# ON SPLIT FEASIBILITY AND FIXED POINT EQUALITY PROBLEMS

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**Abstract.** This paper introduces a new mathematical problem, termed the split feasibility and fixed point equality problem (SFFPEP), which generalizes several well-known problems, including the split feasibility problem (SFP), the split equality problem (SEP), the split feasibility and fixed point problem (SFFPP), and the split equality fixed point problem (SEFPP). To solve the SFFPEP, we propose a novel iterative algorithm within the framework of quasi-nonexpansive mappings in Hilbert spaces. We establish the convergence properties of the algorithm and provide a numerical example to demonstrate its effectiveness, thereby validating our theoretical findings. The proposed approach contributes to the broader study of split problems and fixed point theory, offering new insights and practical tools for solving complex optimization problems.

## 1. Introduction

In recent years, the study of split problems and fixed point theory has gained significant attention due to their wide-ranging applications in fields such as optimization, signal processing, medical imaging, and intensity modulation radiation therapy; see [2, 6, 7, 8, 19]. Among the most well-known problems in

this domain are the split feasibility problem (SFP), the split equality problem (SEP), and their variants, which have been extensively studied and applied in various practical contexts; see [3, 4, 5, 10, 11, 12, 14, 15, 16, 17, 18, 20, 21, 23, 24, 27].

The SFP is formulated as follows:

$$\text{Find } x^* \in C \text{ such that } Ax^* \in Q, \tag{1.1}$$

where $C$ and $Q$ are nonempty closed convex subsets of Hilbert spaces $H_1$ and $H_2$, respectively, and $A : H_1 \to H_2$ is a bounded linear operator.

The SEP, introduced by Moudafi [20], generalizes the SFP, and it involves finding points

$$x^* \in C \text{ and } y^* \in Q \text{ such that } A_1 x^* = A_2 y^*, \tag{1.2}$$

where $C$ and $Q$ are defined as in equation (1.1), and $A_1 : H_1 \to H_3$ and $A_2 : H_2 \to H_3$ are bounded linear operators.

Since every nonempty, closed, and convex subset of a Hilbert space can be represented as the fixed-point set of its associated projection operator, problem (1.2) can be simplified to:

$$\text{Find } x^* \in \text{Fix}(T_1) \text{ and } y^* \in \text{Fix}(T_2) \text{ such that } A_1 x^* = A_2 y^*, \tag{1.3}$$

where $T_i : C_i \to H_i$, $i = 1, 2$, are nonlinear mappings with nonempty fixed-point sets $\text{Fix}(T_i) \neq \emptyset$, $i = 1, 2$. Equation (1.3) is referred to as the "split equality fixed point problem (SEFPP)", as introduced by Moudafi and El-Shemas [21].

The split feasibility and fixed point problem (SFFPP), introduced by Hamdi et al., [12], extends the split feasibility problem (SFP) by requiring the identification of a vector:

$$x^* \in C \cap \text{Fix}(U) \text{ such that } Ax^* \in Q \cap \text{Fix}(T), \tag{1.4}$$

where $U : H_1 \to H_1$ and $T : H_2 \to H_2$ are nonlinear mappings, and $A : H_1 \to H_2$ is a bounded linear operator. Notably, problem (1.4) reduces to Problem (1.1) when $C = \text{Fix}(U)$ and $Q = \text{Fix}(T)$. Numerous algorithms for solving the SFP, SEP, SEFPP, and SFFPP can be found in [3, 5, 10, 11, 17, 18, 21] and references therein.

This paper introduces a novel problem termed "split feasibility and fixed point equality problems (SFFPEP)". The problem is formulated as follows:

$$\text{Find } x^* \in C \cap \text{Fix}(U) \text{ and } y^* \in Q \cap \text{Fix}(T) \text{ such that } A_1 x^* = A_2 y^*, \tag{1.5}$$

where $U : H_1 \to H_1$ and $T : H_2 \to H_2$ are nonlinear mappings, and $A_i : H_i \to H_2$ $(i = 1, 2)$ are bounded linear operators.

Notably, since all components $C, Q, \text{Fix}(U), \text{Fix}(T), A_1$, and $A_2$ are clearly specified, the problem is mathematically well-defined. Furthermore, this problem generalizes and unifies several existing problems, including the SFP, SEP, SEFPP, and SFFPP. For example:

(1) When $A_2 = I$, problem (1.5) reduces to the SFFPP (1.4).
(2) If $C = \text{Fix}(U)$ and $Q = \text{Fix}(T)$, the problem reduces to the SEFPP (1.3) and SEP (1.2), respectively.
(3) When $A_2 = I$, $C = \text{Fix}(U)$, and $Q = \text{Fix}(T)$, the problem reduces to the SFP (1.1).

The SFFPEP framework is designed to address scenarios where solutions must satisfy both feasibility and fixed point conditions across multiple spaces, making it a powerful tool for modeling complex real-world problems.

An example of such a scenario is medical image reconstruction (MIR), for more details about the MIR, see [1, 9, 25, 26, 28]. In medical imaging, particularly in Computed Tomography (CT), the goal is to reconstruct a high-quality image of a patient's internal structures from a set of noisy and incomplete projection data (sinograms). The reconstruction process involves solving an ill-posed inverse problem, where the solution must satisfy both data consistency (agreement with measured projections) and prior-based constraints (e.g., smoothness, sparsity, or learned priors).

The SFFPEP framework provides a unified approach to balance these competing requirements.

1.1. **Application of SFFPEP in MIR.** The SFFPEP formulation can be applied to CT image reconstruction as follows:

(1) Let $C$ represent the set of images that are consistent with the measured sinogram data $g$. This is typically expressed as:

$$C = \{x \in H_1 \mid \|Ax - g\| \leq \delta\},$$

where $A : H_1 \rightarrow H_2$ is the Radon transform (a bounded linear operator modeling the CT projection process), $g \in H_2$ is the measured sinogram data, and $\delta$ is a tolerance parameter accounting for noise and measurement errors.
(2) Let $Q$ represent the set of images satisfying prior-based constraints, such as sparsity in a wavelet domain or smoothness enforced by Total Variation (TV) regularization. This can be written as:

$$Q = \{y \in H_2 \mid R(y) \leq \lambda\},$$

where $R(y)$ is a regularization function (e.g., TV norm or wavelet sparsity) and $\lambda$ is a regularization parameter controlling the strength of the constraint.

(3) Let $U : H_1 \rightarrow H_1$ be a nonlinear operator enforcing physical constraints on the image, such as non-negativity or bounded attenuation, and values. For example, $U$ could be a projection operator ensuring that the reconstructed image $x$ lies within a physically feasible range. Let $T : H_2 \rightarrow H_2$ be a nonlinear operator enforcing regularization constraints, such as a denoising operator based on Total Variation or a deep learning-based prior.

(4) The equality constraint $A_1 x^* = A_2 y^*$ ensures that the reconstructed image $x^*$ and the regularized image $y^*$ are consistent in a shared space, where $A_1 : H_1 \rightarrow H_2$ and $A_2 : H_2 \rightarrow H_2$ are bounded linear operators mapping the image and regularized image spaces to a common space (e.g., the sinogram space).

Thus, the SFFPEP for CT reconstruction is formulated as:

Find $x^* \in C \cap \mathrm{Fix}(U)$ and $y^* \in Q \cap \mathrm{Fix}(T)$ such that $A_1 x^* = A_2 y^*$,

where

(i) $x^*$ is the reconstructed image that satisfies both data consistency and physical constraints,
(ii) $y^*$ is the regularized image that satisfies prior-based constraints,
(iii) $A_1 x^* = A_2 y^*$ ensures that the solutions in the image and regularization spaces are consistent.

To solve the SFFPEP, we propose a new iterative algorithm tailored for quasi-nonexpansive mappings in Hilbert spaces. Quasi-nonexpansive mappings are a broad class of operators that include nonexpansive, firmly nonexpansive, and other important mappings, making our approach widely applicable. We rigorously analyze the convergence properties of the proposed algorithm, establishing its theoretical validity under appropriate conditions. Furthermore, we provide a numerical example to illustrate the algorithm's efficacy and demonstrate its practical utility.

The contributions of this work are twofold. First, we introduce the SFF-PEP as a generalization of several well-known problems, thereby extending the scope of existing research. Second, we develop and analyze a computationally efficient algorithm to solve the SFFPEP, supported by both theoretical convergence guarantees and numerical evidence. Our results not only advance the theoretical understanding of split problems and fixed point theory but also provide a practical tool for solving complex optimization problems in applied mathematics and engineering.

The remainder of this paper is organized as follows. In Section 2, we review relevant mathematical preliminaries and definitions. Section 3 presents the

proposed iterative algorithm and its convergence analysis. Section 4 provides a numerical example, and Section 5 provided the a conclusion.

## 2. Preliminaries

In this section, we present some definitions and lemmas that will be used to prove our main results.

Let $H$ be a Hilbert space, and $T : H \to H$ be a mapping with $\mathrm{Fix}(T) \neq \emptyset$, where $\mathrm{Fix}(T) = \{x \in H : Tx = x\} \neq \emptyset$ denote the set of fixed point of T. The mapping $T$ is said to be:

(a) Nonexpansive, if
$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in H.$$

(b) Quasi-nonexpansive, if
$$\|Tx - q\| \leq \|x - q\|, \quad \forall x \in H, q \in \mathrm{Fix}(T).$$

(c) Firmly quasi-nonexpansive, if
$$\|Tx - q\|^2 \leq \|x - q\|^2 - \|Tx - x\|^2, \quad \forall x \in H, q \in \mathrm{Fix}(T).$$

(d) Demiclosed at 0, if for any sequence $\{x_n\}$ in $H$ such that $x_n$ converges weakly to $x$ and $Tx_n$ converges strongly to 0, it follows that $Tx = 0$.

(e) Semi-compact, if for any bounded sequence $\{x_n\} \subset H$ with $(I - T)x_n$ converging strongly to 0, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to 0.

**Lemma 2.1.** ([22]) *Let $\{x_n\}$ be a sequence in $H$ such that there exists a nonempty set $C \subset H$ satisfying the following conditions:*

(a) *For each $x \in C$, $\lim\limits_{n \to \infty} \|x_n - x\|$ exists.*

(b) *Any weak-cluster point of the sequence $\{x_n\}$ belongs to $C$.*

*Then, there exists $y \in C$ such that $\{x_n\}$ converges weakly to $y$.*

For any element $x \in H$, there exists a unique point in $C$, denoted by $P_C x$, called the metric projection of $x$ onto $C$. This projection satisfies the property:
$$\|x - P_C x\| = \inf_{y \in C} \|x - y\|,$$
where $P_C x$ is the nearest point to $x$ within the set $C$.

**Remark 2.2.** The metric projection $P_C$ from $H$ onto $C$ has the following properties:

(a) $\langle x - P_C x, y - P_C x \rangle \leq 0$ for all $x \in H$ and $y \in C$;

(b) $\|x - P_C x\|^2 \leq \langle x - P_C x, x - y \rangle$ for all $x \in G_1$ and $y \in C$;

(c) $\|P_C x - y\|^2 \leq \|x - y\|^2 - \|(I - P_C)x\|^2$ for all $x \in H$ and $y \in C$.

In the sequel, we adopt the following notations:

(a) $I$: The identity operator on a Hilbert space $H$.

(b) " $\to$ " and " $\rightharpoonup$ ": Strong and weak convergence, respectively.

(c) $\omega_\omega(x_n)$: The set of weak cluster points of $\{x_n\}$, that is,

$$\omega_\omega(x_n) = \{x \in H : \exists \{x_{n_k}\} \subset \{x_n\} \text{ such that } x_{n_k} \rightharpoonup x\}.$$

(d) $\Omega$: The solution set of Problem (1.5), that is,

$$\Omega = \left\{ \text{Find } x^* \in C \cap \text{Fix}(U) \text{ and } y^* \in Q \cap \text{Fix}(T), \ Ax^* = By^* \right\}. \quad (2.1)$$

## 3. MAIN RESULTS

To approximate equation (2.1), we make the following assumptions:

$(B_1)$ $H_1$, $H_2$, $H_3$, are real Hilbert spaces, $C$ and $Q$ are two nonempty closed convex subset of $H_1$ and $H_2$, respectively.

$(B_2)$ $U : H_1 \to H_1$ and $T : H_2 \to H_2$ are two quasi-nonexpansive mappings with $Fix(U) \neq \emptyset$ and $Fix(T) \neq \emptyset$.

$(B_3)$ $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are two bounded linear operators with their adjoints $A^*$ and $B^*$, respectively.

$(B_4)$ $(U - I)$ and $(T - I)$ are demiclosed at zero.

$(B_5)$ $P_C$ and $P_Q$ are metric projection of $H_1$ and $H_2$ onto $C$ and $Q$, respectively.

$(B_6)$ For arbitrary $x_1 \in H_1$ and $y_1 \in H_2$, define a sequence $\{(x_n, y_n)\}$ by:

$$\begin{cases} w_n = P_C\left(x_n - \lambda_n A^*(Ax_n - By_n)\right), \\ x_{n+1} = (1 - \alpha_n)w_n + \alpha_n U(w_n), \\ \\ u_n = P_Q\left(y_n + \lambda_n B^*(Ax_n - By_n)\right), \\ y_{n+1} = (1 - \alpha_n)u_n + \alpha_n T(u_n), \quad \forall n \geq 1, \end{cases} \quad (3.1)$$

where $0 < b < \alpha_n < 1$, and $\lambda_n \in \left(0, \frac{2}{L_1 + L_2}\right)$, where $L_1$ and $L_2$ denote the spectral radius of the operators $\|A^*A\|$ and $\|B^*B\|$, respectively.

We are now in the position to state and prove our main result.

**Theorem 3.1.** *Suppose that assumptions $(B_1) - (B_6)$ are satisfied, and in addition, assume that the solution set $\Omega \neq \emptyset$. Then, the sequence $\{(x_n, y_n)\}$ generated by algorithm (3.1) converges to $(x^*, y^*) \in \Omega$.*

*Proof.* Let $(x^*, y^*) \in \Omega$. By (3.1), and the fact that U is quasi-nonexpansive, we have

$$\|x_{n+1} - x^*\|^2 = \|(1 - \alpha_n)(w_n - x^*) + \alpha_n(Uw_n - x^*)\|^2$$
$$= (1 - \alpha_n)\|w_n - x^*\|^2 + \alpha_n\|Uw_n - x^*\|^2$$
$$- \alpha_n(1 - \alpha_n)\|Uw_n - w_n\|^2. \quad (3.2)$$

On the other hand,

$$
\begin{aligned}
\|w_n - x^*\|^2 &= \left\|P_C(U(x_n - \lambda_n A^*(Ax_n - By_n))) - x^*\right\|^2 \\
&\leq \left\|U(x_n - \lambda_n A^*(Ax_n - By_n)) - x^*\right\|^2 \\
&\leq \left\|x_n - \lambda_n A^*(Ax_n - By_n) - x^*\right\|^2 \\
&= \|x_n - x^*\|^2 - 2\lambda_n \langle Ax_n - Ax^*, Ax_n - By_n \rangle \\
&\quad + \lambda_n^2 L_1 \|Ax_n - By_n\|^2 .
\end{aligned}
\tag{3.3}
$$

From (3.2)-(3.3), we obtain that

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \|x_n - x^*\|^2 - 2\lambda_n \langle Ax_n - Ax^*, Ax_n - By_n \rangle \\
&\quad + \lambda_n^2 L_1 \|Ax_n - By_n\|^2 - \alpha_n(1-\alpha_n)\|U(w_n) - w_n\|^2 .
\end{aligned}
\tag{3.4}
$$

Similarly, the second equation of (3.1) gives

$$
\begin{aligned}
\|y_{n+1} - y^*\|^2 &\leq \|y_n - y^*\|^2 + 2\lambda_n \langle By_n - By^*, Ax_n - By_n \rangle \\
&\quad + \lambda_n^2 L_2 \|Ax_n - By_n\|^2 - \alpha_n(1-\alpha_n)\|T(u_n) - u_n\|^2 .
\end{aligned}
\tag{3.5}
$$

By (3.4), (3.5), and noticing that $Ax^* = By^*$, we deduce that

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 + \|y_{n+1} - y^*\|^2 &\leq \|x_n - x^*\|^2 + \|y_n - y^*\|^2 \\
&\quad - 2\lambda_n \|Ax_n - By_n\|^2 + \lambda_n^2(L_1 + L_2)\|Ax_n - By_n\|^2 \\
&\quad - \alpha_n(1-\alpha_n)\left(\|U(w_n) - w_n\|^2 + \|T(u_n) - u_n\|^2\right).
\end{aligned}
\tag{3.6}
$$

Thus, we deduce that

$$
\begin{aligned}
\Omega_{n+1} &\leq \Omega_n - \lambda_n\left(2 - \lambda_n(L_1 + L_2)\right)\|Ax_n - By_n\|^2 \\
&\quad - \alpha_n(1-\alpha_n)\left(\|U(w_n) - w_n\|^2 + \|T(u_n) - u_n\|^2\right),
\end{aligned}
\tag{3.7}
$$

where

$$
\Omega_n := \|x_n - x^*\|^2 + \|y_n - y^*\|^2 .
$$

Thus, $\{\Omega_n\}$ is a non-increasing sequence and bounded below by 0, therefore, it converges. From (3.7) and the fact that $\{\Omega_n\}$ converges, we deduce that

$$
\lim_{n\to\infty} \|Ax_n - By_n\| = 0,
\tag{3.8}
$$

$$
\lim_{n\to\infty} \|Uw_n - w_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|Tu_n - u_n\| = 0.
\tag{3.9}
$$

Furthermore, since $\{\Omega_n\}$ converges, this ensures that $\{x_n\}$ and $\{y_n\}$ also converge.

Now, let $(x, y) \in \Omega$. This implies that $x \in C \cap \text{Fix}(U)$ and $y \in Q \cap \text{Fix}(T)$ such that $Ax = By$. The fact that $x_n \rightharpoonup x$ and $\lim_{n \to \infty} \|Ax_n - By_n\| = 0$ together with

$$z_n = P_C\Big(x_n - \lambda_n A^*(Ax_n - By_n)\Big),$$

we deduce that $z_n \rightharpoonup P_C x$. Since $x \in C$, by the projection theorem, we obtain that $P_C x = x$. Hence, $z_n \rightharpoonup x$.

Similarly, the fact that $y_n \rightharpoonup y$ and $\lim_{n \to \infty} \|Ax_n - By_n\| = 0$ together with

$$u_n = P_Q(y_n + \lambda_n B^*(Ax_n - By_n)),$$

we deduce that $u_n \rightharpoonup P_Q y$. Since $y \in Q$, by the projection theorem, we obtain that $P_Q y = y$. Hence, $u_n \rightharpoonup y$.

Now, $z_n \rightharpoonup x$ and $\lim_{n \to \infty} \|Uz_n - z_n\| = 0$ together with the demiclosedness of $(U - I)$ at zero, we deduce that $Ux = x$, which implies that $x \in \text{Fix}(U)$.

On the other hand, $u_n \rightharpoonup y$ and $\lim_{n \to \infty} \|Tu_n - u_n\| = 0$ together with the demiclosedness of $(T - I)$ at zero, we deduce that $Ty = y$, which implies that $y \in \text{Fix}(T)$.

Since $x_n \rightharpoonup x$, $y_n \rightharpoonup y$, and the fact that $A$ and $B$ are bounded linear operators, we have

$$Ax_n \rightharpoonup Ax \quad \text{and} \quad By_n \rightharpoonup By,$$

which implies that

$$Ax_n - By_n \rightharpoonup Ax - By,$$

which in turn implies that

$$\|Ax - By\| \le \liminf_{n \to \infty} \|Ax_n - By_n\| = 0,$$

which further implies that $Ax = By$.

Noticing that $x \in C$, $x \in \text{Fix}(U)$, $y \in Q$, and $y \in \text{Fix}(T)$, we have that $x \in C \cap \text{Fix}(U)$ and $y \in Q \cap \text{Fix}(T)$. Hence, we conclude that $(x, y) \in \Omega$.

Summing up, we have proved that:

(i) For each $(x^*, y^*) \in \Omega$, the limit $\lim_{n \to \infty} \Big( \|x_n - x^*\|^2 + \|y_n - y^*\|^2 \Big)$ exists.

(ii) Each weak cluster of the sequence $(x_n, y_n)$ belongs to $\Omega$.

Thus, by Lemma 2.1, we conclude that the sequences $(x_n, y_n)$ converge weakly to $(x^*, y^*) \in \Omega$, which completed the proof. $\qquad \square$

**Theorem 3.2.** *Suppose that all the hypotheses of Theorem 3.1 are satisfied, and in addition, $U$ and $T$ are semi-compact. Then, the sequence $\{(x_n, y_n)\}$ converges strongly to $(x^*, y^*) \in \Omega$.*

*Proof.* As in the proof of Theorem 3.1, $\{x_n\}$ and $\{y_n\}$ are bounded. By (3.9) and the fact that $U$ and $T$ are semi-compact, there exist subsequences $\{x_{n_k}\}$ and $\{y_{n_k}\}$ (suppose without loss of generality) of $\{x_n\}$ and $\{y_n\}$ such that $x_{n_k} \to x$ and $y_{n_k} \to y$. Since $x_n \rightharpoonup x^*$ and $y_n \rightharpoonup y^*$, we have $x = x^*$ and $y = y^*$. By (3.8) and the fact that $x_{n_k} \to x^*$ and $y_{n_k} \to y^*$, we have

$$\lim_{n\to\infty} \|Ax^* - Ay^*\| = \lim_{n\to\infty} \|Ax_{n_k} - By_{n_k}\| = 0,$$

which in turn implies that $Ax^* = Ay^*$. Hence, $(x^*, y^*) \in \Omega$. Thus, the iterative algorithm of Theorem 3.1 converges strongly to the solution of (2.1).   □

**Corollary 3.3.** *Suppose that conditions* $(B_1) - (B_6)$ *are satisfied, and let the sequence* $\{(x_n, y_n)\}$ *be generated by Algorithm (3.1). Assume that* $\Omega \neq \emptyset$, *and let* $U$ *and* $T$ *be firmly quasi-nonexpansive mappings. Then, the sequence* $\{(x_n, y_n)\}$ *generated by Algorithm (3.1) converges weakly to the solution set of* (2.1).

**Corollary 3.4.** *Suppose that conditions* $(B_1) - (B_5)$ *are satisfied, and let the sequence* $\{(x_n, y_n)\}$ *be generated by*

$$
\begin{cases}
z_n = x_n - \lambda_n A^*(Ax_n - By_n), \\
x_{n+1} = (1 - \alpha_n)z_n + \alpha_n U(z_n), \\
u_n = y_n + \lambda_n B^*(Ax_n - By_n), \\
y_{n+1} = (1 - \alpha_n)u_n + \alpha_n T(u_n), \quad \forall n \geq 0,
\end{cases}
\tag{3.10}
$$

*where* $0 < a < \beta_n < 1$, *and* $\lambda_n \in \left(0, \frac{2}{L_1+L_2}\right)$, *and* $L_1$ *and* $L_2$ *denote the spectral radius of the operators* $A^*A$ *and* $B^*B$, *respectively. Assume that* $\Omega \neq \emptyset$. *Then, the sequence* $\{(x_n, y_n)\}$ *generated by Algorithm (3.10) converges weakly to the solution of SEFPP (1.4).*

*Proof.* Trivially, Algorithm (3.1) reduces to Algorithm (3.10) as $P_C = P_Q = I$, and SFFPEP (1.5) reduces to SEFPP (1.4) as $C := \text{Fix}(U)$ and $Q := \text{Fix}(T)$. Therefore, all the hypotheses of Theorem 3.1 are satisfied. Hence, the proof of this corollary follows directly from Theorem 3.1.   □

## 4. NUMERICAL EXAMPLE

In this section, we give a numerical example to illustrate our theoretical results.

**Example 4.1.** Let $H_1 = \Re$ with the inner product defined by $\langle x, y \rangle = xy$ for all $x, y \in \Re$ and $\|.\|$ stand for the corresponding norm. Let $C := [0, \infty)$, $Q := [0, \infty)$ and define a mappings $T : C \to \Re$ and $S : Q \to \Re$ by $Tx = \frac{x^2+5}{1+x}$, for all $x \in C$ and $Sy = \frac{y+5}{5}$, for all $y \in Q$. Then $T$ and $S$ are quasi-nonexpansive mappings with $Fix(T) = 5$ and $Fix(S) = \frac{5}{4}$.

**Corollary 4.2.** *Let $H_1 = \Re$ with the inner product defined by $\langle x, y \rangle = xy$ for all $x, y \in \Re$ and $\|.\|$ stand for the corresponding norm. Let $C := [0, \infty)$, $Q := [0, \infty)$ and define a mappings $U : C \to \Re$ and $T : Q \to \Re$ by $Ux = \frac{x^2+5}{1+x}$, for all $x \in C$ and $Ty = \frac{y+5}{5}$, for all $y \in Q$. And also let $P_C = P_Q = I$, $Ax = x$, $By = 4y$, $\lambda_n = 1$, $\alpha_n = \frac{1}{5}$, and $\{(x_n, y_n)\}$ be the sequence generated by*

$$\begin{cases} x_0 \in C \text{ and } y_0 \in Q, \\ w_n = P_C(x_n - A^*(x_n - 4y_n)), \\ x_{n+1} = (1 - \frac{1}{5})z_n + \frac{1}{5}U(w_n), \\ \\ u_n = P_Q(y_n + B^*(x_n - 4y_n)), \\ y_{n+1} = (1 - \frac{1}{5})u_n + \frac{1}{5}T(u_n), \quad \forall n \geq 0. \end{cases} \tag{4.1}$$

*Then, $\{(x_n, y_n)\}$ converges to $(5, 5/4) \in \Omega$.*

*Proof.* By Example 4.1 $U$ and $T$ are quasi-nonexpansive mappings. Clearly, $A$ and $B$ are bounded linear operator on $\Re$ with $A = A^* = 1$ and $B = B^* = 4$, respectively. Furthermore, it is easy to see that $Fix(U) = 5$ and $Fix(T) = \frac{5}{4}$. Hence,

$$\Omega = \left\{ 5 \in C \cap Fix(U) \text{ and } 5/4 \in Q \cap Fix(T) \text{ such that } A(5) = B(5/4) \right\}.$$

| n | $x_n$ | $y_n$ |
|---|-------|-------|
| 0 | 1.000000000 | 1.000000000 |
| 1 | 1.400000000 | 1.040000000 |
| 2 | 1.700000000 | 1.073600000 |
| 3 | 1.944444444 | 1.101824000 |
| . | . | . |
| . | . | . |
| . | . | . |
| 147 | 4.986338418 | 1.249999997 |
| 148 | 4.986794843 | 1.249999997 |
| 149 | 4.987235985 | 1.249999997 |

TABLE 1. The numerical results of algorithm (3.1),starting with the initial values $x_0 = 1$ and $y_0 = 1$, demonstrated how the sequence $(x_n, y_n)$ converges to $(5, 5/4)$.
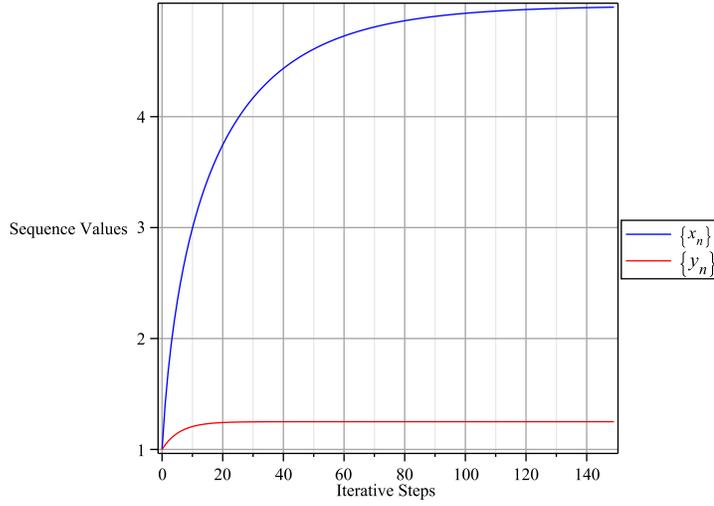
FIGURE 1. Graphical results presentation of algorithm (3.1), starting with the initial values $x_0 = 1$ and $y_0 = 1$, demonstrated how the sequence $(x_n, y_n)$ converges to $(5, 5/4)$.

| n | $x_n$ | $y_n$ |
|---|---|---|
| 0 | 15.00000000 | 15.00000000 |
| 1 | 14.87500000 | 12.80000000 |
| 2 | 14.75059055 | 10.95200000 |
| . | . | . |
| . | . | . |
| . | . | . |
| 147 | 5.356844435 | 1.250000003 |
| 148 | 5.345617341 | 1.250000003 |
| 149 | 5.334724238 | 1.250000003 |

TABLE 2. The numerical results of algorithm (3.1),starting with the initial values $x_0 = 15$ and $y_0 = 15$, demonstrated how the sequence $(x_n, y_n)$ converges to $(5, 5/4)$.
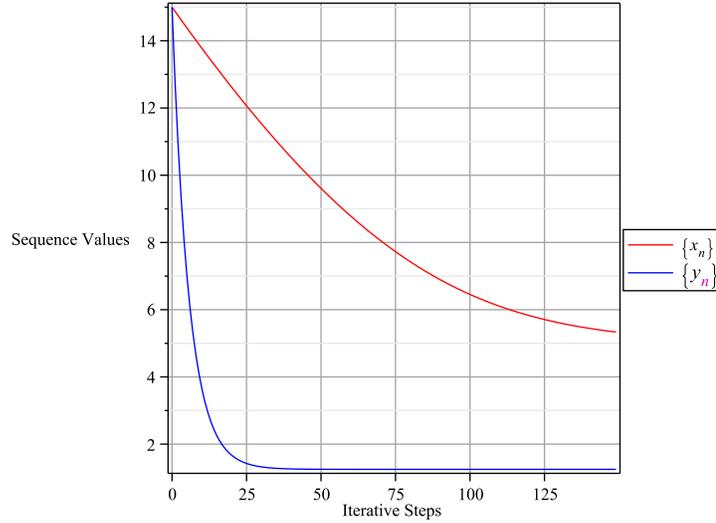
FIGURE 2. Graphical results of algorithm (3.1),starting with the initial values $x_0 = 5$ and $y_0 = 1/4$, demonstrated how the sequence $(x_n, y_n)$ converges to $(5, 5/4)$.

$\Box$

## 5. CONCLUSION

In this paper, we introduce a novel problem called the Split Feasibility and Fixed Point Equality Problem (SFFPEP) and investigate it for the class of quasi-nonexpansive mappings in Hilbert spaces. Under suitable assumptions on the involved parameters and operators, we establish a weak and strong convergence theorems for the proposed problem. Additionally, we provide a numerical example to illustrate our theoretical findings.

The SFFPEP is a significant generalization of several well-known problems, including the Split Feasibility Problem (SFP), the Fixed Point Problem (FPP), the Split Feasibility and Fixed Point Problem (SFFPP), and the Split Equality Fixed Point Problem (SEFPP). Consequently, all results and conclusions derived for the SFFPEP naturally extend to these special cases, demonstrating both the broad applicability and theoretical importance of our proposed problem.

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