \alpha_5}(\varphi(t) - \varphi(a))^{\alpha_3}}{\Delta\Gamma(\alpha_3 + 1)\Gamma(\alpha_2 + \alpha_3 - \alpha_4 + 1)} \\
& \times \left( \mathcal{I}_{a^+}^{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5, \varphi}(h(b)) - \mu \mathcal{I}_{a^+}^{\alpha_3 + \alpha_5, \varphi}u(b) \right) \\
& + \frac{M^{\alpha_2 + \alpha_4}(\varphi(t) - \varphi(a))^{\alpha_3}}{\Gamma(\alpha_3 + 1)\Gamma(\alpha_2 + \alpha_3 + \alpha_5 + 1)} \\
& \times \left( \mathcal{I}_{a^+}^{\alpha_1 + \alpha_2 + \alpha_3 - \alpha_4, \varphi}(h(b)) - \mu \mathcal{I}_{a^+}^{\alpha_3 - \alpha_4, \varphi}u(b) \right) \\
& - \frac{M^{\alpha_4}(\varphi(t) - \varphi(a))^{\alpha_2 + \alpha_3}}{\Delta\Gamma(\alpha_2 + \alpha_3 + 1)\Gamma(\alpha_3 + \alpha_5 + 1)} \\
& \times \left( \mathcal{I}_{a^+}^{\alpha_1 + \alpha_2 + \alpha_3 - \alpha_4, \varphi}(h(b)) - \mu \mathcal{I}_{a^+}^{\alpha_3 - \alpha_4, \varphi}u(b) \right)
\end{aligned}$$

and

$$\begin{aligned}
{}^c\mathcal{D}_{a^+}^{\alpha_4, \varphi}u(t) & = \mathcal{I}_{a^+}^{\alpha_1 + \alpha_2 + \alpha_3 - \alpha_4, \varphi}(h(t)) - \mu \mathcal{I}_{a^+}^{\alpha_3 - \alpha_4, \varphi}u(t) \\
& + \left( \frac{M^{\alpha_2 - \alpha_5}(\varphi(t) - \varphi(a))^{\alpha_3 - \alpha_4}}{\Delta\Gamma(\alpha_3 - \alpha_4 + 1)\Gamma(\alpha_2 + \alpha_3 - \alpha_4 + 1)} \right. \\
& \left. - \frac{M^{-\alpha_5}(\varphi(t) - \varphi(a))^{\alpha_2 + \alpha_3 - \alpha_4}}{\Delta\Gamma(\alpha_3 - \alpha_4 + 1)\Gamma(\alpha_2 + \alpha_3 - \alpha_4 + 1)} \right) \mathcal{I}_{a^+}^{\alpha_5, \varphi}u(b) \\
& + \frac{M^{-\alpha_5}(\varphi(t) - \varphi(a))^{\alpha_2 + \alpha_3 - \alpha_4}}{\Delta\Gamma(\alpha_3 - \alpha_4 + 1)\Gamma(\alpha_2 + \alpha_3 - \alpha_4 + 1)} \\
& \times \left( \mathcal{I}_{a^+}^{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5, \varphi}(h(b)) - \mu \mathcal{I}_{a^+}^{\alpha_3 + \alpha_5, \varphi}u(b) \right) \\
& - \frac{M^{\alpha_2 - \alpha_5}(\varphi(t) - \varphi(a))^{\alpha_3 - \alpha_4}}{\Delta\Gamma(\alpha_3 - \alpha_4 + 1)\Gamma(\alpha_2 + \alpha_3 - \alpha_4 + 1)} \\
& \times \left( \mathcal{I}_{a^+}^{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5, \varphi}(h(b)) - \mu \mathcal{I}_{a^+}^{\alpha_3 + \alpha_5, \varphi}u(b) \right) \\
& + \frac{M^{\alpha_2 + \alpha_4}(\varphi(t) - \varphi(a))^{\alpha_3 - \alpha_4}}{\Gamma(\alpha_3 - \alpha_4 + 1)\Gamma(\alpha_2 + \alpha_3 + \alpha_5 + 1)} \\
& \times \left( \mathcal{I}_{a^+}^{\alpha_1 + \alpha_2 + \alpha_3 - \alpha_4, \varphi}(h(b)) - \mu \mathcal{I}_{a^+}^{\alpha_3 - \alpha_4, \varphi}u(b) \right)
\end{aligned}$$

$$\begin{aligned}
 & - \frac{M^{\alpha_4}(\varphi(t) - \varphi(a))^{\alpha_2 + \alpha_3 - \alpha_4}}{\Delta \Gamma(\alpha_3 + \alpha_5 + 1) \Gamma(\alpha_2 + \alpha_3 - \alpha_4 + 1)} \\
 & \times \left( \mathcal{I}_{a^+}^{\alpha_1 + \alpha_2 + \alpha_3 - \alpha_4, \varphi}(h(b)) - \mu \mathcal{I}_{a^+}^{\alpha_3 - \alpha_4, \varphi} u(b) \right),
 \end{aligned}$$

where,

$$\begin{aligned}
 \Delta &= \frac{M^{\alpha_2 + \alpha_3}}{\Gamma(\alpha_3 + \alpha_5 + 1) \Gamma(\alpha_2 + \alpha_3 - \alpha_4 + 1)} \\
 & - \frac{M^{\alpha_2 + \alpha_3}}{\Gamma(\alpha_3 - \alpha_4 + 1) \Gamma(\alpha_2 + \alpha_3 + \alpha_5 + 1)}.
 \end{aligned}$$

*Proof.* We take  $0 < \alpha_i < 1$ ,  $i = \overline{1, 3}$ . Then, we have

$$\left( {}^c \mathcal{D}_{a^+}^{\alpha_2, \varphi} \left( {}^c \mathcal{D}_{a^+}^{\alpha_3, \varphi} + \mu \right) \right) u(t) = \mathcal{I}_{a^+}^{\alpha_1, \varphi}(h(t)) + c_1$$

and

$$\left( {}^c \mathcal{D}_{a^+}^{\alpha_3, \varphi} + \mu \right) u(t) = \mathcal{I}_{a^+}^{\alpha_1 + \alpha_2, \varphi}(h(t)) + \mathcal{I}_{a^+}^{\alpha_2, \varphi} c_1 + c_2.$$

Consequently,

$$u(t) = \mathcal{I}_{a^+}^{\alpha_1 + \alpha_2 + \alpha_3, \varphi}(h(t)) - \mu \mathcal{I}_{a^+}^{\alpha_3, \varphi} u(t) + \mathcal{I}_{a^+}^{\alpha_2 + \alpha_3, \varphi} c_1 + \mathcal{I}_{a^+}^{\alpha_3, \varphi} c_2 + c_3.$$

Some easy calculations allow us to write

$$u(t) = \mathcal{I}_{a^+}^{\alpha_1 + \alpha_2 + \alpha_3, \varphi}(h(t)) - \mu \mathcal{I}_{a^+}^{\alpha_3, \varphi} u(t) + \mathcal{I}_{a^+}^{\alpha_2 + \alpha_3, \varphi} c_1 + \mathcal{I}_{a^+}^{\alpha_3, \varphi} c_2. \quad (2.4)$$

Therefore,

$$\begin{aligned}
 u(t) &= \mathcal{I}_{a^+}^{\alpha_1 + \alpha_2 + \alpha_3, \varphi}(h(t)) - \mu \mathcal{I}_{a^+}^{\alpha_3, \varphi} u(t) \\
 &+ c_1 \frac{(\varphi(t) - \varphi(a))^{\alpha_2 + \alpha_3}}{\Gamma(\alpha_2 + \alpha_3 + 1)} + c_2 \frac{(\varphi(t) - \varphi(a))^{\alpha_3}}{\Gamma(\alpha_3 + 1)}.
 \end{aligned} \quad (2.5)$$

According to Lemma 2.7, we have

$$\begin{aligned}
 {}^c \mathcal{D}_{a^+}^{\alpha_4, \varphi} u(t) &= \mathcal{I}_{a^+}^{\alpha_1 + \alpha_2 + \alpha_3 - \alpha_4, \varphi}(h(t)) - \mu \mathcal{I}_{a^+}^{\alpha_3 - \alpha_4, \varphi} u(t) \\
 &+ \mathcal{I}_{a^+}^{\alpha_2 + \alpha_3 - \alpha_4, \varphi} c_1 + \mathcal{I}_{a^+}^{\alpha_3 - \alpha_4, \varphi} c_2
 \end{aligned}$$

and

$$\begin{aligned}
 \mathcal{I}_{a^+}^{\alpha_5, \varphi} u(t) &= \mathcal{I}_{a^+}^{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5, \varphi}(h(t)) - \mu \mathcal{I}_{a^+}^{\alpha_3 + \alpha_5, \varphi} u(t) \\
 &+ \mathcal{I}_{a^+}^{\alpha_2 + \alpha_3 + \alpha_5, \varphi} c_1 + \mathcal{I}_{a^+}^{\alpha_3 + \alpha_5, \varphi} c_2.
 \end{aligned}$$

So, we have

$$\begin{aligned}
 {}^c \mathcal{D}_{a^+}^{\alpha_4, \varphi} u(t) &= \mathcal{I}_{a^+}^{\alpha_1 + \alpha_2 + \alpha_3 - \alpha_4, \varphi}(h(t)) - \mu \mathcal{I}_{a^+}^{\alpha_3 - \alpha_4, \varphi} u(t) \\
 &+ c_1 \frac{(\varphi(t) - \varphi(a))^{\alpha_2 + \alpha_3 - \alpha_4}}{\Gamma(\alpha_2 + \alpha_3 - \alpha_4 + 1)} + c_2 \frac{(\varphi(t) - \varphi(a))^{\alpha_3 - \alpha_4}}{\Gamma(\alpha_3 - \alpha_4 + 1)}
 \end{aligned}$$

and

$$\begin{aligned} \mathcal{I}_{a^+}^{\alpha_5, \varphi} u(t) &= \mathcal{I}_{a^+}^{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5, \varphi} (h(t)) - \mu \mathcal{I}_{a^+}^{\alpha_3 + \alpha_5, \varphi} u(t) \\ &\quad + c_1 \frac{(\varphi(t) - \varphi(a))^{\alpha_2 + \alpha_3 + \alpha_5}}{\Gamma(\alpha_2 + \alpha_3 + \alpha_5 + 1)} + c_2 \frac{(\varphi(t) - \varphi(a))^{\alpha_3 + \alpha_5}}{\Gamma(\alpha_3 + \alpha_5 + 1)}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &c_1 \frac{M^{\alpha_2 + \alpha_3 - \alpha_4}}{\Gamma(\alpha_2 + \alpha_3 - \alpha_4 + 1)} + c_2 \frac{M^{\alpha_3 - \alpha_4}}{\Gamma(\alpha_3 - \alpha_4 + 1)} \\ &= \mu \mathcal{I}_{a^+}^{\alpha_3 - \alpha_4, \varphi} u(b) - \mathcal{I}_{a^+}^{\alpha_1 + \alpha_2 + \alpha_3 - \alpha_4, \varphi} (h(b)) \end{aligned}$$

and

$$\begin{aligned} &c_1 \frac{M^{\alpha_2 + \alpha_3 + \alpha_5}}{\Gamma(\alpha_2 + \alpha_3 + \alpha_5 + 1)} + c_2 \frac{M^{\alpha_3 + \alpha_5}}{\Gamma(\alpha_3 + \alpha_5 + 1)} \\ &= \mathcal{I}_{a^+}^{\alpha_5, \varphi} u(b) + \mu \mathcal{I}_{a^+}^{\alpha_3 + \alpha_5, \varphi} u(b) - \mathcal{I}_{a^+}^{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5, \varphi} (h(b)). \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} c_1 &= -\frac{M^{-\alpha_5}}{\Delta \Gamma(\alpha_3 - \alpha_4 + 1)} \sum_{i=1}^n \rho_i u(\zeta_i) \\ &\quad + \frac{M^{-\alpha_5}}{\Delta \Gamma(\alpha_3 - \alpha_4 + 1)} \left( \mathcal{I}_{a^+}^{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5, \varphi} (h(b)) - \mu \mathcal{I}_{a^+}^{\alpha_3 + \alpha_5, \varphi} u(b) \right) \\ &\quad - \frac{M^{\alpha_4}}{\Delta \Gamma(\alpha_3 + \alpha_5 + 1)} \left( \mathcal{I}_{a^+}^{\alpha_1 + \alpha_2 + \alpha_3 - \alpha_4, \varphi} (h(b)) - \mu \mathcal{I}_{a^+}^{\alpha_3 - \alpha_4, \varphi} u(b) \right) \end{aligned}$$

and

$$\begin{aligned} c_2 &= \frac{M^{\alpha_2 - \alpha_5}}{\Delta \Gamma(\alpha_2 + \alpha_3 - \alpha_4 + 1)} \sum_{i=1}^n \rho_i u(\zeta_i) \\ &\quad + \frac{M^{\alpha_2 + \alpha_4}}{\Delta \Gamma(\alpha_2 + \alpha_3 + \alpha_5 + 1)} \left( \mathcal{I}_{a^+}^{\alpha_1 + \alpha_2 + \alpha_3 - \alpha_4, \varphi} (h(b)) - \mu \mathcal{I}_{a^+}^{\alpha_3 - \alpha_4, \varphi} u(b) \right) \\ &\quad - \frac{M^{\alpha_2 - \alpha_5}}{\Delta \Gamma(\alpha_2 + \alpha_3 - \alpha_4 + 1)} \left( \mathcal{I}_{a^+}^{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5, \varphi} (h(b)) - \mu \mathcal{I}_{a^+}^{\alpha_3 + \alpha_5, \varphi} u(b) \right) \end{aligned}$$

such that

$$\begin{aligned} \Delta &= \frac{M^{\alpha_2 + \alpha_3}}{\Gamma(\alpha_3 + \alpha_5 + 1) \Gamma(\alpha_2 + \alpha_3 - \alpha_4 + 1)} \\ &\quad - \frac{M^{\alpha_2 + \alpha_3}}{\Gamma(\alpha_3 - \alpha_4 + 1) \Gamma(\alpha_2 + \alpha_3 + \alpha_5 + 1)}. \end{aligned}$$

This completes the proof.  $\square$

### 3. EXISTENCE AND UNIQUENESS OF THE SOLUTIONS

Let us, in this section, consider the space

$$\mathbb{X} = \{u : u \in C[a, b], {}^c\mathcal{D}_{a^+}^{\alpha_4, \varphi} u \in C[a, b]\}$$

with

$$\|u\|_{\mathbb{X}} = \max \left\{ \|u\|_{\infty}, \|{}^c\mathcal{D}_{a^+}^{\alpha_4, \varphi} u\|_{\infty} \right\}, \quad (3.1)$$

where

$$\|u\|_{\infty} = \sup_{t \in [a, b]} |u(t)| \quad \text{and} \quad \|{}^c\mathcal{D}_{a^+}^{\alpha_4, \varphi} u\|_{\infty} = \sup_{t \in [a, b]} |{}^c\mathcal{D}_{a^+}^{\alpha_4, \varphi} u(t)|.$$

To ensure clarity and precision, it is necessary to establish the following conditions to demonstrate the existence and uniqueness.

- (H<sub>1</sub>) We impose for  $f, g : [a, b] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^+$  to be continuous.
- (H<sub>2</sub>) There are positive constants  $\delta_{1,1}, \delta_{1,2}, \delta_{2,1}, \delta_{2,2}$  such that for all  $t \in [a, b]$ ,  $u, v, x, y \in \mathbb{R}$ ,

$$|g(t, u, v) - g(t, x, y)| \leq \delta_{1,1} |u - x| + \delta_{1,2} |v - y|,$$

and

$$|f(t, u, v) - f(t, x, y)| \leq \delta_{2,1} |u - x| + \delta_{2,2} |v - y|$$

with  $\delta_1 = \max(\delta_{1,1}, \delta_{1,2}), \delta_2 = \max(\delta_{2,1}, \delta_{2,2})$ .

- (H<sub>3</sub>) There are a nonnegative functions  $\phi_1, \phi_2 \in L[0, 1]$  that satisfy

$$|g(t, u, v)| \leq \phi_1(t) + \rho_1 |u(t)| + \rho_2 |v(t)|$$

and

$$|f(t, u, v)| \leq \phi_2(t) + \rho_3 |u(t)| + \rho_4 |v(t)|,$$

where  $\rho_1, \rho_2, \rho_3, \rho_4 \in \mathbb{R}_+^*$ .

Now, we put

$$\beta = \frac{M^{\alpha_5}}{\Gamma(1 + \alpha_5)},$$

$$\begin{aligned}
\mathcal{N}_1 = & \frac{\mu M^{\alpha_3}}{\Gamma(\alpha_3 + 1)} + \frac{2\mu M^{\alpha_2 + 2\alpha_3}}{|\Delta| \Gamma(\alpha_2 + \alpha_3 + 1) \Gamma(\alpha_3 - \alpha_4 + 1) \Gamma(\alpha_3 + \alpha_5 + 1)} \\
& + \frac{n \left( \max_{i=1, n} \rho_i \right) M^{\alpha_2 + \alpha_3 - \alpha_5}}{|\Delta| \Gamma(\alpha_2 + \alpha_3 + 1) \Gamma(\alpha_3 - \alpha_4 + 1)} \\
& + \frac{\mu M^{\alpha_2 + 2\alpha_3}}{|\Delta| \Gamma(\alpha_3 + 1) \Gamma(\alpha_2 + \alpha_3 - \alpha_4 + 1) \Gamma(\alpha_3 + \alpha_5 + 1)} \\
& + \frac{n \left( \max_{i=1, n} \rho_i \right) M^{\alpha_2 + \alpha_3 - \alpha_5}}{|\Delta| \Gamma(\alpha_3 + 1) \Gamma(\alpha_2 + \alpha_3 - \alpha_4 + 1)} \\
& + \frac{\mu M^{\alpha_2 + 2\alpha_3}}{|\Delta| \Gamma(\alpha_3 + 1) \Gamma(\alpha_2 + \alpha_3 + \alpha_5 + 1) \Gamma(\alpha_3 - \alpha_4 + 1)},
\end{aligned}$$

$$\begin{aligned}
\mathcal{N}_2 = & \frac{2M^{\alpha_1 + 2\alpha_2 + 2\alpha_3}}{|\Delta| \Gamma(\alpha_2 + \alpha_3 + 1) \Gamma(\alpha_3 - \alpha_4 + 1) \Gamma(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + 1)} \\
& + \frac{2M^{\alpha_1 + \alpha_2 + \alpha_3}}{\Gamma(\alpha_1 + \alpha_2 + \alpha_3 + 1)} \\
& + \frac{2M^{\alpha_1 + 2\alpha_2 + 2\alpha_3}}{|\Delta| \Gamma(\alpha_3 + 1) \Gamma(\alpha_2 + \alpha_3 - \alpha_4 + 1) \Gamma(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + 1)} \\
& + \frac{2M^{\alpha_1 + 2\alpha_2 + 2\alpha_3}}{|\Delta| \Gamma(\alpha_2 + \alpha_3 + 1) \Gamma(\alpha_3 + \alpha_5 + 1) \Gamma(\alpha_1 + \alpha_2 + \alpha_3 - \alpha_4 + 1)} \\
& + \frac{2M^{\alpha_1 + 2\alpha_2 + 2\alpha_3}}{|\Delta| \Gamma(\alpha_3 + 1) \Gamma(\alpha_2 + \alpha_3 + \alpha_5 + 1) \Gamma(\alpha_1 + \alpha_2 + \alpha_3 - \alpha_4 + 1)},
\end{aligned}$$

$$\begin{aligned}
\mathcal{N}_3 = & \frac{\mu M^{\alpha_3 - \alpha_4}}{\Gamma(\alpha_3 - \alpha_4 + 1)} + \frac{2n \left( \max_{i=1, n} \rho_i \right) M^{\alpha_2 + \alpha_3 - \alpha_4 - \alpha_5}}{|\Delta| \Gamma(\alpha_3 - \alpha_4 + 1) \Gamma(\alpha_2 + \alpha_3 - \alpha_4 + 1)} \\
& + \frac{\mu M^{\alpha_2 + 2\alpha_3 - \alpha_4}}{|\Delta| \Gamma(\alpha_3 - \alpha_4 + 1) \Gamma(\alpha_2 + \alpha_3 - \alpha_4 + 1) \Gamma(\alpha_3 + \alpha_5 + 1)} \\
& + \frac{\mu M^{\alpha_2 + 2\alpha_3 - \alpha_4}}{|\Delta| \Gamma(\alpha_3 - \alpha_4 + 1) \Gamma(\alpha_2 + \alpha_3 + \alpha_5 + 1) \Gamma(\alpha_3 - \alpha_4 + 1)} \\
& + \frac{2\mu M^{\alpha_2 + 2\alpha_3 - \alpha_4}}{|\Delta| \Gamma(\alpha_2 + \alpha_3 - \alpha_4 + 1) \Gamma(\alpha_3 - \alpha_4 + 1) \Gamma(\alpha_3 + \alpha_5 + 1)}
\end{aligned}$$

and

$$\begin{aligned} \mathcal{N}_4 &= \frac{2M^{\alpha_1+\alpha_2+\alpha_3-\alpha_4}}{\Gamma(\alpha_1+\alpha_2+\alpha_3-\alpha_4+1)} \\ &+ \frac{2M^{\alpha_1+2\alpha_2+2\alpha_3-\alpha_4}}{|\Delta|\Gamma(\alpha_2+\alpha_3-\alpha_4+1)\Gamma(\alpha_3-\alpha_4+1)\Gamma(\alpha_1+\alpha_2+\alpha_3+\alpha_5+1)} \\ &+ \frac{2M^{\alpha_1+2\alpha_2+2\alpha_3-\alpha_4}}{|\Delta|\Gamma(\alpha_2+\alpha_3-\alpha_4+1)\Gamma(\alpha_3+\alpha_5+1)\Gamma(\alpha_1+\alpha_2+\alpha_3-\alpha_4+1)} \\ &+ \frac{2M^{\alpha_1+2\alpha_2+2\alpha_3-\alpha_4}}{|\Delta|\Gamma(\alpha_3-\alpha_4+1)\Gamma(\alpha_2+\alpha_3+\alpha_5+1)\Gamma(\alpha_1+\alpha_2+\alpha_3-\alpha_4+1)}. \end{aligned}$$

Let us demonstrate the validity of proposition.

**Proposition 3.1.** *Let  $q > 0$  and  $\varphi, u, v \in C^n([a, b])$ , we have*

$$(\varphi(t) - \varphi(a))^q \leq M^q, \quad (3.2)$$

$$|\mathcal{I}_{a^+}^{q,\varphi} u(t)| \leq \frac{M^q \|u\|_{\mathbb{X}}}{\Gamma(q+1)}, \quad (3.3)$$

$$|g(s, u(s), {}^c \mathcal{D}_{a^+}^{\alpha_4, \varphi} u(s))| \leq \|\phi_1\|_{\infty} + (\rho_1 + \rho_2) \|u\|_{\mathbb{X}}, \quad (3.4)$$

$$|f(s, u(s), {}^c \mathcal{I}_{a^+}^{\alpha_5, \varphi} u(s))| \leq \|\phi_2\|_{\infty} + (\rho_3 + \rho_4 \beta) \|u\|_{\mathbb{X}}, \quad (3.5)$$

$$|g(s, u(s), {}^c \mathcal{D}_{a^+}^{\alpha_4, \varphi} u(s)) - g(s, v(s), {}^c \mathcal{D}_{a^+}^{\alpha_4, \varphi} v(s))| \leq 2\delta_1 \|u - v\|_{\mathbb{X}}, \quad (3.6)$$

$$|f(s, u(s), {}^c \mathcal{I}_{a^+}^{\alpha_5, \varphi} u(s)) - f(s, v(s), {}^c \mathcal{I}_{a^+}^{\alpha_5, \varphi} v(s))| \leq (1 + \beta) \delta_2 \|u - v\|_{\mathbb{X}}, \quad (3.7)$$

$$|\mathcal{I}_{a^+}^{q,\varphi} u(t) - \mathcal{I}_{a^+}^{q,\varphi} v(t)| \leq \frac{M^q \|u - v\|_{\mathbb{X}}}{\Gamma(q+1)} \quad (3.8)$$

and

$$|h_u(s)| \leq (d_1 \|\phi_1\|_{\infty} + d_2 \|\phi_2\|_{\infty}) + (d_1 \rho_1 + d_1 \rho_2 + d_2 \rho_3 + d_2 \rho_4 \beta) \|u\|_{\mathbb{X}}. \quad (3.9)$$

*Proof.* We have  $\varphi'(t) > 0$  for all  $t = [a, b]$ . Then, we have

$$(\varphi(t) - \varphi(a))^q \leq (\varphi(b) - \varphi(a))^q = M^q$$

and

$$\begin{aligned} |\mathcal{I}_{a^+}^{q,\varphi} u(t)| &\leq \frac{1}{\Gamma(q)} \int_a^t \varphi'(s) (\varphi(t) - \varphi(s))^{q-1} |u(s)| ds \\ &\leq \frac{\sup_{s \in [a, b]} |u(s)|}{\Gamma(q)} \int_a^t \varphi'(s) (\varphi(t) - \varphi(s))^{q-1} ds \\ &\leq \frac{M^q \|u\|_{\mathbb{X}}}{\Gamma(q+1)}. \end{aligned}$$

By  $(\mathbb{H}_3)$ , we obtain

$$\begin{aligned} |g(s, u(s), {}^c\mathcal{D}_{a^+}^{\alpha_4, \varphi} u(s))| &\leq \phi_1(t) + \rho_1 \sup_{s \in [a, b]} |u(s)| + \rho_2 \sup_{s \in [a, b]} |{}^c\mathcal{D}_{a^+}^{\alpha_4, \varphi} u(s)| \\ &\leq \|\phi_1\|_\infty + \rho_1 \|u\|_\infty + \rho_2 \|{}^c\mathcal{D}_{a^+}^{\alpha_4, \varphi} u\|_\infty \\ &\leq \|\phi_1\|_\infty + (\rho_1 + \rho_2) \|u\|_{\mathbb{X}}. \end{aligned}$$

Using  $(\mathbb{H}_3)$ , we get

$$\begin{aligned} |f(s, u(s), \mathcal{I}_{a^+}^{\alpha_5, \varphi} u(s))| &\leq \phi_2(t) + \rho_3 \sup_{s \in [a, b]} |u(s)| + \rho_4 \sup_{s \in [a, b]} |\mathcal{I}_{a^+}^{\alpha_5, \varphi} u(s)| \\ &\leq \phi_2(t) + \rho_3 \left( \sup_{s \in [a, b]} |u(s)| \right) + \beta \rho_4 \left( \sup_{s \in [a, b]} |u(s)| \right) \\ &\leq \|\phi_2\|_\infty + (\rho_3 + \beta \rho_4) \|u\|_{\mathbb{X}}. \end{aligned}$$

And by  $(\mathbb{H}_2)$ , we get

$$\begin{aligned} &|g(s, u(s), {}^c\mathcal{D}_{a^+}^{\alpha_4, \varphi} u(s)) - g(s, v(s), {}^c\mathcal{D}_{a^+}^{\alpha_4, \varphi} v(s))| \\ &\leq \delta_1 \left( \sup_{s \in [a, b]} |u(s) - v(s)| + \sup_{s \in [a, b]} |{}^c\mathcal{D}_{a^+}^{\alpha_4, \varphi} u(s) - {}^c\mathcal{D}_{a^+}^{\alpha_4, \varphi} v(s)| \right) \\ &\leq \delta_1 \left( \|u - v\|_\infty + \|{}^c\mathcal{D}_{a^+}^{\alpha_4, \varphi} (u - v)\|_\infty \right) \\ &\leq 2\delta_1 \|u - v\|_{\mathbb{X}}. \end{aligned}$$

Also,

$$\begin{aligned} &|f(s, u(s), \mathcal{I}_{a^+}^{\alpha_5, \varphi} u(s)) - f(s, v(s), \mathcal{I}_{a^+}^{\alpha_5, \varphi} v(s))| \\ &\leq \delta_2 (|u(s) - v(s)| + |\mathcal{I}_{a^+}^{\alpha_5, \varphi} u(s) - \mathcal{I}_{a^+}^{\alpha_5, \varphi} v(s)|) \\ &\leq \delta_2 \left( \sup_{s \in [a, b]} |u(s) - v(s)| + \sup_{s \in [a, b]} |\mathcal{I}_{a^+}^{\alpha_5, \varphi} (u(s) - v(s))| \right) \\ &\leq (1 + \beta) \delta_2 \sup_{s \in [a, b]} |u(s) - v(s)| \\ &\leq (1 + \beta) \delta_2 \|u - v\|_{\mathbb{X}}. \end{aligned}$$

By  $(\mathbb{H}_3)$ , (3.3) and (3.4), we see that

$$\begin{aligned} &|d_1 g(s, u(s), {}^c\mathcal{D}_{a^+}^{\alpha_4, \varphi} u(s)) + d_2 f(s, u(s), \mathcal{I}_{a^+}^{\alpha_5, \varphi} u(s))| \\ &\leq |d_1 g(s, u(s), {}^c\mathcal{D}_{a^+}^{\alpha_4, \varphi} u(s))| + d_2 |f(s, u(s), \mathcal{I}_{a^+}^{\alpha_5, \varphi} u(s))| \\ &\leq d_1 \|\phi_1\|_\infty + d_1 \rho_1 \|u\|_\infty + d_1 \rho_2 \|{}^c\mathcal{D}_{a^+}^{\alpha_4, \varphi} u\|_\infty + d_2 \|\phi_2\|_\infty \\ &\quad + d_2 (\rho_3 + \beta \rho_4) \|u\|_\infty \\ &\leq (d_1 \|\phi_1\|_\infty + d_2 \|\phi_2\|_\infty) + (d_1 \rho_1 + d_1 \rho_2 + d_2 \rho_3 + d_2 \rho_4 \beta) \|u\|_{\mathbb{X}}. \end{aligned}$$

□

The following key findings compellingly establish the existence of solutions.

**Theorem 3.2.** *If  $(\mathbb{H}_1)$  and  $(\mathbb{H}_3)$  are satisfied, then, (1.2) has at least one solution.*

*Proof.* We define mapping  $\mathbb{T} : \mathbb{X} \rightarrow \mathbb{X}$  as follows:

$$\begin{aligned}
 (\mathbb{T}u)(t) &= \mathcal{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3,\varphi} h_u(t) - \mu \mathcal{I}_{a^+}^{\alpha_3,\varphi} u(t) \\
 &+ \left( \frac{M^{\alpha_2-\alpha_5}(\varphi(t) - \varphi(a))^{\alpha_3}}{\Delta\Gamma(\alpha_3+1)\Gamma(\alpha_2+\alpha_3-\alpha_4+1)} \right. \\
 &\quad \left. - \frac{M^{-\alpha_5}(\varphi(t) - \varphi(a))^{\alpha_2+\alpha_3}}{\Delta\Gamma(\alpha_2+\alpha_3+1)\Gamma(\alpha_3-\alpha_4+1)} \right) \mathcal{I}_{a^+}^{\alpha_5,\varphi} u(b) \\
 &+ \frac{M^{-\alpha_5}(\varphi(t) - \varphi(a))^{\alpha_2+\alpha_3}}{\Delta\Gamma(\alpha_2+\alpha_3+1)\Gamma(\alpha_3-\alpha_4+1)} \\
 &\times \left( \mathcal{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3+\alpha_5,\varphi} h_u(b) - \mu \mathcal{I}_{a^+}^{\alpha_3+\alpha_5,\varphi} u(b) \right) \\
 &- \frac{M^{\alpha_2-\alpha_5}(\varphi(t) - \varphi(a))^{\alpha_3}}{\Delta\Gamma(\alpha_3+1)\Gamma(\alpha_2+\alpha_3-\alpha_4+1)} \\
 &\times \left( \mathcal{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3+\alpha_5,\varphi} h_u(b) - \mu \mathcal{I}_{a^+}^{\alpha_3+\alpha_5,\varphi} u(b) \right) \\
 &+ \frac{M^{\alpha_2+\alpha_4}(\varphi(t) - \varphi(a))^{\alpha_3}}{\Delta\Gamma(\alpha_3+1)\Gamma(\alpha_2+\alpha_3+\alpha_5+1)} \\
 &\times \left( \mathcal{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3-\alpha_4,\varphi} h_u(b) - \mu \mathcal{I}_{a^+}^{\alpha_3-\alpha_4,\varphi} u(b) \right) \\
 &- \frac{M^{\alpha_4}(\varphi(t) - \varphi(a))^{\alpha_2+\alpha_3}}{\Delta\Gamma(\alpha_2+\alpha_3+1)\Gamma(\alpha_3+\alpha_5+1)} \\
 &\times \left( \mathcal{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3-\alpha_4,\varphi} h_u(b) - \mu \mathcal{I}_{a^+}^{\alpha_3-\alpha_4,\varphi} u(b) \right)
 \end{aligned}$$

and

$$\begin{aligned}
 {}^c\mathcal{D}_{a^+}^{\alpha_4,\varphi} (\mathbb{T}u)(t) &= \mathcal{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3-\alpha_4,\varphi} h_u(t) - \mu \mathcal{I}_{a^+}^{\alpha_3-\alpha_4,\varphi} u(t) \\
 &+ \left( \frac{M^{\alpha_2-\alpha_5}(\varphi(t) - \varphi(a))^{\alpha_3-\alpha_4}}{\Delta\Gamma(\alpha_3-\alpha_4+1)\Gamma(\alpha_2+\alpha_3-\alpha_4+1)} \right. \\
 &\quad \left. - \frac{M^{-\alpha_5}(\varphi(t) - \varphi(a))^{\alpha_2+\alpha_3-\alpha_4}}{\Delta\Gamma(\alpha_3-\alpha_4+1)\Gamma(\alpha_2+\alpha_3-\alpha_4+1)} \right) \mathcal{I}_{a^+}^{\alpha_5,\varphi} u(b) \\
 &+ \frac{M^{-\alpha_5}(\varphi(t) - \varphi(a))^{\alpha_2+\alpha_3-\alpha_4}}{\Delta\Gamma(\alpha_3-\alpha_4+1)\Gamma(\alpha_2+\alpha_3-\alpha_4+1)} \\
 &\times \left( \mathcal{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3+\alpha_5,\varphi} h_u(b) - \mu \mathcal{I}_{a^+}^{\alpha_3+\alpha_5,\varphi} u(b) \right)
 \end{aligned}$$

$$\begin{aligned}
& - \frac{M^{\alpha_2 - \alpha_5} (\varphi(t) - \varphi(a))^{\alpha_3 - \alpha_4}}{\Delta \Gamma(\alpha_3 - \alpha_4 + 1) \Gamma(\alpha_2 + \alpha_3 - \alpha_4 + 1)} \\
& \times \left( \mathcal{I}_{a^+}^{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5, \varphi} h_u(b) - \mu \mathcal{I}_{a^+}^{\alpha_3 + \alpha_5, \varphi} u(b) \right) \\
& + \frac{M^{\alpha_2 + \alpha_4} (\varphi(t) - \varphi(a))^{\alpha_3 - \alpha_4}}{\Delta \Gamma(\alpha_3 - \alpha_4 + 1) \Gamma(\alpha_2 + \alpha_3 + \alpha_5 + 1)} \\
& \times \left( \mathcal{I}_{a^+}^{\alpha_1 + \alpha_2 + \alpha_3 - \alpha_4, \varphi} h_u(b) - \mu \mathcal{I}_{a^+}^{\alpha_3 - \alpha_4, \varphi} u(b) \right) \\
& - \frac{M^{\alpha_4} (\varphi(t) - \varphi(a))^{\alpha_2 + \alpha_3 - \alpha_4}}{\Delta \Gamma(\alpha_3 + \alpha_5 + 1) \Gamma(\alpha_2 + \alpha_3 - \alpha_4 + 1)} \\
& \times \left( \mathcal{I}_{a^+}^{\alpha_1 + \alpha_2 + \alpha_3 - \alpha_4, \varphi} h_u(b) - \mu \mathcal{I}_{a^+}^{\alpha_3 - \alpha_4, \varphi} u(b) \right).
\end{aligned}$$

If  $e \in \{\alpha_3, \alpha_3 - \alpha_4, \alpha_3 + \alpha_5, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3 - \alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_5\}$  and  $x \in \{t, b\}$ , then

$$\mathcal{I}_{a^+}^{e, \varphi} u(x) = \frac{1}{\Gamma(e)} \int_a^x \varphi'(s) (\varphi(x) - \varphi(s))^{e-1} u(s) ds,$$

and

$$\begin{aligned}
\mathcal{I}_{a^+}^{e, \varphi} h_u(x) &= \frac{1}{\Gamma(e)} \int_a^x \varphi'(s) (\varphi(y) - \varphi(s))^{e-1} \\
& \times \left( d_1 g(s, u(s), {}^c \mathcal{D}_{a^+}^{\alpha_4, \varphi} u(s)) + d_2 f(s, u(s), {}^c \mathcal{I}_{a^+}^{\alpha_5, \varphi} u(s)) \right) ds.
\end{aligned}$$

Considering the set  $\mathbb{U}_r = \{u \in \mathbb{X}, \|u\|_{\mathbb{X}} \leq r\}$ , we have

$$\max\{(\mathcal{N}_1 + \omega), (\mathcal{N}_3 + \omega)\} \leq \frac{1}{2} \quad \text{and} \quad \max\{\mathcal{N}_2, \mathcal{N}_4\} \leq \frac{r}{2\lambda},$$

where,

$$d_1 \|\phi_1\|_{\infty} + d_2 \|\phi_2\|_{\infty} = \lambda, \quad d_1 \rho_1 + d_1 \rho_2 + d_2 \rho_3 + d_2 \rho_4 \beta = \omega.$$

For any  $u \in \mathbb{U}_r$  and by  $(\mathbb{H}_3)$ , we show that  $\mathbb{T}\mathbb{U}_r \subset \mathbb{U}_r$ . So, one observes that

$$\begin{aligned}
\|\mathbb{T}u\|_{\infty} &\leq \sup \left\{ \mathcal{I}_{a^+}^{\alpha_1 + \alpha_2 + \alpha_3, \varphi} |h_u(t)| + \mu \mathcal{I}_{a^+}^{\alpha_3, \varphi} |u(t)| \right. \\
& + \left( \frac{M^{\alpha_2 - \alpha_5} (\varphi(t) - \varphi(a))^{\alpha_3}}{|\Delta| \Gamma(\alpha_3 + 1) \Gamma(\alpha_2 + \alpha_3 - \alpha_4 + 1)} \right. \\
& \left. \left. + \frac{M^{-\alpha_5} (\varphi(t) - \varphi(a))^{\alpha_2 + \alpha_3}}{|\Delta| \Gamma(\alpha_2 + \alpha_3 + 1) \Gamma(\alpha_3 - \alpha_4 + 1)} \right) \mathcal{I}_{a^+}^{\alpha_5, \varphi} |u(b)| \right\}
\end{aligned}$$

$$\begin{aligned}
 & + \frac{M^{-\alpha_5}(\varphi(t) - \varphi(a))^{\alpha_2 + \alpha_3}}{|\Delta| \Gamma(\alpha_2 + \alpha_3 + 1) \Gamma(\alpha_3 - \alpha_4 + 1)} \\
 & \times \left( \mathcal{I}_{a^+}^{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5, \varphi} |h_u(b)| + \mu \mathcal{I}_{a^+}^{\alpha_3 + \alpha_5, \varphi} |u(b)| \right) \\
 & + \frac{M^{\alpha_2 - \alpha_5}(\varphi(t) - \varphi(a))^{\alpha_3}}{|\Delta| \Gamma(\alpha_3 + 1) \Gamma(\alpha_2 + \alpha_3 - \alpha_4 + 1)} \\
 & \times \left( \mathcal{I}_{a^+}^{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5, \varphi} |h_u(b)| + \mu \mathcal{I}_{a^+}^{\alpha_3 + \alpha_5, \varphi} |u(b)| \right) \\
 & + \frac{M^{\alpha_4}(\varphi(t) - \varphi(a))^{\alpha_2 + \alpha_3}}{|\Delta| \Gamma(\alpha_2 + \alpha_3 + 1) \Gamma(\alpha_3 + \alpha_5 + 1)} \\
 & \times \left( \mathcal{I}_{a^+}^{\alpha_1 + \alpha_2 + \alpha_3 - \alpha_4, \varphi} |h_u(b)| + \mu \mathcal{I}_{a^+}^{\alpha_3 - \alpha_4, \varphi} |u(b)| \right) \\
 & + \frac{M^{\alpha_2 + \alpha_4}(\varphi(t) - \varphi(a))^{\alpha_3}}{|\Delta| \Gamma(\alpha_3 + 1) \Gamma(\alpha_2 + \alpha_3 + \alpha_5 + 1)} \\
 & \times \left( \mathcal{I}_{a^+}^{\alpha_1 + \alpha_2 + \alpha_3 - \alpha_4, \varphi} |h_u(b)| + \mu \mathcal{I}_{a^+}^{\alpha_3 - \alpha_4, \varphi} |u(b)| \right) \Big\}.
 \end{aligned}$$

Exploiting Lemma 2.7, and Proposition 3.1, we obtain

$$\begin{aligned}
 \|Tu\|_\infty & \leq \|u\|_{\mathbb{X}} \left\{ \frac{\mu M^{\alpha_3}}{\Gamma(\alpha_3 + 1)} \right. \\
 & + \frac{\mu 2M^{\alpha_2 + 2\alpha_3}}{|\Delta| \Gamma(\alpha_2 + \alpha_3 + 1) \Gamma(\alpha_3 - \alpha_4 + 1) \Gamma(\alpha_3 + \alpha_5 + 1)} \\
 & + \frac{n \left( \max_{i=1, n} \rho_i \right) M^{\alpha_2 + \alpha_3 - \alpha_5}}{|\Delta| \Gamma(\alpha_2 + \alpha_3 + 1) \Gamma(\alpha_3 - \alpha_4 + 1)} \\
 & + \frac{\mu M^{\alpha_2 + 2\alpha_3}}{|\Delta| \Gamma(\alpha_3 + 1) \Gamma(\alpha_2 + \alpha_3 - \alpha_4 + 1) \Gamma(\alpha_3 + \alpha_5 + 1)} \\
 & + \frac{n \left( \max_{i=1, n} \rho_i \right) M^{\alpha_2 + \alpha_3 - \alpha_5}}{|\Delta| \Gamma(\alpha_3 + 1) \Gamma(\alpha_2 + \alpha_3 - \alpha_4 + 1)} \\
 & \left. + \frac{\mu M^{\alpha_2 + 2\alpha_3}}{|\Delta| \Gamma(\alpha_3 + 1) \Gamma(\alpha_2 + \alpha_3 + \alpha_5 + 1) \Gamma(\alpha_3 - \alpha_4 + 1)} \right\} \\
 & + (\lambda + \omega \|u\|_{\mathbb{X}}) \left\{ \frac{2M^{\alpha_1 + \alpha_2 + \alpha_3}}{\Gamma(\alpha_1 + \alpha_2 + \alpha_3 + 1)} \right. \\
 & \left. + \frac{2M^{\alpha_1 + 2\alpha_2 + 2\alpha_3}}{|\Delta| \Gamma(\alpha_2 + \alpha_3 + 1) \Gamma(\alpha_3 - \alpha_4 + 1) \Gamma(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + 1)} \right\}
 \end{aligned}$$

$$\begin{aligned}
& + \frac{2M^{\alpha_1+2\alpha_2+2\alpha_3}}{|\Delta|\Gamma(\alpha_3+1)\Gamma(\alpha_2+\alpha_3-\alpha_4+1)\Gamma(\alpha_1+\alpha_2+\alpha_3+\alpha_5+1)} \\
& + \frac{2M^{\alpha_1+2\alpha_2+2\alpha_3}}{|\Delta|\Gamma(\alpha_2+\alpha_3+1)\Gamma(\alpha_3+\alpha_5+1)\Gamma(\alpha_1+\alpha_2+\alpha_3-\alpha_4+1)} \\
& + \left. \frac{2M^{\alpha_1+2\alpha_2+2\alpha_3}}{|\Delta|\Gamma(\alpha_3+1)\Gamma(\alpha_2+\alpha_3+\alpha_5+1)\Gamma(\alpha_1+\alpha_2+\alpha_3-\alpha_4+1)} \right\} \\
& \leq \mathcal{N}_1 \|u\|_{\mathbb{X}} + \mathcal{N}_2 (\lambda + \omega \|u\|_{\mathbb{X}}) \\
& \leq (\mathcal{N}_1 + \omega) \|u\|_{\mathbb{X}} + \lambda \mathcal{N}_2.
\end{aligned}$$

Additionally, we can write

$$\begin{aligned}
\|{}^c\mathcal{D}_{a^+}^{\alpha_4,\varphi}(\mathbb{T}u)(t)\|_{\infty} & \leq \sup \left\{ \mathcal{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3-\alpha_4,\varphi} |h_u(t)| + \mu \mathcal{I}_{a^+}^{\alpha_3-\alpha_4,\varphi} |u(t)| \right. \\
& + \left( \frac{M^{\alpha_2-\alpha_5}(\varphi(t) - \varphi(a))^{\alpha_3-\alpha_4}}{|\Delta|\Gamma(\alpha_3-\alpha_4+1)\Gamma(\alpha_2+\alpha_3-\alpha_4+1)} \right. \\
& + \left. \frac{M^{-\alpha_5}(\varphi(t) - \varphi(a))^{\alpha_2+\alpha_3-\alpha_4}}{|\Delta|\Gamma(\alpha_3-\alpha_4+1)\Gamma(\alpha_2+\alpha_3-\alpha_4+1)} \right) \mathcal{I}_{a^+}^{\alpha_5,\varphi} |u(b)| \\
& + \frac{M^{-\alpha_5}(\varphi(t) - \varphi(a))^{\alpha_2+\alpha_3-\alpha_4}}{|\Delta|\Gamma(\alpha_3-\alpha_4+1)\Gamma(\alpha_2+\alpha_3-\alpha_4+1)} \\
& \times \left( \mathcal{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3+\alpha_5,\varphi} |h_u(b)| + \mu \mathcal{I}_{a^+}^{\alpha_3+\alpha_5,\varphi} |u(b)| \right) \\
& + \frac{M^{\alpha_2-\alpha_5}(\varphi(t) - \varphi(a))^{\alpha_3-\alpha_4}}{|\Delta|\Gamma(\alpha_3-\alpha_4+1)\Gamma(\alpha_2+\alpha_3-\alpha_4+1)} \\
& \times \left( \mathcal{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3+\alpha_5,\varphi} |h_u(b)| + \mu \mathcal{I}_{a^+}^{\alpha_3+\alpha_5,\varphi} |u(b)| \right) \\
& + \frac{M^{\alpha_4}(\varphi(t) - \varphi(a))^{\alpha_2+\alpha_3-\alpha_4}}{|\Delta|\Gamma(\alpha_3+\alpha_5+1)\Gamma(\alpha_2+\alpha_3-\alpha_4+1)} \\
& \times \left( \mathcal{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3-\alpha_4,\varphi} |h_u(b)| + \mu \mathcal{I}_{a^+}^{\alpha_3-\alpha_4,\varphi} |u(b)| \right) \\
& + \left. \frac{M^{\alpha_2+\alpha_4}(\varphi(t) - \varphi(a))^{\alpha_3-\alpha_4}}{|\Delta|\Gamma(\alpha_3-\alpha_4+1)\Gamma(\alpha_2+\alpha_3+\alpha_5+1)} \right. \\
& \times \left. \left( \mathcal{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3-\alpha_4,\varphi} |h_u(b)| + \mu \mathcal{I}_{a^+}^{\alpha_3-\alpha_4,\varphi} |u(b)| \right) \right\}.
\end{aligned}$$

Utilizing both Lemma 2.7 and Proposition 3.1 once more, we can confidently conclude that

$$\begin{aligned}
 & \left\| {}^c\mathcal{D}_{a^+}^{\alpha_4, \varphi} \mathbb{T}u \right\|_{\infty} \\
 & \leq \|u\|_{\mathbb{X}} \left\{ \frac{\mu M^{\alpha_3 - \alpha_4}}{\Gamma(\alpha_3 - \alpha_4 + 1)} + \frac{2n \left( \max_{i=1, n} \rho_i \right) M^{\alpha_2 + \alpha_3 - \alpha_4 - \alpha_5}}{|\Delta| \Gamma(\alpha_3 - \alpha_4 + 1) \Gamma(\alpha_2 + \alpha_3 - \alpha_4 + 1)} \right. \\
 & \quad + \frac{\mu M^{\alpha_2 + 2\alpha_3 - \alpha_4}}{|\Delta| \Gamma(\alpha_3 - \alpha_4 + 1) \Gamma(\alpha_2 + \alpha_3 - \alpha_4 + 1) \Gamma(\alpha_3 + \alpha_5 + 1)} \\
 & \quad + \frac{\mu M^{\alpha_2 + 2\alpha_3 - \alpha_4}}{|\Delta| \Gamma(\alpha_3 - \alpha_4 + 1) \Gamma(\alpha_2 + \alpha_3 + \alpha_5 + 1) \Gamma(\alpha_3 - \alpha_4 + 1)} \\
 & \quad \left. + \frac{2\mu M^{\alpha_2 + 2\alpha_3 - \alpha_4}}{|\Delta| \Gamma(\alpha_2 + \alpha_3 - \alpha_4 + 1) \Gamma(\alpha_3 - \alpha_4 + 1) \Gamma(\alpha_3 + \alpha_5 + 1)} \right\} \\
 & \quad + (\lambda + \omega \|u\|_{\mathbb{X}}) \left\{ \frac{2M^{\alpha_1 + \alpha_2 + \alpha_3 - \alpha_4}}{\Gamma(\alpha_1 + \alpha_2 + \alpha_3 - \alpha_4 + 1)} \right. \\
 & \quad + \frac{2M^{\alpha_1 + 2\alpha_2 + 2\alpha_3 - \alpha_4}}{|\Delta| \Gamma(\alpha_2 + \alpha_3 - \alpha_4 + 1) \Gamma(\alpha_3 - \alpha_4 + 1) \Gamma(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + 1)} \\
 & \quad + \frac{2M^{\alpha_1 + 2\alpha_2 + 2\alpha_3 - \alpha_4}}{|\Delta| \Gamma(\alpha_2 + \alpha_3 - \alpha_4 + 1) \Gamma(\alpha_3 + \alpha_5 + 1) \Gamma(\alpha_1 + \alpha_2 + \alpha_3 - \alpha_4 + 1)} \\
 & \quad \left. + \frac{2M^{\alpha_1 + 2\alpha_2 + 2\alpha_3 - \alpha_4}}{|\Delta| \Gamma(\alpha_3 - \alpha_4 + 1) \Gamma(\alpha_2 + \alpha_3 + \alpha_5 + 1) \Gamma(\alpha_1 + \alpha_2 + \alpha_3 - \alpha_4 + 1)} \right\} \\
 & \leq \mathcal{N}_3 \|u\|_{\mathbb{X}} + \mathcal{N}_4 (\lambda + \omega \|u\|_{\mathbb{X}}) \\
 & \leq (\mathcal{N}_3 + \omega) \|u\|_{\mathbb{X}} + \lambda \mathcal{N}_4.
 \end{aligned}$$

As a result, it follows that

$$\begin{aligned}
 \|\mathbb{T}u\|_{\mathbb{X}} &= \max \left\{ \|\mathbb{T}u\|_{\infty}, \left\| {}^c\mathcal{D}_{a^+}^{\alpha_4, \varphi} \mathbb{T}u \right\|_{\infty} \right\} \\
 &\leq \|u\|_{\mathbb{X}} \max \{ (\mathcal{N}_1 + \omega), (\mathcal{N}_3 + \omega) \} + \lambda \max \{ \mathcal{N}_2, \mathcal{N}_4 \} \\
 &\leq \frac{r}{2} + \frac{r}{2} = r.
 \end{aligned}$$

In the upcoming sections, we will present a thorough demonstration to the reader regarding the complete continuity of  $\mathbb{T}$ .

Taking  $u \in \mathbb{U}_r$ ,  $t_1, t_2 \in [a, b]$ ,  $t_1 < t_2$ . Proposition 3.1 and Lemma 2.8 allow us to write

$$\begin{aligned}
|(\mathbb{T}u)(t_2) - (\mathbb{T}u)(t_1)| &\leq \sup \left\{ \mathcal{I}_{a^+}^{\alpha_1 + \alpha_2 + \alpha_3, \varphi} (|h_u(t_2) - h_u(t_1)|) \right. \\
&\quad + \mu \mathcal{I}_{a^+}^{\alpha_3, \varphi} |u(t_2) - u(t_1)| \\
&\quad + \left( \frac{M^{\alpha_2 - \alpha_5} (\varphi(t_2) - \varphi(t_1))^{\alpha_3}}{|\Delta| \Gamma(\alpha_3 + 1) \Gamma(\alpha_2 + \alpha_3 - \alpha_4 + 1)} \right. \\
&\quad \left. + \frac{M^{-\alpha_5} (\varphi(t) - \varphi(a))^{\alpha_2 + \alpha_3}}{|\Delta| \Gamma(\alpha_2 + \alpha_3 + 1) \Gamma(\alpha_3 - \alpha_4 + 1)} \right) \mathcal{I}_{a^+}^{\alpha_5, \varphi} |u(b)| \\
&\quad + \frac{M^{-\alpha_5} (\varphi(t_2) - \varphi(t_1))^{\alpha_2 + \alpha_3}}{|\Delta| \Gamma(\alpha_2 + \alpha_3 + 1) \Gamma(\alpha_3 - \alpha_4 + 1)} \\
&\quad \times \left( \mathcal{I}_{a^+}^{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5, \varphi} (|h_u(b) + h_u(b)|) + \mu \mathcal{I}_{a^+}^{\alpha_3 + \alpha_5, \varphi} |u(b)| \right) \\
&\quad + \frac{M^{\alpha_2 - \alpha_5} (\varphi(t_2) - \varphi(t_1))^{\alpha_3}}{|\Delta| \Gamma(\alpha_3 + 1) \Gamma(\alpha_2 + \alpha_3 - \alpha_4 + 1)} \\
&\quad \times \left( \mathcal{I}_{a^+}^{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5, \varphi} (|h_u(b) + h_u(b)|) + \mu \mathcal{I}_{a^+}^{\alpha_3 + \alpha_5, \varphi} |u(b)| \right) \\
&\quad + \frac{M^{\alpha_4} (\varphi(t_2) - \varphi(t_1))^{\alpha_2 + \alpha_3}}{|\Delta| \Gamma(\alpha_2 + \alpha_3 + 1) \Gamma(\alpha_3 + \alpha_5 + 1)} \\
&\quad \times \left( \mathcal{I}_{a^+}^{\alpha_1 + \alpha_2 + \alpha_3 - \alpha_4, \varphi} (|h_u(b) + h_u(b)|) + \mu \mathcal{I}_{a^+}^{\alpha_3 - \alpha_4, \varphi} |u(b)| \right) \\
&\quad + \frac{M^{\alpha_2 + \alpha_4} (\varphi(t_2) - \varphi(t_1))^{\alpha_3}}{|\Delta| \Gamma(\alpha_3 + 1) \Gamma(\alpha_2 + \alpha_3 + \alpha_5 + 1)} \\
&\quad \left. \times \left( \mathcal{I}_{a^+}^{\alpha_1 + \alpha_2 + \alpha_3 - \alpha_4, \varphi} (|h_u(b) + h_u(b)|) + \mu \mathcal{I}_{a^+}^{\alpha_3 - \alpha_4, \varphi} |u(b)| \right) \right\}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
|(\mathbb{T}u)(t_2) - (\mathbb{T}u)(t_1)| &\leq \frac{(\lambda + \omega \|u\|_{\mathbb{X}})}{\Gamma(\alpha_1 + \alpha_2 + \alpha_3 + 1)} (\varphi(t_2) - \varphi(t_1))^{\alpha_1 + \alpha_2 + \alpha_3} \quad (3.10) \\
&\quad + \mathcal{A}_1 (\varphi(t_2) - \varphi(t_1))^{\alpha_2 + \alpha_3} + \mathcal{A}_2 (\varphi(t_2) - \varphi(t_1))^{\alpha_3},
\end{aligned}$$

where

$$\mathcal{A}_1 = \sup \left\{ \frac{M^{\alpha_2 - \alpha_5} \mathcal{I}_{a^+}^{\alpha_5, \varphi} |u(b)|}{|\Delta| \Gamma(\alpha_2 + \alpha_3 + 1) \Gamma(\alpha_3 - \alpha_4 + 1)} \right. \\ \left. + \frac{M^{\alpha_2 - \alpha_5} \left( \mathcal{I}_{a^+}^{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5, \varphi} |h_u(b)| + \mu \mathcal{I}_{a^+}^{\alpha_3 + \alpha_5, \varphi} |u(b)| \right)}{|\Delta| \Gamma(\alpha_2 + \alpha_3 + 1) \Gamma(\alpha_3 - \alpha_4 + 1)} \right. \\ \left. + \frac{M^{\alpha_4} \left( \mathcal{I}_{a^+}^{\alpha_1 + \alpha_2 + \alpha_3 - \alpha_4, \varphi} |h_u(b)| + \mu \mathcal{I}_{a^+}^{\alpha_3 - \alpha_4, \varphi} |u(b)| \right)}{|\Delta| \Gamma(\alpha_2 + \alpha_3 + 1) \Gamma(\alpha_3 + \alpha_5 + 1)} \right\}$$

and

$$\mathcal{A}_2 = \sup \left\{ \frac{M^{-\alpha_5} \mathcal{I}_{a^+}^{\alpha_5, \varphi} |u(b)|}{|\Delta| \Gamma(\alpha_3 + 1) \Gamma(\alpha_2 + \alpha_3 - \alpha_4 + 1)} + \frac{\mu \|u\|_\infty}{\Gamma(\alpha_3 + 1)} \right. \\ \left. + \frac{M^{-\alpha_5} \left( \mathcal{I}_{a^+}^{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5, \varphi} |h_u(b)| + \mu \mathcal{I}_{a^+}^{\alpha_3 + \alpha_5, \varphi} |u(b)| \right)}{|\Delta| \Gamma(\alpha_3 + 1) \Gamma(\alpha_2 + \alpha_3 - \alpha_4 + 1)} \right. \\ \left. + \frac{M^{\alpha_2 + \alpha_4} \left( \mathcal{I}_{a^+}^{\alpha_1 + \alpha_2 + \alpha_3 - \alpha_4, \varphi} |h_u(b)| + \mu \mathcal{I}_{a^+}^{\alpha_3 - \alpha_4, \varphi} |u(b)| \right)}{|\Delta| \Gamma(\alpha_3 + 1) \Gamma(\alpha_2 + \alpha_3 + \alpha_5 + 1)} \right\}.$$

Additionally, it is evident that

$$\begin{aligned} & \left| {}^c \mathcal{D}_{a^+}^{\alpha_4, \varphi} (\mathbb{T}u)(t_2) - {}^c \mathcal{D}_{a^+}^{\alpha_4, \varphi} (\mathbb{T}u)(t_1) \right| \tag{3.11} \\ & \leq \frac{(\lambda + \omega \|u\|_{\mathbb{X}})}{\Gamma(\alpha_1 + \alpha_2 + \alpha_3 - \alpha_4 + 1)} (\varphi(t_2) - \varphi(t_1))^{\alpha_1 + \alpha_2 + \alpha_3 - \alpha_4} \\ & \quad + \mathcal{B}_1 (\varphi(t_2) - \varphi(t_1))^{\alpha_2 + \alpha_3 - \alpha_4} + \mathcal{B}_2 (\varphi(t_2) - \varphi(t_1))^{\alpha_3 - \alpha_4}, \end{aligned}$$

where

$$\mathcal{B}_1 = \sup \left\{ \frac{M^{\alpha_4} \left( \mathcal{I}_{a^+}^{\alpha_1 + \alpha_2 + \alpha_3 - \alpha_4, \varphi} |h_u(b)| + \mu \mathcal{I}_{a^+}^{\alpha_3 - \alpha_4, \varphi} |u(b)| \right)}{|\Delta| \Gamma(\alpha_3 + \alpha_5 + 1) \Gamma(\alpha_2 + \alpha_3 - \alpha_4 + 1)} \right. \\ \left. + \frac{M^{-\alpha_5} \left( \mathcal{I}_{a^+}^{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5, \varphi} |h_u(b)| + \mu \mathcal{I}_{a^+}^{\alpha_3 + \alpha_5, \varphi} |u(b)| + \mathcal{I}_{a^+}^{\alpha_5, \varphi} |u(b)| \right)}{|\Delta| \Gamma(\alpha_3 - \alpha_4 + 1) \Gamma(\alpha_2 + \alpha_3 - \alpha_4 + 1)} \right\}$$

and

$$\mathcal{B}_2 = \sup \left\{ \frac{\mu \|u\|_\infty}{\Gamma(\alpha_3 - \alpha_4 + 1)} + \frac{M^{\alpha_2 - \alpha_5} \left( \mathcal{I}_{a^+}^{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5, \varphi} |h_u(b)| + \mu \mathcal{I}_{a^+}^{\alpha_3 + \alpha_5, \varphi} |u(b)| + \mathcal{I}_{a^+}^{\alpha_5, \varphi} |u(b)| \right)}{|\Delta| \Gamma(\alpha_3 - \alpha_4 + 1) \Gamma(\alpha_2 + \alpha_3 - \alpha_4 + 1)} + \frac{M^{\alpha_2 + \alpha_4} \left( \mathcal{I}_{a^+}^{\alpha_1 + \alpha_2 + \alpha_3 - \alpha_4, \varphi} |h_u(b)| + \mu \mathcal{I}_{a^+}^{\alpha_3 - \alpha_4, \varphi} |u(b)| \right)}{|\Delta| \Gamma(\alpha_3 - \alpha_4 + 1) \Gamma(\alpha_2 + \alpha_3 + \alpha_5 + 1)} \right\}.$$

We can state now that (3.10) and (3.11) are independent of  $u$  and tend to zero if  $t_2 \rightarrow t_1$ . Therefore,  $\mathbb{T}(\mathbb{U}_r)$  is equicontinuous. By Arzela–Ascoli theorem, we state that  $\overline{\mathbb{T}(\mathbb{U}_r)}$  is compact. Therefore,  $\mathbb{T} : \mathbb{U}_r \rightarrow \mathbb{U}_r$  is completely continuous. At the end, according to Schauder theorem, we conclude that (1.2) has at least one solution.  $\square$

Now, let us proceed to prove the following main theorem.

**Theorem 3.3.** *Let us suppose that  $(\mathbb{H}_1$ – $\mathbb{H}_2$ ) are verified. Then (1.2) has a unique solution, in the case where*

$$\vartheta = \max \{ \mathcal{N}_1 + \mathcal{N}_2 (2d_1\delta_1 + d_2\delta_2 (1 + \beta)), \mathcal{N}_3 + \mathcal{N}_4 (2d_1\delta_1 + d_2\delta_2 (1 + \beta)) \} < 1.$$

*Proof.* Let us taking  $u, v \in \mathbb{U}, t \in [a, b]$ . Hence, using  $(\mathbb{H}_2)$  and according to the Proposition 3.1, we write

$$\begin{aligned} \|\mathbb{T}u - \mathbb{T}v\|_\infty &\leq \sup \left\{ \mathcal{I}_{a^+}^{\alpha_1 + \alpha_2 + \alpha_3, \varphi} (|h_u(t) - h_v(t)|) + \mu \mathcal{I}_{a^+}^{\alpha_3, \varphi} |u(t) - v(t)| \right. \\ &\quad + \left( \frac{M^{\alpha_2 - \alpha_5} (\varphi(t) - \varphi(a))^{\alpha_3}}{|\Delta| \Gamma(\alpha_3 + 1) \Gamma(\alpha_2 + \alpha_3 - \alpha_4 + 1)} \right. \\ &\quad \left. + \frac{M^{-\alpha_5} (\varphi(t) - \varphi(a))^{\alpha_2 + \alpha_3}}{|\Delta| \Gamma(\alpha_2 + \alpha_3 + 1) \Gamma(\alpha_3 - \alpha_4 + 1)} \right) \times \mathcal{I}_{a^+}^{\alpha_5, \varphi} |u(b) - v(b)| \\ &\quad + \frac{M^{-\alpha_5} (\varphi(t) - \varphi(a))^{\alpha_2 + \alpha_3}}{|\Delta| \Gamma(\alpha_2 + \alpha_3 + 1) \Gamma(\alpha_3 - \alpha_4 + 1)} \\ &\quad \times \left( \mathcal{I}_{a^+}^{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5, \varphi} (|h_u(b) - h_v(b)|) + \mu \mathcal{I}_{a^+}^{\alpha_3 + \alpha_5, \varphi} |u(b) - v(b)| \right) \\ &\quad + \frac{M^{\alpha_2 - \alpha_5} (\varphi(t) - \varphi(a))^{\alpha_3}}{|\Delta| \Gamma(\alpha_3 + 1) \Gamma(\alpha_2 + \alpha_3 - \alpha_4 + 1)} \\ &\quad \left. \times \left( \mathcal{I}_{a^+}^{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5, \varphi} (|h_u(b) - h_v(b)|) + \mu \mathcal{I}_{a^+}^{\alpha_3 + \alpha_5, \varphi} |u(b) - v(b)| \right) \right\} \end{aligned}$$

$$\begin{aligned}
 & + \frac{M^{\alpha_4}(\varphi(t) - \varphi(a))^{\alpha_2 + \alpha_3}}{|\Delta| \Gamma(\alpha_2 + \alpha_3 + 1) \Gamma(\alpha_3 + \alpha_5 + 1)} \\
 & \times \left( \mathcal{I}_{a^+}^{\alpha_1 + \alpha_2 + \alpha_3 - \alpha_4, \varphi} (|h_u(b) - h_v(b)|) + \mu \mathcal{I}_{a^+}^{\alpha_3 - \alpha_4, \varphi} |u(b) - v(b)| \right) \\
 & + \frac{M^{\alpha_2 + \alpha_4}(\varphi(t) - \varphi(a))^{\alpha_3}}{|\Delta| \Gamma(\alpha_3 + 1) \Gamma(\alpha_2 + \alpha_3 + \alpha_5 + 1)} \\
 & \times \left( \mathcal{I}_{a^+}^{\alpha_1 + \alpha_2 + \alpha_3 - \alpha_4, \varphi} (|h_u(b) - h_v(b)|) + \mu \mathcal{I}_{a^+}^{\alpha_3 - \alpha_4, \varphi} |u(b) - v(b)| \right) \Big\}.
 \end{aligned}$$

The Lemma 2.7 implies both

$$\|\mathbb{T}u - \mathbb{T}v\|_\infty \leq [\mathcal{N}_1 + \mathcal{N}_2 (2d_1\delta_1 + d_2\delta_2 (1 + \beta))] \|u - v\|_{\mathbb{X}} \quad (3.12)$$

and

$$\|{}^c\mathcal{D}_{a^+}^{\alpha_4, \varphi} \mathbb{T}u - {}^c\mathcal{D}_{a^+}^{\alpha_4, \varphi} \mathbb{T}v\|_\infty \leq [\mathcal{N}_3 + \mathcal{N}_4 (2d_1\delta_1 + d_2\delta_2 (1 + \beta))] \|u - v\|_{\mathbb{X}}. \quad (3.13)$$

According to (3.12) and (3.13), we get

$$\|\mathbb{T}u - \mathbb{T}v\|_{\mathbb{X}} \leq \vartheta \|u - v\|_{\mathbb{X}}$$

for  $\vartheta < 1$ . Hence  $\mathbb{T}$  is contraction. According to Banach principle, the problem (1.2) has a unique solution.  $\square$

**Example 3.4.** Consider the example

$$\begin{aligned}
 & {}^c\mathcal{D}_{a^+}^{0,6,\varphi} \left( {}^c\mathcal{D}_{a^+}^{0,7,\varphi} \left( {}^c\mathcal{D}_{a^+}^{0,6,\varphi} + \pi \right) \right) u(t) = \frac{2}{5}g(t, u(t), {}^c\mathcal{D}_{a^+}^{\frac{1}{3},\varphi} u(t)) \\
 & \quad + \frac{3}{7}f(t, u(t), \mathcal{I}_{a^+}^{\frac{1}{5},\varphi} u(t)), t \in \left[ \frac{1}{2}, 1 \right], \\
 & \quad \varphi(t) = t^2, \\
 & g(t, u(t), {}^c\mathcal{D}_{a^+}^{\frac{1}{3},\varphi} u(t)) = \frac{t}{4} + \left( \sin \frac{\pi t}{2} - \frac{1}{8} \right)^2 u(t) + \frac{t^2}{5} {}^c\mathcal{D}_{a^+}^{\frac{1}{3},\varphi} u(t), \\
 & f(t, u(t), \mathcal{I}_{a^+}^{\frac{1}{5},\varphi} u(t)) = \frac{t^2}{5} + \frac{1225}{64} \left( \frac{t}{5} - \cos \pi t \right)^2 u(t) + \frac{t}{5} \mathcal{I}_{a^+}^{\frac{1}{5},\varphi} u(t), \\
 & \quad \phi_1(t) = \frac{t}{4}, \quad \phi_2(t) = \frac{t^2}{5}.
 \end{aligned} \quad (3.14)$$

Since  $f$  and  $g$  are continuous and

$$\begin{aligned}
 \left| g(t, u(t), {}^c\mathcal{D}_{a^+}^{\frac{1}{3},\varphi} u(t)) \right| & \leq \frac{1}{4} + \frac{49}{64} |u(t)| + \frac{1}{5} \left| {}^c\mathcal{D}_{a^+}^{\frac{1}{3},\varphi} u(t) \right|, \\
 \left| f(t, u(t), \mathcal{I}_{a^+}^{\frac{1}{5},\varphi} u(t)) \right| & \leq \frac{1}{5} + \left( \frac{49}{64} + \frac{1}{5\Gamma(\frac{6}{5})} \sqrt[5]{\frac{3}{4}} \right) |u(t)|,
 \end{aligned}$$

and by calculate, we obtain

$$\vartheta \simeq 0.8665841 < 1.$$

Hence,  $(\mathbb{H}_1)$ ,  $(\mathbb{H}_2)$  are satisfied. Thus, Theorem 3.3 allows us to state that (3.14) has unique solution.

**Discussion of the Example.** The example presented in this work serves to illustrate the applicability of the theoretical results established through Banach fixed point theorem. Specifically, we consider a particular form of the proposed sequential Langevin equation with appropriate parameter values and kernel functions satisfying the assumptions imposed in Theorem 3.3. By substituting these values, we verify that the data of the example fulfill the conditions required for this result. This example confirms the theoretical framework and also demonstrates the consistency and reliability of the obtained analytical results.

#### 4. CONCLUSION AND FUTURE WORK

In this paper, we have investigated a new class of sequential Langevin equations involving three  $\varphi$ -Caputo fractional derivatives combined with a  $\varphi$ -Riemann–Liouville integral term. This formulation generalizes several existing fractional Langevin problems and provides a flexible framework for studying nonlocal and memory-dependent phenomena. By employing fixed point techniques, we established sufficient conditions ensuring both the existence and uniqueness of solutions. The application of Schauders fixed point theorem guaranteed the existence of at least one solution, and the Banach contraction principle provided uniqueness under suitable Lipschitz assumptions. Furthermore, the presented example demonstrated the validity of the theoretical results and illustrated the effectiveness of the proposed approach.

The results obtained show the importance of the  $\varphi$ -Caputo derivative in capturing nonuniform temporal effects and nonlinear deformations in time-dependent systems. This generalization enables an understanding of complex dynamical processes.

For future research, several directions can be explored. One possible extension is to study the Ulam–Hyers or Mittag-Leffler stability of the proposed Langevin system under perturbations. Another promising direction is to consider systems with time delays, impulses, or stochastic effects driven by  $\varphi$ -Caputo operators. It would also be of interest to develop efficient numerical algorithms to approximate the analytical results obtained here.

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