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COUPLED FIXED POINT RESULTS IN PARTIALLY ORDERED MODULAR METRIC SPACES AND APPLICATION TO THE VOLTERRA TYPE INTEGRAL EQUATIONS

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Abstract. In this work, we prove some coupled fixed point theorems for mixed monotone mappings in the setting of partially ordered modular metric spaces and provide some consequences of the established results. Moreover, we give some examples in support of the results. At the end of the paper, we give application to the Volterra type integral equations. The results proved in this paper extend, generalize and enrich several results in the existing literature (see, for example, *Bhaskar and Lakshmikantham* [6] and *Mutlu et al.* [21]).

1. Introduction

Fixed point theory is a growing area in mathematics with significant properties. In 1922, Banach [5] established the Banach contraction principle in the setting of complete metric spaces. Because of its simplicity and structure, it has become a very popular tool in solving existence problems in many branches of mathematical analysis. Many mathematicians have worked remarkably on fixed point results for partially ordered metric spaces. The very first result in

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this direction was taken by Ran and Reurings [30], it was a combination of the Banach contraction principle and the Knaster-Tarski fixed point theorem.

The concept of coupled fixed point was introduced by Guo and Lakshmikantham [14] in 1987. After this result, Bhaskar and Lakshmikantham [6] studied applications of coupled fixed point theorems for binary mappings. They introduced the concept of the mixed monotone property and proved certain coupled fixed point theorems. These theorems are among the most interesting coupled fixed point theorems for mappings in partially ordered metric spaces having the mixed monotone property. Also, they proved the existence of a unique solution for a periodic boundary value problem.

Nakano [23] presented the concept of modular space. Modular spaces were studied by Musielak and Orlicz and other authors, see [16, 19, 20, 27, 28]. The modular metric space was introduced by Chistyakov [8] who constructed a theory of this space via F-modular mappings in 2008. He also obtained some results in [9, 10]. Mongkolkeha et al. [18] explored contraction mappings in the setting of modular metric spaces to find fixed point theorems. Dehghan et al. [12] gave an example concerning some results obtained in [18]. Several fixed point theorems on modular metric spaces were proved and a homotopy application was given in [13]. The theory has evolved with works done by authors like Abdou [1], Abobaker and Ryan [2], Mitrovic et al. [17] and Hussain [15]. Many researchers have been pursuing the study of fixed points in the field of modular metric spaces (see, [3], [7], [22], [24], [25], [26], [29], [32]).

Recently, Mutlu et al. [21] extended certain coupled fixed point theorems which was introduced the mappings having the mixed monotone property in various metric spaces to partially ordered modular metric spaces and proved the existence of a unique solution for a given nonlinear integral equation (see, also Sharma and Jain [31]).

Inspired and motivated by the above research studies, the aim of this work is to prove certain coupled fixed point theorems for mixed monotone mappings in partially ordered modular metric spaces and provide some consequences of the main results. Also, we give some examples in support of the established results. Finally, an application to the Volterra type integral equation is included. Our results extend, generalize and enrich several results in the existing literature.

2. Preliminaries

In this section, we give a series of definitions of some fundamental concepts related to modular metric spaces.

Definition 2.1. ([19]) Let $\mathcal{M}$ be a vector space on $\mathbb{R}$ and $\rho\colon \mathcal{M} \to [0,\infty]$ be a function. If ρ satisfies the following axioms, we call that ρ is a modular on $\mathcal{M}$:
(1) $\rho(0) = 0$;
(2) If $a \in \mathcal{M}$ and $\rho(\beta\, a) = 0$ for all numbers $\beta > 0$, then $a = 0$;
(3) $\rho(-a) = \rho(a)$, for all $a \in \mathcal{M}$;
(4) $\rho(\beta\, a + \gamma\, b) \le \rho(a) + \rho(b)$, for all $\beta, \gamma \ge 0$ with $\beta + \gamma = 1$ and $a, b \in \mathcal{M}$.

Let $\mathcal{M} \neq \emptyset$ and $\omega\colon (0,\infty) \times \mathcal{M} \times \mathcal{M} \to [0,\infty]$ be a function where $\lambda \in (0,\infty)$. Throughout this paper, the value $\omega(\lambda, a, b)$ is denoted by $\omega_\lambda(a,b)$ for all $a, b \in \mathcal{M}$ and $\lambda > 0$.

Definition 2.2. ([9]) Let $\mathcal{M} \neq \emptyset$ and $\omega\colon (0,\infty) \times \mathcal{M} \times \mathcal{M} \to [0,\infty]$ be a function. If ω satisfies the following axioms for all $a, b, c \in \mathcal{M}$, we call that ω is a metric modular on $\mathcal{M}$:
(m_1) $\omega_\lambda(a,b) = 0$ for all $\lambda > 0 \Leftrightarrow a = b$;
(m_2) $\omega_\lambda(a,b) = \omega_\lambda(b,a)$ for all $\lambda > 0$;
(m_3) $\omega_{\lambda+\mu}(a,b) \le \omega_\lambda(a,c) + \omega_\mu(c,b)$ for all $\lambda, \mu > 0$.

If instead of condition (m_1), we have the condition (m_{1^*}),
(m_{1^*}) $\omega_\lambda(a,a) = 0$ for all $\lambda > 0$, $a \in \mathcal{M}$,
then ω is said to be pseudo-modular on $\mathcal{M}$. The main property of a (pseudo) modular function ω on a set $\mathcal{M}$ is that given $a, b \in \mathcal{M}$, the function $0 < \lambda \to \omega_\lambda(a,b) \in [0,\infty]$ is non-increasing on $(0,\infty)$.

In fact, if $0 < \mu < \lambda$, then (m_3), (m_{1^*}) and (m_2) imply

$$\omega_\lambda(a,b) \le \omega_{\lambda-\mu}(a,a) + \omega_\mu(a,b) = \omega_\mu(a,b). \tag{2.1}$$

A modular (pseudo-modular, strict modular) function ω on $\mathcal{M}$ is said to be convex if, instead of (m_3), for all $\lambda, \mu > 0$ and $a, b, c \in \mathcal{M}$ it satisfies the inequality

$$(m_4) \qquad \omega_{\lambda+\mu}(a,b) \le \frac{\lambda}{\lambda+\mu}\omega_\lambda(a,c) + \frac{\mu}{\lambda+\mu}\omega_\mu(c,b). \tag{2.2}$$

From [9, 10], we know that as fix $a_0 \in \mathcal{M}$, the two sets

$$\mathcal{M}_\omega = \mathcal{M}_\omega(a_0) = \{a \in \mathcal{M} : \omega_\lambda(a, a_0) \to 0 \,\text{as}\, \lambda \to \infty\} \tag{2.3}$$

and

$$\mathcal{M}^*{}_\omega = \mathcal{M}^*{}_\omega(a_0) = \{a \in \mathcal{M} : \exists\, \lambda = \lambda(a) > 0 \text{ such that } \omega_\lambda(a, a_0) < \infty\} \tag{2.4}$$

are each known as modular sets around a_0.

Moreover, from [9, 10], the modular space $\mathcal{M}_\omega$ can be equipped by a metric

$$d_\omega(a,b) = \inf\{\lambda > 0 : \omega_\lambda(a,b) \le \lambda\}, \tag{2.5}$$

which is generated by ω for any $a, b \in \mathcal{M}_\omega$, where ω is a modular on $\mathcal{M}$.

Definition 2.3. ([10, 18]) Let $\mathcal{M}_\omega$ be a modular metric space and $\{a_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}_\omega$ and $A \subset \mathcal{M}_\omega$. Then

(1) $\{a_n\}_{n\in\mathbb{N}}$ is called modular convergent to an element $a \in \mathcal{M}_\omega$, if $\omega_\lambda(a_n, a) \to 0$ as $n \to \infty$ for all $\lambda > 0$ and any such an element a will be called a modular limit of the sequence $\{a_n\}_{n\in\mathbb{N}}$.

(ii) $\{a_n\}_{n\in\mathbb{N}}$ is called a modular Cauchy sequence (ω-Cauchy) if and only if for all $\varepsilon > 0$ there exists $n(\varepsilon) \in \mathbb{N}$ such that $\omega_\lambda(a_n, a_m) < \varepsilon$ for each $m, n \geq n(\varepsilon)$ and $\lambda > 0$.

(iii) A is called complete modular if every modular Cauchy sequence $\{a_n\}$ in A is modular convergent in A.

Mongkolkeha et al. [18] introduced Banach contraction in modular metric spaces. The definition is as follows:

Definition 2.4. ([18]) Let $\mathcal{M}_\omega$ be a modular metric space. A self mapping T on $\mathcal{M}_\omega$ is said to be a contraction if there exists $0 \leq h < 1$ such that

$$\omega_\lambda(Ta, Tb) \leq h\, \omega_\lambda(a, b) \tag{2.6}$$

for all $a, b \in \mathcal{M}_\omega$ and $\lambda > 0$.

Definition 2.5. ([6]) Let $(\mathcal{M}, \leq)$ be a partially ordered set. The mapping $F\colon \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ is said to have the mixed monotone property if $F(a, b)$ is monotone non-decreasing in a and is monotone non-increasing in b, that is, for any $a, b \in \mathcal{M}$,

$$a_1, a_2 \in \mathcal{M}, \;\; a_1 \leq a_2 \;\Rightarrow\; F(a_1, b) \leq F(a_2, b)$$

and

$$b_1, b_2 \in \mathcal{M}, \;\; b_1 \leq b_2 \;\Rightarrow\; F(a, b_1) \geq F(a, b_2).$$

Definition 2.6. ([6, 11]) An element $(a, b) \in \mathcal{M} \times \mathcal{M}$ is said to be a coupled fixed point of the mapping $F\colon \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ if $F(a, b) = a$ and $F(b, a) = b$.

Example 2.7. ([4]) Let $\mathcal{M} = [0, +\infty)$ and $F\colon \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ be defined by $F(a, b) = \frac{a+b}{6}$ for all $a, b \in \mathcal{M}$. Then one can easily see that F has a unique coupled fixed point $(0, 0)$.

Example 2.8. ([4]) Let $\mathcal{M} = [0, +\infty)$ and $F\colon \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ be defined by $F(a, b) = \frac{a+b}{2}$ for all $a, b \in \mathcal{M}$. Then we see that F has two coupled fixed point $(0, 0)$ and $(1, 1)$, that is, the coupled fixed point is not unique.

3. Main results

In this section, we shall prove some coupled fixed point theorems in the setting of partially ordered modular metric spaces.

Theorem 3.1. *Let $(\mathcal{M}_\omega, \leq)$ be a partially ordered complete modular metric space and the mapping $F\colon \mathcal{M}_\omega \times \mathcal{M}_\omega \to \mathcal{M}_\omega$ have the mixed monotone property in $\mathcal{M}_\omega$. Suppose that the following condition is satisfies:*

$$\omega_\lambda(F(a,b), F(s,t)) \leq \Delta(a,b,s,t) \tag{3.1}$$

for all $a,b,s,t \in \mathcal{M}_\omega$ and $\lambda > 0$, where $a \geq s$, $b \leq t$,

$$\begin{aligned}\Delta(a,b,s,t) &= \alpha\,\omega_\lambda(a,s) + \beta\,\omega_\lambda(b,t) + \gamma\,\omega_\lambda(a,F(a,b)) \\ &\quad + \delta\,\omega_\lambda(s,F(s,t)) + \theta\,\omega_\lambda(s,F(a,b)),\end{aligned}$$

and $\alpha,\beta,\gamma,\delta,\theta$ are nonnegative reals such that $\alpha+\beta+\gamma+\delta+\theta < 1$. If there exist two elements $a_0, b_0 \in \mathcal{M}_\omega$ with $a_0 \leq F(a_0,b_0)$ and $b_0 \geq F(b_0,a_0)$, then F has a unique coupled fixed point in $\mathcal{M}_\omega$.

Proof. Let $a_0, b_0 \in \mathcal{M}_\omega$ be such that $a_0 \leq F(a_0,b_0)$ and $b_0 \geq F(b_0,a_0)$. Let $a_1 = F(a_0,b_0)$ and $b_1 = F(b_0,a_0)$. Then $a_0 \leq a_1$ and $b_0 \geq b_1$. Again, let $a_2 = F(a_1,b_1)$ and $b_2 = F(b_1,a_1)$. Since F has the mixed monotone property on $\mathcal{M}_\omega$, we have $a_1 \leq a_2$ and $b_1 \geq b_2$.

Repeating the above process, we get two sequences $\{a_n\}$ and $\{b_n\}$ in $\mathcal{M}_\omega$ such that $a_{n+1} = F(a_n,b_n)$ and $b_{n+1} = F(b_n,a_n)$ for all $n \geq 0$ and

$$a_0 \leq a_1 \leq \cdots \leq a_n \leq a_{n+1} \leq \ldots, \qquad b_0 \geq b_1 \geq \cdots \geq b_n \geq b_{n+1} \geq \ldots. \tag{3.2}$$

Now, using the condition (3.1) with $a = a_{n-1}$, $b = b_{n-1}$, $s = a_n$ and $t = b_n$, we have

$$\begin{aligned}\omega_\lambda(a_n, a_{n+1}) &= \omega_\lambda\big(F(a_{n-1},b_{n-1}), F(a_n,b_n)\big) \\ &\leq \Delta(a_{n-1},b_{n-1},a_n,b_n)\end{aligned} \tag{3.3}$$

for all $n \geq 0$ and $\lambda > 0$, where

$$\begin{aligned}\Delta(a_{n-1},b_{n-1},a_n,b_n) &= \alpha\,\omega_\lambda(a_{n-1},a_n) + \beta\,\omega_\lambda(b_{n-1},b_n) \\ &\quad + \gamma\,\omega_\lambda(a_{n-1},F(a_{n-1},b_{n-1})) + \delta\,\omega_\lambda(a_n,F(a_n,b_n)) \\ &\quad + \theta\,\omega_\lambda(a_n,F(a_{n-1},b_{n-1})) \\ &= \alpha\,\omega_\lambda(a_{n-1},a_n) + \beta\,\omega_\lambda(b_{n-1},b_n) + \gamma\,\omega_\lambda(a_{n-1},a_n) \\ &\quad + \delta\,\omega_\lambda(a_n,a_{n+1}) + \theta\,\omega_\lambda(a_n,a_n) \\ &= (\alpha+\gamma)\,\omega_\lambda(a_{n-1},a_n) + \beta\,\omega_\lambda(b_{n-1},b_n) + \delta\,\omega_\lambda(a_n,a_{n+1}).\end{aligned}$$

Using this in the equation (3.3), we obtain

$$\omega_\lambda(a_n,a_{n+1}) \leq (\alpha+\gamma)\,\omega_\lambda(a_{n-1},a_n) + \beta\,\omega_\lambda(b_{n-1},b_n) + \delta\,\omega_\lambda(a_n,a_{n+1}), \tag{3.4}$$

for all $n \geq 0$ and $\lambda > 0$.

By a similar fashion, we can obtain

$$\omega_\lambda(b_n, b_{n+1}) \leq (\alpha+\gamma)\,\omega_\lambda(b_{n-1}, b_n) + \beta\,\omega_\lambda(a_{n-1}, a_n) + \delta\,\omega_\lambda(b_n, b_{n+1}) \quad (3.5)$$

for all $n \geq 0$ and $\lambda > 0$. Adding equations (3.4) and (3.5), we obtain

$$\begin{aligned} \omega_\lambda(a_n, a_{n+1}) + \omega_\lambda(b_n, b_{n+1}) &\leq (\alpha+\gamma)[\omega_\lambda(a_{n-1}, a_n) + \omega_\lambda(b_{n-1}, b_n)] \\ &\quad + \beta\,[\omega_\lambda(a_{n-1}, a_n) + \omega_\lambda(b_{n-1}, b_n)] \\ &\quad + \delta\,[\omega_\lambda(a_n, a_{n+1}) + \omega_\lambda(b_n, b_{n+1})] \\ &= (\alpha+\beta+\gamma)[\omega_\lambda(a_{n-1}, a_n) + \omega_\lambda(b_{n-1}, b_n)] \\ &\quad + \delta\,[\omega_\lambda(a_n, a_{n+1}) + \omega_\lambda(b_n, b_{n+1})]. \qquad (3.6) \end{aligned}$$

Let $\Psi_n = \omega_\lambda(a_n, a_{n+1}) + \omega_\lambda(b_n, b_{n+1})$ for all $n \geq 0$ and $\lambda > 0$. Then from equation (3.6), we obtain

$$\Psi_n \leq (\alpha+\beta+\gamma)\Psi_{n-1} + \delta\,\Psi_n.$$

This implies that

$$\begin{aligned} \Psi_n &\leq \Big(\frac{\alpha+\beta+\gamma}{1-\delta}\Big)\Psi_{n-1} \\ &= \kappa\,\Psi_{n-1}, \end{aligned}$$

where

$$\kappa = \Big(\frac{\alpha+\beta+\gamma}{1-\delta}\Big).$$

Since by the assumption of the theorem $\alpha+\beta+\gamma+\delta+\theta < 1$, we have $\kappa < 1$.

Continuing the same way, we obtain

$$0 \leq \Psi_n \leq \kappa\,\Psi_{n-1} \leq \kappa^2\,\Psi_{n-2} \leq \cdots \leq \kappa^n\,\Psi_0. \quad (3.7)$$

If $\Psi_0 = 0$, then $\omega_\lambda(a_0, a_1) + \omega_\lambda(b_0, b_1) = 0$. Therefore, we get $\omega_\lambda(a_0, a_1) = 0$ and $\omega_\lambda(b_0, b_1) = 0$. Thus, from the condition (m_1) of modular metric space, we get $a_0 \leq a_1 = F(a_0, b_0)$ and $b_0 \geq b_1 = F(b_0, a_0)$. This implies that (a_0, b_0) is a coupled fixed point of F.

Now, we show that $\{a_n\}$ and $\{b_n\}$ are modular Cauchy sequences. Now, taking the limit as $n \to \infty$ in equation (3.7), we obtain

$$\lim_{n\to\infty} \Psi_n = \lim_{n\to\infty} (\omega_\lambda(a_n, a_{n+1}) + \omega_\lambda(b_n, b_{n+1})) = 0, \quad (3.8)$$

since $\kappa < 1$. Therefore, for a given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\omega_\lambda(a_n, a_{n+1}) + \omega_\lambda(b_n, b_{n+1}) < \varepsilon \text{ for all } n \geq n_0, \lambda > 0. \quad (3.9)$$

Without loss of generality, suppose that for $m, n \in \mathbb{N}$ and $n < m$, there exists $n_{\frac{\lambda}{m-n}} \in \mathbb{N}$ satisfying

$$\omega_{\frac{\lambda}{m-n}}(a_n, a_{n+1}) + \omega_{\frac{\lambda}{m-n}}(b_n, b_{n+1}) < \frac{\varepsilon}{m-n} \text{ for all } n \geq n_{\frac{\lambda}{m-n}}. \quad (3.10)$$

Thus, we obtain

$$\omega_\lambda(a_n, a_m) \leq \omega_{\frac{\lambda}{m-n}}(a_n, a_{n+1}) + \omega_{\frac{\lambda}{m-n}}(a_{n+1}, a_{n+2}) + \cdots + \omega_{\frac{\lambda}{m-n}}(a_{m-1}, a_m) \tag{3.11}$$

and

$$\omega_\lambda(b_n, b_m) \leq \omega_{\frac{\lambda}{m-n}}(b_n, b_{n+1}) + \omega_{\frac{\lambda}{m-n}}(b_{n+1}, b_{n+2}) + \cdots + \omega_{\frac{\lambda}{m-n}}(b_{m-1}, b_m) \tag{3.12}$$

for each $n < m$. Thus, from equations (3.9)-(3.11), we have

$$\begin{aligned}\omega_\lambda(a_n, a_m) + \omega_\lambda(b_n, b_m) &\leq [\omega_{\frac{\lambda}{m-n}}(a_n, a_{n+1}) + \omega_{\frac{\lambda}{m-n}}(b_n, b_{n+1})] \\ &\quad + \cdots + [\omega_{\frac{\lambda}{m-n}}(a_{m-1}, a_m) + \omega_{\frac{\lambda}{m-n}}(b_{m-1}, b_m)] \\ &< \frac{\varepsilon}{m-n} + \frac{\varepsilon}{m-n} + \cdots + \frac{\varepsilon}{m-n} = \varepsilon \end{aligned} \tag{3.13}$$

for all $n \geq n_{\frac{\lambda}{m-n}}$ and $\lambda > 0$. The above inequality shows that

$$\omega_\lambda(a_n, a_m) < \varepsilon, \ \ \omega_\lambda(b_n, b_m) < \varepsilon \tag{3.14}$$

for all $n \geq n_{\frac{\lambda}{m-n}}$. This shows that $\{a_n\}$ and $\{b_n\}$ are modular Cauchy sequences in $\mathcal{M}_\omega$. By the completeness of $\mathcal{M}_\omega$, there exist $a, b \in \mathcal{M}_\omega$, we have

$$\lim_{n\to\infty} a_n = a \text{ and } \lim_{n\to\infty} b_n = b.$$

Hence, there exists $n_0 \in \mathbb{N}$ with $\omega_{\frac{\lambda}{2}}(a_n, a) < \frac{\varepsilon}{2}$, $\omega_{\frac{\lambda}{2}}(b_n, b) < \frac{\varepsilon}{2}$, $\omega_{\frac{\lambda}{2}}(a_n, a_{n+1}) < \frac{\varepsilon}{2}$ and $\omega_{\frac{\lambda}{2}}(a_n, F(a, b)) < \frac{\varepsilon}{2}$ for all $n \geq n_0$, $\lambda > 0$ and every $\varepsilon > 0$. So, from equation (3.1), we obtain

$$\begin{aligned}\omega_\lambda(F(a, b), a) &\leq \omega_{\frac{\lambda}{2}}(F(a, b), a_{n+1}) + \omega_{\frac{\lambda}{2}}(a_{n+1}, a) \\ &= \omega_{\frac{\lambda}{2}}(F(a, b), F(a_n, b_n)) + \omega_{\frac{\lambda}{2}}(a_{n+1}, a) \\ &\leq \Delta(a, b, a_n, b_n) + \omega_{\frac{\lambda}{2}}(a_{n+1}, a),\end{aligned} \tag{3.15}$$

where

$$\begin{aligned}\Delta(a, b, a_n, b_n) &= \alpha\, \omega_{\frac{\lambda}{2}}(a, a_n) + \beta\, \omega_{\frac{\lambda}{2}}(b, b_n) + \gamma\, \omega_{\frac{\lambda}{2}}(a, F(a, b)) \\ &\quad + \delta\, \omega_{\frac{\lambda}{2}}(a_n, F(a_n, b_n)) + \theta\, \omega_{\frac{\lambda}{2}}(a_n, F(a, b)) \\ &= \alpha\, \omega_{\frac{\lambda}{2}}(a, a_n) + \beta\, \omega_{\frac{\lambda}{2}}(b, b_n) + \gamma\, \omega_{\frac{\lambda}{2}}(a, F(a, b)) \\ &\quad + \delta\, \omega_{\frac{\lambda}{2}}(a_n, a_{n+1}) + \theta\, \omega_{\frac{\lambda}{2}}(a_n, F(a, b)).\end{aligned}$$

Using this in equation (3.15), we obtain

$$\begin{aligned}\omega_\lambda(F(a,b),a) &\le \alpha\,\omega_{\frac{\lambda}{2}}(a,a_n)+\beta\,\omega_{\frac{\lambda}{2}}(b,b_n)+\gamma\,\omega_{\frac{\lambda}{2}}(a,F(a,b))\\ &\quad+\delta\,\omega_{\frac{\lambda}{2}}(a_n,a_{n+1})+\theta\,\omega_{\frac{\lambda}{2}}(a_n,F(a,b))+\omega_{\frac{\lambda}{2}}(a_{n+1},a)\\ &<\alpha\frac{\varepsilon}{2}+\beta\frac{\varepsilon}{2}+\gamma\,\omega_{\frac{\lambda}{2}}(a,F(a,b))+\delta\frac{\varepsilon}{2}+\theta\frac{\varepsilon}{2}+\frac{\varepsilon}{2}\\ &=(\alpha+\beta+\delta+\theta)\frac{\varepsilon}{2}+\gamma\,\omega_{\frac{\lambda}{2}}(a,F(a,b))+\frac{\varepsilon}{2}.\end{aligned}$$

This implies that

$$(1-\gamma)\omega_\lambda(F(a,b),a)<(\alpha+\beta+\delta+\theta)\frac{\varepsilon}{2}+\frac{\varepsilon}{2}$$

or

$$\begin{aligned}\omega_\lambda(F(a,b),a) &< \Big(\frac{\alpha+\beta+\delta+\theta}{1-\gamma}\Big)\frac{\varepsilon}{2}+\Big(\frac{1}{1-\gamma}\Big)\frac{\varepsilon}{2}\\ &<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon,\end{aligned}$$

since by hypothesis $\alpha+\beta+\gamma+\delta+\theta<1$. Thus, we get $F(a,b)=a$.

By a similar fashion, we can show that $F(b,a)=b$. This shows that (a,b) is a coupled fixed point of F.

Now, we prove the uniqueness of coupled fixed point. Assume that F has another coupled fixed point (a_m,b_m) such that $(a,b)\neq(a_m,b_m)$. Then for $\lambda>0$, we obtain

$$\begin{aligned}\omega_\lambda(a_m,a) &= \omega_\lambda(F(a_m,b_m),F(a,b))\\ &\le \Delta(a_m,b_m,a,b),\end{aligned}\tag{3.16}$$

where

$$\begin{aligned}\Delta(a_m,b_m,a,b) &= \alpha\,\omega_\lambda(a_m,a)+\beta\,\omega_\lambda(b_m,b)+\gamma\,\omega_\lambda(a_m,F(a_m,b_m))\\ &\quad+\delta\,\omega_\lambda(a,F(a,b))+\theta\,\omega_\lambda(a,F(a_m,b_m))\\ &= \alpha\,\omega_\lambda(a_m,a)+\beta\,\omega_\lambda(b_m,b)+\gamma\,\omega_\lambda(a_m,a_m)\\ &\quad+\delta\,\omega_\lambda(a,a)+\theta\,\omega_\lambda(a,a_m)\\ &= (\alpha+\theta)\omega_\lambda(a_m,a)+\beta\,\omega_\lambda(b_m,b).\end{aligned}$$

Using this in equation (3.16), we obtain

$$\omega_\lambda(a_m,a)\le(\alpha+\theta)\omega_\lambda(a_m,a)+\beta\,\omega_\lambda(b_m,b).\tag{3.17}$$

Similarly, we can show that

$$\omega_\lambda(b_m,b)\le(\alpha+\theta)\omega_\lambda(b_m,b)+\beta\,\omega_\lambda(a_m,a).\tag{3.18}$$

Adding equations (3.17) and (3.18), we obtain

$$\begin{aligned}\omega_\lambda(a_m,a)+\omega_\lambda(b_m,b) &\le (\alpha+\theta)[\omega_\lambda(a_m,a)+\omega_\lambda(b_m,b)]\\ &\quad+\beta\,[\omega_\lambda(a_m,a)+\omega_\lambda(b_m,b)]\\ &=(\alpha+\beta+\theta)[\omega_\lambda(a_m,a)+\omega_\lambda(b_m,b)]\\ &<\omega_\lambda(a_m,a)+\omega_\lambda(b_m,b),\end{aligned}$$

which is a contradiction, since $\alpha+\beta+\theta<1$. Hence, we get $\omega_\lambda(a_m,a)+\omega_\lambda(b_m,b)=0$ for all $\lambda>0$. Thus, we obtain $\omega_\lambda(a_m,a)=0$, that is, $a_m=a$ and $\omega_\lambda(b_m,b)=0$, that is, $b_m=b$. Therefore, (a,b) is a unique coupled fixed point of F. This completes the proof. □

Remark 3.2. Theorem 3.1 generalizes Theorem 3.3 of [21] for more general contractive condition.

Remark 3.3. Theorem 3.1 also extends and generalizes Theorem 2.1 of [6] from partially ordered complete metric spaces to partially ordered complete modular metric spaces and more general contractive condition.

If we take $\alpha=k$, $\beta=l$ and $\gamma=\delta=\theta=0$ where $k,l\in[0,1)$ in Theorem 3.1, then we have the following result.

Corollary 3.4. (Theorem 3.3, [21]) *Let $(\mathcal{M}_\omega,\le)$ be a partially ordered complete modular metric space and the mapping $F\colon\mathcal{M}_\omega\times\mathcal{M}_\omega\to\mathcal{M}_\omega$ have the mixed monotone property in $\mathcal{M}_\omega$. Suppose that the following condition is satisfied:*

$$\omega_\lambda(F(a,b),F(s,t))\le k\,\omega_\lambda(a,s)+l\,\omega_\lambda(b,t), \tag{3.19}$$

for all $a,b,s,t\in\mathcal{M}_\omega$ and $\lambda>0$, where $a\ge s$, $b\le t$ and k,l are nonnegative reals such that $k+l<1$. If there exist two elements $a_0,b_0\in\mathcal{M}_\omega$ with $a_0\le F(a_0,b_0)$ and $b_0\ge F(b_0,a_0)$, then F has a unique coupled fixed point in $\mathcal{M}_\omega$.

If we take $\gamma=k$, $\delta=l$ and $\alpha=\beta=\theta=0$ where $k,l\in[0,1)$ in Theorem 3.1, then we have the following result.

Corollary 3.5. *Let $(\mathcal{M}_\omega,\le)$ be a partially ordered complete modular metric space and the mapping $F\colon\mathcal{M}_\omega\times\mathcal{M}_\omega\to\mathcal{M}_\omega$ have the mixed monotone property in $\mathcal{M}_\omega$. Suppose that the following condition is satisfied:*

$$\omega_\lambda(F(a,b),F(s,t))\le k\,\omega_\lambda(a,F(a,b))+l\,\omega_\lambda(s,F(s,t)), \tag{3.20}$$

for all $a,b,s,t\in\mathcal{M}_\omega$ and $\lambda>0$, where $a\ge s$, $b\le t$ and k,l are nonnegative reals such that $k+l<1$. If there exist two elements $a_0,b_0\in\mathcal{M}_\omega$ with $a_0\le F(a_0,b_0)$ and $b_0\ge F(b_0,a_0)$, then F has a unique coupled fixed point in $\mathcal{M}_\omega$.

If we take $k = l$ in Corollary 3.4 and Corollary 3.5, then we have the following results.

Corollary 3.6. (Corollary 3.4, [21]) *Let $(\mathcal{M}_\omega, \leq)$ be a partially ordered complete modular metric space and the mapping $F\colon \mathcal{M}_\omega \times \mathcal{M}_\omega \to \mathcal{M}_\omega$ have the mixed monotone property in $\mathcal{M}_\omega$. Suppose that the following condition is satisfied:*

$$\omega_\lambda(F(a,b), F(s,t)) \leq \frac{k}{2}\,[\omega_\lambda(a,s) + \omega_\lambda(b,t)], \tag{3.21}$$

for all $a, b, s, t \in \mathcal{M}_\omega$ and $\lambda > 0$, where $a \geq s$, $b \leq t$ and $k \in [0,1)$ is a constant. If there exist two elements $a_0, b_0 \in \mathcal{M}_\omega$ with $a_0 \leq F(a_0, b_0)$ and $b_0 \geq F(b_0, a_0)$, then F has a unique coupled fixed point in $\mathcal{M}_\omega$.

Remark 3.7. Corollary 3.6 extends and generalizes Theorem 2.1 of [6] from partially ordered complete metric spaces to partially ordered complete modular metric spaces.

Corollary 3.8. *Let $(\mathcal{M}_\omega, \leq)$ be a partially ordered complete modular metric space and the mapping $F\colon \mathcal{M}_\omega \times \mathcal{M}_\omega \to \mathcal{M}_\omega$ have the mixed monotone property in $\mathcal{M}_\omega$. Suppose that the following condition is satisfied:*

$$\omega_\lambda(F(a,b), F(s,t)) \leq \frac{k}{2}\,[\omega_\lambda(a, F(a,b)) + \omega_\lambda(s, F(s,t))], \tag{3.22}$$

for all $a, b, s, t \in \mathcal{M}_\omega$ and $\lambda > 0$, where $a \geq s$, $b \leq t$ and $k \in [0,1)$ is a constant. If there exist two elements $a_0, b_0 \in \mathcal{M}_\omega$ with $a_0 \leq F(a_0, b_0)$ and $b_0 \geq F(b_0, a_0)$, then F has a unique coupled fixed point in $\mathcal{M}_\omega$.

If we take $\alpha = h$, $\beta = \gamma = \delta = \theta = 0$ and define $F(a,a) = \mathcal{T}a$ in Theorem 3.1, then we have the following result.

Corollary 3.9. (Theorem 3.2, [18]) *Let ω be a metric modular on $\mathcal{M}$ and $\mathcal{M}_\omega$ be a modular metric space induced by ω. If $\mathcal{M}_\omega$ is complete, then the mapping $\mathcal{T}\colon \mathcal{M}_\omega \to \mathcal{M}_\omega$ satisfies the contractive condition:*

$$\omega_\lambda(\mathcal{T}a, \mathcal{T}b) \leq h\,\omega_\lambda(a,b) \tag{3.23}$$

for all $a, b \in \mathcal{M}_\omega$ and all $\lambda > 0$, where $h \in [0,1)$ is a constant. Then $\mathcal{T}$ has a unique fixed point in $\mathcal{M}_\omega$.

Theorem 3.10. *Let $(\mathcal{M}_\omega, \leq)$ be a partially ordered complete modular metric space and the mapping $F\colon \mathcal{M}_\omega \times \mathcal{M}_\omega \to \mathcal{M}_\omega$ has the mixed monotone property in $\mathcal{M}_\omega$. Suppose that the following condition is satisfies:*

$$\omega_\lambda(F(a,b), F(s,t)) \leq \sigma\,\Psi(a,b,s,t) \tag{3.24}$$

for all $a, b, s, t \in \mathcal{M}_\omega$ *and* $\lambda > 0$*, where* $a \geq s$*,* $b \leq t$*,*

$$\Psi(a,b,s,t) = \max\Big\{\omega_\lambda(a,s), \omega_\lambda(b,t), \omega_\lambda(a, F(a,b)), \omega_\lambda(s, F(s,t)), \omega_\lambda(s, F(a,b))\Big\}$$

and $\sigma \in (0,1)$ *is a constant. If there exist two elements* $a_0, b_0 \in \mathcal{M}_\omega$ *with* $a_0 \leq F(a_0, b_0)$ *and* $b_0 \geq F(b_0, a_0)$*, then* F *has a unique coupled fixed point in* $\mathcal{M}_\omega$*.*

Proof. It follows from Theorem 3.1 by noting that

$$\begin{aligned}
&\alpha\,\omega_\lambda(a,s) + \beta\,\omega_\lambda(b,t) + \gamma\,\omega_\lambda(a,F(a,b)) + \delta\,\omega_\lambda(s,F(s,t)) + \theta\,\omega_\lambda(s,F(a,b)) \\
&\quad \leq (\alpha+\beta+\gamma+\delta+\theta)\max\Big\{\omega_\lambda(a,s), \omega_\lambda(b,t), \omega_\lambda(a,F(a,b)), \\
&\qquad\qquad \omega_\lambda(s,F(s,t)), \omega_\lambda(s,F(a,b))\Big\} \\
&\quad = \sigma\ \max\Big\{\omega_\lambda(a,s), \omega_\lambda(b,t), \omega_\lambda(a,F(a,b)), \omega_\lambda(s,F(s,t)), \omega_\lambda(s,F(a,b))\Big\},
\end{aligned}$$

where $\sigma = \alpha+\beta+\gamma+\delta+\theta < 1$. □

Example 3.11. Let $\mathcal{M}_\omega = \mathbb{R}$. If we take the usual partial order " $\leq$ " in $\mathbb{R}$, then $(\mathcal{M}_\omega, \leq)$ is a partially ordered set. We define a mapping $\omega\colon (0,\infty) \times \mathcal{M}_\omega \times \mathcal{M}_\omega \to [0,\infty)$ by $\omega_\lambda(a,b) = \frac{|a-b|}{\lambda}$ for all $a, b \in \mathcal{M}_\omega$ and $\lambda > 0$. It can said that $\mathcal{M}_\omega$ is a complete modular metric space. We consider the mapping $F\colon \mathcal{M}_\omega \times \mathcal{M}_\omega \to \mathcal{M}_\omega$ such that $F(a,b) = \frac{a+b}{6}$. It is easy to see that F has the mixed monotone property. Then, we have

$$\begin{aligned}
\omega_\lambda(F(a,b), F(s,t)) &= \omega_\lambda\Big(\frac{a+b}{6}, \frac{s+t}{6}\Big) \\
&= \frac{1}{\lambda}\Big|\frac{a+b}{6} - \frac{s+t}{6}\Big| \\
&= \frac{1}{6}\Big|\frac{a-s+b-t}{\lambda}\Big| \\
&\leq \frac{1}{6}\Big(\frac{|a-s|}{\lambda} + \frac{|b-t|}{\lambda}\Big) \\
&= \frac{1}{6}\Big(\omega_\lambda(a,s) + \omega_\lambda(b,t)\Big)
\end{aligned}$$

for any $a, b, s, t \in \mathcal{M}_\omega$ and $\lambda > 0$. Thus, the inequality (3.21) of Corollary 3.6 is satisfied for $k = \frac{1}{3} < 1$. Therefore, from Corollary 3.6, F has a unique coupled fixed point. Also, there are $a_0 = 0 \leq F(0,0) = F(a_0, b_0)$ and $b_0 = 0 \geq F(0,0) = F(b_0, a_0)$. It is obvious that $(0,0)$ is the unique coupled fixed point of F. Similarly, we can verify the conditions of Corollary 3.8.

On the other hand if $F\colon \mathcal{M}_\omega \times \mathcal{M}_\omega \to \mathcal{M}_\omega$ is defined by $F(a,b) = \frac{a+b}{2}$, then F satisfies the the inequality (3.21) of Corollary 3.6 for $k = 1$. Then, we obtain

$$\begin{aligned} \omega_\lambda(F(a,b),F(s,t)) &= \omega_\lambda\Big(\frac{a+b}{2},\frac{s+t}{2}\Big) \\ &= \frac{1}{\lambda}\Big|\frac{a+b}{2}-\frac{s+t}{2}\Big| \\ &= \frac{1}{2}\Big|\frac{a-s+b-t}{\lambda}\Big| \\ &\le \frac{1}{2}\Big(\frac{|a-s|}{\lambda}+\frac{|b-t|}{\lambda}\Big) \\ &= \frac{1}{2}\Big(\omega_\lambda(a,s)+\omega_\lambda(b,t)\Big). \end{aligned}$$

In this case, the coupled fixed points of F are $(0,0)$ and $(1,1)$. Thus, the coupled fixed point of F is not unique. Hence, the conditions $k < 1$ in Corollary 3.6 and Corollary 3.8 and $k+l < 1$ in Corollary 3.4 and Corollary 3.5 are the most appropriate conditions for satisfying the uniqueness of coupled fixed point.

Example 3.12. Let $\mathcal{M}_\omega = \mathbb{R}$. If we take the usual partial order " $\le$ " in $\mathbb{R}$. Then $(\mathcal{M}_\omega, \le)$ is a partially ordered set. We define a mapping $\omega\colon (0,\infty) \times \mathcal{M}_\omega \times \mathcal{M}_\omega \to [0,\infty)$ by $\omega_\lambda(a,b) = \frac{|a-b|}{\lambda}$ for all $a,b \in \mathcal{M}_\omega$ and $\lambda > 0$. It can said that $\mathcal{M}_\omega$ is a complete modular metric space. We consider the mapping $F\colon \mathcal{M}_\omega \times \mathcal{M}_\omega \to \mathcal{M}_\omega$ such that $F(a,b) = \frac{3a-b}{5}$. It is easy to see that F has the mixed monotone property. Let us take $a \ne s$ and $b = t$ in the inequality of Corollary 3.6. Hence $\mu = |a-s| > 0$.

Now, using inequality of Corollary 3.6, we have

$$\begin{aligned} \frac{3}{5\lambda}\mu &= \frac{3|a-s|}{5\lambda} \\ &= \omega_\lambda(F(a,b),F(s,t)) \\ &\le \frac{k}{2}\big(\omega_\lambda(a,s)+\omega_\lambda(b,t)\big) \\ &= \frac{k}{2}\Big(\frac{|a-s|}{\lambda}+\frac{|b-t|}{\lambda}\Big) \\ &= \frac{k}{2}\Big(\frac{|a-s|}{\lambda}\Big) \\ &= \frac{k}{2\lambda}\mu, \end{aligned}$$

that is,

$$\frac{3}{5\lambda}\mu \leq \frac{k}{2\lambda}\mu \ \text{ or } \ k \geq \frac{6}{5},$$

which is a contradiction. Hence the Corollary 3.6 is not applicable to the operator F in order to prove that $(0,0)$ is a unique coupled fixed point of F.

4. An application to the Volterra type integral equations

In this section, we show that there exists a unique solution of a Volterra type integral equations using the Corollary 3.6.

We consider the following Volterra type integral equations:

$$\begin{aligned} a(u) &= h(u) + \int_0^U k(u,p)g(u,a(p),b(p))dp, \ u \in [0,U] = I, \\ b(u) &= h(u) + \int_0^U k(u,p)g(u,b(p),a(p))dp, \ u \in [0,U] = I, \end{aligned} \tag{4.1}$$

where $U \in \mathbb{R}^+$ with $U > 0$, $g\colon I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $k\colon I \times \mathbb{R} \to \mathbb{R}$ and $h\colon I \to \mathbb{R}$, $I = [0,U]$ and $k(u,p)$ is called the kernel function.

Let $\mathcal{M}_\omega = C(I,\mathbb{R})$ and $\mathcal{M}_\omega$ be a partially ordered set. We define the order relation as follows:

$$a \leq b \ \Leftrightarrow \ a(u) \leq b(u),$$

for all $a, b \in \mathcal{M}_\omega$ and all $u \in I$. We can easily see that $\mathcal{M}_\omega$ is a complete modular metric space such that

$$\omega_\lambda(a,b) = \sup_{u \in I} \frac{|a(u) - b(u)|}{\lambda},$$

for all $a, b \in \mathcal{M}_\omega$ and $\lambda > 0$.

We consider the following assumptions:

(i) the mappings $g\colon I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $k\colon I \times \mathbb{R} \to \mathbb{R}$ and $h\colon I \to \mathbb{R}$ are continuous;

(ii) there exists a non-negative constant $0 \leq k < 1$ such that

$$0 \leq g(u,a,b) - g(u,s,t) \leq \frac{k}{2}\Big(\frac{|a-s| + |b-t|}{\lambda}\Big), \tag{4.2}$$

for all $a, b, s, t \in \mathcal{M}_\omega$, $u \in I$ and $\lambda > 0$ where $a \geq s$, $b \leq t$;

(iii)

$$\sup_{u \in I} \int_0^U |k(u,p)|dp \leq 1;$$

(iv) there exist $x_0, y_0 \in \mathcal{M}_\omega$ such that

$$x_0(u) \geq h(u) + \int_0^U k(u,p)g(u, x_0(p), y_0(p))dp,$$
$$y_0(u) \leq h(u) + \int_0^U k(u,p)g(u, y_0(p), x_0(p))dp.$$

Definition 4.1. $(\eta, \theta) \in C(I, \mathbb{R}) \times C(I, \mathbb{R})$ is called a coupled lower and upper solution of the integral equations (4.1) if $\eta(u) \leq \theta(u)$ and

$$\eta(u) \leq h(u) + \int_0^U k(u,p)g(u, \eta(p), \theta(p))dp,$$
$$\theta(u) \leq h(u) + \int_0^U k(u,p)g(u, \theta(p), \eta(p))dp$$

for all $u \in I$.

Theorem 4.2. *Consider the Corollary* 3.6 *and assume that the conditions* (*i*) - (*iv*) *are satisfied. The integral equations* (4.1) *has a unique solution in* $\mathcal{M}_\omega = C(I, \mathbb{R})$ *if there exists a coupled lower and upper solution for equations* (4.1).

Proof. Let $\mathcal{M}_\omega = C(I, \mathbb{R})$, $\mathcal{M}_\omega$ is a partially ordered set if we define the order relation such that $a, b \in C(I, \mathbb{R})$ and all $u \in I$,

$$a \leq b \Leftrightarrow a(u) \leq b(u).$$

It is clear that $\mathcal{M}_\omega$ is a complete modular metric space such that

$$\omega_\lambda(a, b) = \sup_{u \in I} \frac{|a(u) - b(u)|}{\lambda}, \tag{4.3}$$

for all $a, b \in C(I, \mathbb{R})$ and all $\lambda > 0$. Also, we define a partial order on $\mathcal{M}_\omega \times \mathcal{M}_\omega$, where $\mathcal{M}_\omega = C(I, \mathbb{R})$ such that

$$(a, b) \leq (s, t) \Rightarrow a(u) \leq s(u) \text{ and } b(u) \geq t(u),$$

for $(a, b), (s, t) \in \mathcal{M}_\omega \times \mathcal{M}_\omega$ and all $u \in I$.

Now, we define $F\colon \mathcal{M}_\omega \times \mathcal{M}_\omega \to \mathcal{M}_\omega$ with

$$F(a, b)(u) = h(u) + \int_0^U k(u,p)g(u, a(p), b(p))dp, \tag{4.4}$$

for all $a, b \in \mathcal{M}_\omega$ and $u \in I$.

We need to show that F has the mixed monotone property. If $a_1 \leq a_2$, that is, $a_1(u) \leq a_2(u)$ for all $u \in I$, by assumption (ii), we get

$$\begin{aligned} F(a_1, b)(u) - F(a_2, b)(u) &= \int_0^U k(u,p)g(u, a_1(p), b(p))dp \\ &\quad - \int_0^U k(u,p)g(u, a_2(p), b(p))dp \\ &= \int_0^U k(u,p)\big(g(u, a_1(p), b(p)) \\ &\quad - g(u, a_2(p), b(p))\big)dp \\ &\leq 0. \end{aligned}$$

Then, $F(a_1, b)(u) \leq F(a_2, b)(u)$ for all $u \in I$, that is, $F(a_1, b) \leq F(a_2, b)$.

Similarly, if $b_1 \geq b_2$, that is, $b_1(u) \geq b_2(u)$ for all $u \in I$, by assumption (ii), we get

$$\begin{aligned} F(a, b_1)(u) - F(a, b_2)(u) &= \int_0^U k(u,p)g(u, a(p), b_1(p))dp \\ &\quad - \int_0^U k(u,p)g(u, a(p), b_2(p))dp \\ &= \int_0^U k(u,p)\big(g(u, a(p), b_1(p)) \\ &\quad - g(u, a(p), b_2(p))\big)dp \\ &\leq 0. \end{aligned}$$

Then, $F(a, b_1)(u) \leq F(a, b_2)(u)$ for all $u \in I$, that is, $F(a, b_1) \leq F(a, b_2)$. Thus, $F(a, b)$ is monotone non-decreasing in a and monotone non-increasing in b.

Now, we show that F has a coupled fixed point. Let $a \geq s$ and $b \leq t$. Then $a(u) \geq s(u)$ and $b(u) \leq t(u)$ for all $u \in I$. From equations (4.2), (4.3) and (4.4), we obtain

$$\begin{aligned} &\frac{|F(a,b)(u) - F(s,t)(u)|}{\lambda} \\ &= \frac{\left| \int_0^U k(u,p)g(u, a(p), b(p))dp - \int_0^U k(u,p)g(u, s(p), t(p))dp \right|}{\lambda} \\ &= \frac{\left| \int_0^U k(u,p)\Big(g(u, a(p), b(p)) - g(u, s(p), t(p))\Big)dp \right|}{\lambda} \end{aligned}$$

$$
\begin{aligned}
&\le \frac{\int_0^U |k(u,p)|\big|g(u,a(p),b(p)) - g(u,s(p),t(p)\big|dp}{\lambda} \\
&\le \frac{k}{2}\,\frac{\int_0^U |k(u,p)|\big|(a(p)-s(p)) + (b(p)-t(p))\big|dp}{\lambda} \\
&\le \frac{k}{2}\int_0^U |k(u,p)|\Big(\frac{|a(p)-s(p)|}{\lambda} + \frac{|b(p)-t(p)|}{\lambda}\Big)dp \\
&\le \frac{k}{2}\int_0^U \sup_{u\in I}|k(u,p)|\Big(\sup_{u\in I}\frac{|a(p)-s(p)|}{\lambda} + \sup_{u\in I}\frac{|b(p)-t(p)|}{\lambda}\Big)dp \\
&\le \frac{k}{2}\Big(\sup_{u\in I}\frac{|a(u)-s(u)|}{\lambda} + \sup_{u\in I}\frac{|b(u)-t(u)|}{\lambda}\Big)\sup_{u\in I}\int_0^U |k(u,p)|dp \\
&\le \frac{k}{2}\Big(\sup_{u\in I}\frac{|a(u)-s(u)|}{\lambda} + \sup_{u\in I}\frac{|b(u)-t(u)|}{\lambda}\Big),
\end{aligned}
$$

for all $\lambda > 0$ which implies that

$$
\frac{|F(a,b)(u) - F(s,t)(u)|}{\lambda} \le \frac{k}{2}\Big(\sup_{u\in I}\frac{|a(u)-s(u)|}{\lambda} + \sup_{u\in I}\frac{|b(u)-t(u)|}{\lambda}\Big).
$$

Hence, we obtain

$$
\omega_\lambda(F(a,b),F(s,t)) \le \frac{k}{2}\big(\omega_\lambda(a,s) + \omega_\lambda(b,t)\big),
$$

for all $a,b,s,t \in \mathcal{M}_\omega$ and all $\lambda > 0$.

On the other hand, let (η,θ) be a coupled lower and upper solution of the integral equations (4.1). Then we obtain

$$
\eta(u) \le F(\eta,\theta)(u) \text{ and } \theta(u) \ge F(\theta,\eta)(u),
$$

for all $u \in I$, where $\eta,\theta \in \mathcal{M}_\omega$. Hence,

$$
\eta \le F(\eta,\theta) \text{ and } \theta \ge F(\theta,\eta).
$$

Thus, we obtain that all the conditions of Corollary 3.6 are satisfied. Hence, from Corollary 3.6, we get a unique solution $(a,b) \in \mathcal{M}_\omega \times \mathcal{M}_\omega$ of integral equations (4.1), where $\mathcal{M}_\omega = C(I,\mathbb{R})$. □

The aforesaid application is illustrated by the following example.

Example 4.3. Let $\mathcal{M}_\omega = C([0,1],\mathbb{R})$ and $g\colon I\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$. Now consider the following functional integral equation:

$$
\begin{aligned}
a(u) &= \frac{u^3+7}{4} + \int_0^1 \frac{p^2}{24(u+3)}\frac{\sin u}{9}\Big(\frac{|a(p)|}{1+|a(p)|} + \frac{|b(p)|}{1+|b(p)|}\Big)dp, \\
b(u) &= \frac{u^3+7}{4} + \int_0^1 \frac{p^2}{24(u+3)}\frac{\sin u}{9}\Big(\frac{|b(p)|}{1+|b(p)|} + \frac{|a(p)|}{1+|a(p)|}\Big)dp, \qquad (4.5)
\end{aligned}
$$

for all $a, b \in \mathcal{M}_\omega$ and $u \in I = [0,1]$. Observe that the above equation (4.5) is a special case of equations (4.1) with

$$\begin{aligned} g(u,a,b) &= \frac{\sin u}{9}\Big(\frac{|a(p)|}{1+|a(p)|} + \frac{|b(p)|}{1+|b(p)|}\Big), \\ g(u,b,a) &= \frac{\sin u}{9}\Big(\frac{|b(p)|}{1+|b(p)|} + \frac{|a(p)|}{1+|a(p)|}\Big), \end{aligned}$$

$$h(u) = \frac{u^3+7}{4},\ k(u,p) = \frac{p^2}{24(u+3)} \text{ and } U = 1.$$

It is also easily seen that the functions g, h and k are continuous.

Moreover, there exists $k = \frac{1}{2}$ with $0 < k < 1$ such that

$$0 \le g(u,a,b) - g(u,s,t) \le \frac{k}{2}\Big(\frac{|a-s|+|b-t|}{\lambda}\Big),$$

for all $a, b, s, t \in \mathcal{M}_\omega$, $u \in I$ and $\lambda > 0$, where $a \ge s$, $b \le t$ and

$$\begin{aligned} \sup_{u\in[0,U]} \int_0^U k(u,p)dp &= \sup_{u\in[0,1]} \int_0^1 \frac{p^2}{24(u+3)}dp \\ &= \sup_{u\in[0,1]} \Big(\frac{1}{72(u+3)}\Big) < 1. \end{aligned}$$

Thus, the assumptions (i)-(iii) of Theorem 4.2 are satisfied.

Now, for arbitrary $a, b, s, t \in \mathcal{M}_\omega$, $\lambda > 0$ and for all $u \in I = [0,1]$, we have

$$\begin{aligned} \frac{|g(u,a,b) - g(u,s,t)|}{\lambda} &= \frac{1}{\lambda}\Big|\frac{\sin u}{9}\Big(\frac{|a(p)|}{1+|a(p)|} + \frac{|b(p)|}{1+|b(p)|}\Big) \\ &\quad - \frac{\sin u}{9}\Big(\frac{|s(p)|}{1+|s(p)|} + \frac{|t(p)|}{1+|t(p)|}\Big)\Big| \\ &\le \frac{1}{9\lambda}\big(|a-s|+|b-t|\big) = \frac{k}{2\lambda}\big(|a-s|+|b-t|\big). \end{aligned}$$

Therefore, the function g satisfies equation (4.2) with $k = \frac{2}{9} < 1$.

Now, consider $x_0(u) = 1$ and $y_0(u) = 1$. Then we have

$$\begin{aligned} &h(u) + \int_0^U k(u,p) g(u, y_0(p), x_0(p))dp \\ &= \frac{u^3+7}{4} + \int_0^1 \frac{p^2}{24(u+3)}\Big(\frac{1}{2}+\frac{1}{2}\Big)dp \\ &= \frac{u^3+7}{4} + \int_0^1 \frac{p^2}{24(u+3)}dp \\ &= \frac{u^3+7}{4} + \frac{1}{72(u+3)} \ge 1. \end{aligned}$$

That is, $y_0 \leq F(y_0, x_0)$. Similarly, it can be shown that $x_0 \geq F(x_0, y_0)$. Thus all the conditions of Theorem 4.2 are satisfied. It follows that the integral equation (4.5) has a unique solution in $\mathcal{M}_\omega$ with $\mathcal{M}_\omega = C([0,1], \mathbb{R})$.

5. Conclusion

In this article, we establish some coupled fixed point theorems in the setting of partially ordered modular metric spaces and provide some consequences as corollaries of the established results. Furthermore, we give some examples in support of the established results. Finally, an application to the Volterra type integral equation is also included. The results reported in this paper extend, generalize and enrich several results in the existing literature (see, for example [6] and [21]).

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