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ON NONHOMOGENEOUS BIHARMONIC EQUATION WITH CRITICAL SOBOLEV EXPONENT AND PRESCRIBED SINGULARITIES

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Abstract. In this paper, we are concerned with the existence of multiple solutions for the following critical biharmonic problem

$$(\mathcal{P}_\mu)\begin{cases} \Delta^2 u - \mu V(x)u = |u|^{2^*-2}u + \lambda f(x) & \text{in } \Omega, \\ u = \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is an open bounded domain of $\mathbb{R}^N$ $(N \geq 5)$ with smooth boundary $\partial\Omega$, $2^* = \frac{2N}{N-4}$. For $0 < \mu < \bar{\mu} = \left(\frac{N(N-4)}{4}\right)^2$, we establish the relationship between the number of solutions and the profile of V.

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1. Introduction

This work is devoted to the study of the following biharmonic problem with critical nonlinearity

$$(\mathcal{P}_\mu)\begin{cases} \Delta^2 u - \mu V(x)u = |u|^{2^*-2}u + \lambda f(x) & \text{in } \Omega, \\ u = \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

where Δ^2 is the biharmonic operator, Ω is an open bounded domain of $\mathbb{R}^N$ ($N \geq 5$) with smooth boundary $\partial\Omega$, $\lambda > 0$ is a parameter, $2^* = \frac{2N}{N-4}$ is the critical Rellich-Sobolev exponent and $0 < \mu < \bar{\mu}$, where

$$\bar{\mu} = \left(\frac{N(N-4)}{4}\right)^2$$

is the best constant for the Rellich inequality. $f(x)$ is a given function, and $V(x)$ has finitely many singular points, whose detailed descriptions will be provided later.

The study of biharmonic problems with singularities, particularly those involving multipolar potentials, plays a crucial role in physics and elasticity. In thin plate theory, multipolar potentials can model materials with localized defects or inclusions, affecting their mechanical behavior [12]. Moreover, such models appear in quantum physics, especially in the study of multipolar interactions in electrodynamics and fluid mechanics (see [9] and [10]).

Elliptic equations involving the Laplace operator and Sobolev critical nonlinearity have attracted considerable interest during the past several years (see [3], [4], [5], [7], and [20]). Particularly, we point out that when $\mu = 0$ Tarantello in the celebrated paper [20] studied the following problem

$$\begin{cases} -\Delta u = |u|^{2^*-2}u + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Δ is the Laplace operator, Ω is an open bounded domain of $\mathbb{R}^N$ ($N \geq 3$) with smooth boundary $\partial\Omega$, $2^* = \frac{2N}{N-2}$. By using the Nehari variety, she proved the existence of at least two positive solutions for $f \neq 0$ and satisfying a suitable assumption. In the singular case ($\mu \neq 0$).

Deng-Kang [7] extended Tarantellos criticalgrowth analysis by further introducing both a Hardy potential and a singular weight in the nonlinearity, proving the existence of two positive solutions under certain hypotheses. Additionally, in [3], Bouchekif-Messirdi have investigated the effect of the linear weight $V(x)$ with multiple singular points on the existence and multiplicity of positive solutions for $(\mathcal{P}_\mu)$ involving the Laplace operator. The authors proved the existence of multiple positive solutions by introducing suitable unbounded coefficients to overcome the difficulties created by the singular terms. Indeed, another motivation for this work is derived from the interesting paper by Chen-Chen [5], in which the authors considered the singular problem

$$\begin{cases} -\Delta u - \mu V(x)u = |u|^{2^*-2}u + \lambda f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where the linear weight $V(x)$ has finitely many singular points. It is important to note that the assumptions on $V(x)$ in [5] differ from those in [3], making the study more intricate and interesting.

On the other hand, biharmonic problems have been studied by some authors (see, for example, [1], [6], [8], [11], [14], [16], [18], and [19]). We would like to mention the result of Qian et al. [18], where they expanded the study of Tarantello [20] by considering the problem with the biharmonic operator and proving the existence of at least two solutions. When $\mu \neq 0$, we should point out that Huang et al. [14] generalized the singular problem of Deng-Kang [7] by establishing multiplicity results for $(\mathcal{P}_\mu)$ with $V(x) = |x|^{-s}$ for $0 < s \leq 4$. Moreover, biharmonic problems with prescribed singularities have seldom been studied, we only find some results in [2], [13], and [15]. Thus, it is necessary for us to investigate the related biharmonic problem deeply.

Comparing problem $(\mathcal{P}_\mu)$ with previous works, our study presents additional complexities. First, the biharmonic equation is significantly more challenging than the Laplacian case studied in [4] and [5] due to the implicit nature of the associated energy estimates. Unlike in the Laplacian case, the lack of a maximum principle for fourth-order equations introduces additional difficulties in studying the sign of solutions. Furthermore, while the authors in [2] introduced a classical perturbation term to manage the singular behavior, our work deals with a different type of singular term $V(x)$ under more delicate conditions. This structure not only prevents the use of standard perturbation techniques but also requires a more refined variational approach to address the difficulties arising from the critical nonlinearity.

Problem $(\mathcal{P}_\mu)$ is related to the following Rellich inequality [6]:

$$\int_\Omega \frac{u^2}{|x-a|^4} dx \leq \frac{1}{\bar{\mu}} \int_\Omega |\Delta u|^2 dx, \quad \forall a \in \Omega, u \in H_0^2(\Omega),$$

where $H_0^2(\Omega)$ is the completion of $\mathcal{C}_0^\infty(\Omega)$ with respect to the standard norm

$$\| \cdot \| = \left(\int_\Omega |\Delta \cdot |^2 dx \right)^{\frac{1}{2}}.$$

Then, the following best constant:

$$\mathcal{A}_\mu(\Omega) = \inf_{H_0^2(\Omega)\backslash\{0\}} \frac{\int_\Omega \left(|\Delta u|^2 - \mu \frac{u^2}{|x-a|^4}\right) dx}{\left(\int_\Omega |u|^{2^*} dx\right)^{2/2^*}}, \quad \forall a \in \Omega, \mu < \bar{\mu} \tag{1.1}$$

is well-defined and independent of any $\Omega \subset \mathbb{R}^N (N \geq 5)$. Thus, we will simply denote $\mathcal{A}_\mu(\Omega) = \mathcal{A}_\mu(\mathbb{R}^N) = \mathcal{A}_\mu$. The authors in [6] proved that $\mathcal{A}_\mu(\Omega)$ is attained in $\mathbb{R}^{\mathbb{N}}$ by the functions

$$\left\{ y_\varepsilon(x-a) = \varepsilon^{\frac{4-N}{2}} U_\mu(\varepsilon^{-1}(x-a)),\ \varepsilon > 0 \right\},$$

where $U_\mu(x)$ is positive, radially symmetric, radially decreasing, and solves

$$\Delta^2 u - \mu \frac{u}{|x|^4} = u^{2^*-1},\ x \in \mathbb{R}^{\mathbb{N}}\backslash\{0\}, u > 0,$$

satisfying

$$\int_{\mathbb{R}^{\mathbb{N}}} \left(|\Delta y_\varepsilon(x-a)|^2 - \mu \frac{|y_\varepsilon(x-a)|^2}{|x-a|^4}\right) dx = \int_{\mathbb{R}^N} |y_\varepsilon(x-a)|^{2^*} dx = \mathcal{A}_\mu^{\frac{N}{4}}.$$

By setting $\rho = |x|$, we have

$$U_\mu(\rho) = O_1(\rho^{-a(\mu)}),\ as\ \rho \to 0,$$
$$U_\mu(\rho) = O_1(\rho^{-b(\mu)}),\ U'_\mu(\rho) = O_1(\rho^{-b(\mu)-1}),\ as\ \rho \to \infty,$$

where $a(\mu) = \delta\varphi(\mu),\ \ b(\mu) = \delta(2 - \varphi(\mu))$ with $\delta = \frac{N-4}{2}$ and $\varphi : [0, \bar{\mu}] \to [0, 1]$ is defined as

$$\varphi(\mu) = 1 - \frac{\sqrt{N^2 - 4N + 8 - 4\sqrt{(N-2)^2 + \mu}}}{N-4}.$$

For $\mu \in [0, \bar{\mu})$, we have $0 \leq a(\mu) \leq \delta \leq b(\mu) \leq 2\delta$. Furthermore, there exist positive constants $\mathcal{C}_1(\mu)$ and $\mathcal{C}_2(\mu)$ such that

$$0 < \mathcal{C}_1(\mu) \leq U_\mu(x)(|x|^{a(\mu)/\delta} + |x|^{b(\mu)/\delta})^\delta \leq \mathcal{C}_2(\mu), \quad \forall x \in \mathbb{R}^N\backslash\{0\}.$$

Before presenting our main result, we will provide some notations.
Let $\mathcal{D}^{2,2}(\mathbb{R}^{\mathbb{N}})$ is the completion of $\mathcal{C}^\infty(\mathbb{R}^{\mathbb{N}})$ with respect to the norm

$$\|\ .\ \|_{\mathcal{D}^{2,2}(\mathbb{R}^{\mathbb{N}})} = \left(\int_{\mathbb{R}^{\mathbb{N}}} |\Delta\ .\ |^2 dx\right)^{1/2}.$$

$L^p(\Omega)$ be the usual Lebesgue space endowed with the standard norm

$$\|\ .\ \|_p = \left(\int_\Omega |\ .\ |^p dx\right)^{1/p}.$$

$\|\ .\ \|_-$ is the norm in $H^{-2}(\Omega)$ the dual space of $H_0^2(\Omega)$. $B(a, r)$ denotes a ball centered at $a \in \mathbb{R}^{\mathbb{N}}$ with radius $r > 0$. $l_i,\ \nu_i$, and C_i denote positive different constants whose exact values are unimportant. For all $\varepsilon > 0, t > 0, O(\varepsilon^t)$ denotes the quantity satisfying $|O(\varepsilon^t)|/\varepsilon^t \leq C_1,\ O_1(\varepsilon^t)$ denotes $C_2\varepsilon^t \leqslant O_1(\varepsilon^t) \leqslant C_3\varepsilon^t$ and $o(\varepsilon^t)$ means $o(\varepsilon^t)/\varepsilon^t \to 0$ as $\varepsilon \to 0$ and $o(1)$ a generic infinitesimal value, $\to$ and $\rightharpoonup$ denote strong and weak convergences, respectively.

Throughout this paper, we will omit dx in integrals for convenience, and we note

$$\mathcal{A}_0 = \inf\left\{\int_\Omega |\Delta u|^2;\ u \in H_0^2(\Omega),\ \int_\Omega |u|^{2^*} = 1\right\}. \tag{1.2}$$

Now, we are ready to state our assumptions on $f(x)$ and $V(x)$.

(F0) $f \in H^{-2}(\Omega),\ f(x) > 0$ a.e. in Ω.

(V1) There exist $a_1, a_2, ..., a_k \in \Omega$ such that $V(x) \in L^\infty_{loc}(\Omega \setminus \{a_1, a_2, ..., a_k\})$,

$$\lim_{x \to a_j} V(x)|x - a_j|^4 = 1,\ j \in \{1, 2, ..., k\}.$$

Moreover, there are $\alpha, \beta > 2(b(\mu) - \delta), \delta_0 > 0$, such that for $x \in B(a_j, \delta_0)$:

$$|x - a_j|^{-4} - |x - a_j|^{\beta-4} \leq V(x) \leq |x - a_j|^{-4} - |x - a_j|^{\alpha-4}, \tag{1.3}$$

here δ_0 is fixed such that $|a_i - a_j| \geq 4\delta_0$ for $i \neq j$ and $B(a_j, 4\delta_0) \subset \Omega$.

(V2) There exists $0 < C < 1$ such that

$$\mu \int_\Omega V(x)u^2 \leq C \int_\Omega |\Delta u|^2 \text{ for all } u \in H_0^2(\Omega).$$

Inspired by [5] and [20], the main result of this paper is the following theorem:

Theorem 1.1. *Let* $0 < \mu < \bar{\mu}$, *and assume that* $(F0), (V1)$ *and* $(V2)$ *hold. Then there exists* $\lambda^* > 0$ *such that* $(\mathcal{P}_\mu)$ *has at least* k *solutions on* $H_0^2(\Omega)$, *for any* $0 < \lambda < \lambda^*$.

This article is organized as follows. In Section 2, we give preliminary results. In Section 3, we give the proof of Theorem 1.1.

2. Some preliminary results

To prove Theorem 1.1, we will use critical point theory. On $H_0^2(\Omega)$, we define the energy functional corresponding to problem $(\mathcal{P}_\mu)$ by

$$J_\mu(u) = \frac{1}{2}\int_\Omega (|\Delta u|^2 - \mu V(x)u^2) - \frac{1}{2^*}\int_\Omega |u|^{2^*} - \lambda \int_\Omega fu.$$

We say that $u \in H_0^2(\Omega)$ is a weak solution of problem $(\mathcal{P}_\mu)$ if for every $\varphi \in H_0^2(\Omega)$,

$$\int_\Omega (\Delta u \Delta \varphi - \mu V(x) u\varphi) - \int_\Omega |u|^{2^*-2} u\varphi - \lambda \int_\Omega f\varphi = 0.$$

Define the constraint set

$$\mathcal{N}_\mu = \left\{ u \in H_0^2(\Omega);\ \int_\Omega (|\Delta u|^2 - \mu V(x) u^2) - \int_\Omega |u|^{2^*} - \lambda \int_\Omega fu = 0 \right\}.$$

In order to motivate our result, decompose $\mathcal{N}_\mu$ with $\mathcal{N}_\mu^+, \mathcal{N}_\mu^0, \mathcal{N}_\mu^-$ defined as follows

$$\mathcal{N}_\mu^+ = \left\{ u \in N_\mu : \int_\Omega \left(|\Delta u|^2 - \mu V(x) u^2\right) > (2^* - 1) \int_\Omega |u|^{2^*} \right\},$$

$$\mathcal{N}_\mu^0 = \left\{ u \in N_\mu : \int_\Omega \left(|\Delta u|^2 - \mu V(x) u^2\right) = (2^* - 1) \int_\Omega |u|^{2^*} \right\},$$

and

$$\mathcal{N}_\mu^- = \left\{ u \in N_\mu : \int_\Omega \left(|\Delta u|^2 - \mu V(x) u^2\right) < (2^* - 1) \int_\Omega |u|^{2^*} \right\}.$$

Now, we give some basic properties of $\mathcal{N}_\mu^+$, $\mathcal{N}_\mu^0$ and $\mathcal{N}_\mu^-$.

Lemma 2.1. *Assume $(F0), (V1)$ and $(V2)$ are verified, and $0 < \mu < \bar{\mu}$. Then there exists $\lambda_1 > 0$ such that for any $\lambda \in (0, \lambda_1)$,*

(i) *$\mathcal{N}_\mu^\pm \neq \varnothing$, and $\mathcal{N}_\mu^0 = \{0\}$.*
(ii) *$\mathcal{N}_\mu^-$ is a closed set in $H_0^2(\Omega)$.*

Proof. (i) For $u \in H_0^2(\Omega)$, define $\Phi : (0, +\infty) \to \mathbb{R}$ by

$$\Phi(t) = t \int_\Omega (|\Delta u|^2 - \mu V(x) u^2) - t^{2^*-1} \int_\Omega |u|^{2^*},$$

easy computations show that Φ is concave and achieves its maximum at

$$t_0 = \left(\frac{\int_\Omega (|\Delta u|^2 - \mu V(x) u^2)}{(2^* - 1) \int_\Omega |u|^{2^*}} \right)^{\frac{1}{2^*-2}}.$$

Hence, it follows from $\int_\Omega fu > 0$ that there exists $\lambda_1 > 0$ such that for all $\lambda \in (0, \lambda_1)$, one has that $\Phi(t_0) > \lambda \int_\Omega fu > 0$. Thus, there is $0 < t^- < t_0 < t^+$ such that

$$\Phi(t^-) = \Phi(t^+) = \lambda \int_\Omega fu$$

and

$$\Phi'(t^+) < 0 < \Phi'(t^-),$$

it follows that $t^- u \in \mathcal{N}_\mu^+$ and $t^+ u \in \mathcal{N}_\mu^-$. In conclusion, for $\lambda \in (0, \lambda_1)$, we have

$$\mathcal{N}_\mu^+ \neq \varnothing \quad and \quad \mathcal{N}_\mu^- \neq \varnothing.$$

Now, we can claim that $\mathcal{N}_\mu^0 = \{0\}$. Arguing by contradiction, we assume that there exists $u \in \mathcal{N}_\mu$, $u \neq 0$ such that

$$\int_\Omega (|\Delta u|^2 - \mu V(x) u^2) = (2^* - 1) \int_\Omega |u|^{2^*} = \frac{2^* - 1}{2^* - 2} \lambda \int_\Omega fu. \tag{2.1}$$

By (V2), we have $\int_\Omega (|\Delta u|^2 - \mu V(x) u^2) \geq (1 - C) \int_\Omega |\Delta u|^2$, and so

$$\int_\Omega |u|^{2^*} \geq \frac{1 - C}{2^* - 1} \int_\Omega |\Delta u|^2.$$

Therefore, as $\lambda \in (0, \lambda_1)$, we deduce from (2.1) that

$$\begin{aligned}
0 &= \frac{2^* - 2}{2^* - 1} \int_\Omega (|\Delta u|^2 - \mu V(x) u^2) - \lambda \int_\Omega fu \\
&= \left(\left(\frac{1}{2^* - 1} \right)^{\frac{1}{2^* - 2}} - \left(\frac{1}{2^* - 1} \right)^{\frac{2^* - 1}{2^* - 2}} \right) \frac{\left(\int_\Omega (|\Delta u|^2 - \mu V(x) u^2) \right)^{\frac{2^* - 1}{2^* - 2}}}{\left(\int_\Omega |u|^{2^*} \right)^{\frac{1}{2^* - 2}}} - \lambda \int_\Omega fu \\
&= \Phi(t_0) - \lambda \int_\Omega fu \\
&> 0,
\end{aligned}$$

which is a contradiction. Hence $\mathcal{N}_\mu^0 = \{0\}$ for any $\lambda \in (0, \lambda_1)$.

(ii) Now, we claim that $\mathcal{N}_\mu^-$ is a closed set in $H_0^2(\Omega)$. Let $\{u_n\} \subset \mathcal{N}_\mu^-$ with $u_n \to u_0$ in $H_0^2(\Omega)$, in the following we will prove that $u_0 \in \mathcal{N}_\mu^-$. Since $u_n \in \mathcal{N}_\mu^-$, we drive

$$\begin{aligned}
\int_\Omega (|\Delta u_0|^2 - \mu V(x) u_0^2) &= \lim_{n \to +\infty} \int_\Omega (|\Delta u_n|^2 - \mu V(x) u_n^2) \\
&\leq \lim_{n \to +\infty} (2^* - 1) \int_\Omega |u_n|^{2^*} \\
&\leq (2^* - 1) \int_\Omega |u_0|^{2^*},
\end{aligned}$$

thus, $u_0 \in \mathcal{N}_\mu^- \cup \mathcal{N}_\mu^0$. It follows from (V2) and (1.2), that

$$\begin{aligned}\int_\Omega (|\Delta u_n|^2 - \mu V(x) u_n^2) &< (2^* - 1) \int_\Omega |u_n|^{2^*} \\ &\leq (2^* - 1) \left(\frac{\int_\Omega (|\Delta u_n|^2}{\mathcal{A}_0} \right)^{\frac{2^*}{2}} \\ &\leq (2^* - 1) \left(\frac{\int_\Omega (|\Delta u_n|^2 - \mu V(x) u_n^2)}{(1-C)\mathcal{A}_0} \right)^{\frac{2^*}{2}},\end{aligned}$$

therefore, we get

$$\int_\Omega (|\Delta u_n|^2 - \mu V(x) u_n^2) > \left(\frac{((1-C)\mathcal{A}_0)^{\frac{2^*}{2}}}{(2^*-1)} \right)^{\frac{2^*-2}{2}}.$$

Hence, we obtain

$$\int_\Omega (|\Delta u_0|^2 - \mu V(x) u_0^2) \geq \left(\frac{((1-C)\mathcal{A}_0)^{\frac{2^*}{2}}}{(2^*-1)} \right)^{\frac{2^*-2}{2}} > 0.$$

It follows that $u_0 \neq 0$, that is, $u_0 \notin \mathcal{N}_\mu^0$. In turn, $u_0 \in \mathcal{N}_\mu^-$. This concludes the proof. □

We have, now, the following energy estimates for any weak solutions of $(\mathcal{P}_\mu)$.

Lemma 2.2. *Let u be a solution of $(\mathcal{P}_\mu)$, then for any $\lambda > 0$, and $0 < \mu < \bar{\mu}$ there exists $\chi > 0$ such that*

$$J_\mu(u) \geq -\chi \lambda^2.$$

Proof. Let u be a solution of $(\mathcal{P}_\mu)$. Then by easy computations, we obtain

$$\begin{aligned}J_\mu(u) &= \frac{1}{2} \int_\Omega (|\Delta u|^2 - \mu V(x) u^2) - \frac{1}{2^*} \int_\Omega |u|^{2^*} - \lambda \int_\Omega f u \\ &\geq \frac{2}{N}(1-C)\|u\|^2 - \lambda \frac{N+4}{2N} \|u\| \|f\|_-.\end{aligned}$$

It follows from

$$g(t) = \frac{2}{N}(1-C)t^2 - \lambda \frac{N+4}{2N} \|f\|_- t,$$

attaining its absolute minimum for $t > 0$ at the point

$$\bar{t} = \lambda \frac{N+4}{8(1-C)} \|f\|_-,$$

that is

$$g(t) \geq g(\bar{t}) = -\lambda^2 \frac{(N+4)^2}{32N(1-C)} \|f\|_{-}.$$

The conclusion follows by letting

$$\chi = \frac{(N+4)^2}{32N(1-C)} \|f\|_{-}^2.$$

□

For $j \in \{1, 2, ..., k\}$, and $u \in H_0^2(\Omega)$, similar to [17], we denote

$$\beta_j(u) = \frac{\int_\Omega \psi_j(x) |\Delta u|^2}{\int_\Omega |\Delta u|^2},$$

where

$$\psi_j(x) = \min\{\delta_0, |x - a_j|\}.$$

Proposition 2.3. *Let $r_0 = \frac{\delta_0}{3}$, if $\beta_j(u) < r_0$, then*

$$\int_\Omega |\Delta u|^2 \geq 3 \int_{\Omega \setminus B(a_j, \delta_0)} |\Delta u|^2. \tag{2.2}$$

Proof. Let $r_0 = \frac{\delta_0}{3}$. From the definition of ψ_j, we have

$$\begin{aligned}
\delta_0 \int_{\Omega \setminus B(a_j, \delta_0)} |\Delta u|^2 &\leq \int_{\Omega \setminus B(a_j, \delta_0)} \psi_j(x) |\Delta u|^2 \\
&\leq \int_{\Omega \setminus B(a_j, \delta_0)} \psi_j(x) |\Delta u|^2 + \int_{B(a_j, \delta_0)} \psi_j(x) |\Delta u|^2 \\
&= \int_\Omega \psi_j(x) |\Delta u|^2,
\end{aligned}$$

since $\beta_j(u) < r_0$, we obtain

$$\begin{aligned}
\int_\Omega \psi_j(x) |\Delta u|^2 &< r_0 \int_\Omega |\Delta u|^2 \\
&= \frac{\delta_0}{3} \int_\Omega |\Delta u|^2,
\end{aligned}$$

hence, we get

$$\delta_0 \int_{\Omega \setminus B(a_j, \delta_0)} |\Delta u|^2 \leq \frac{\delta_0}{3} \int_\Omega |\Delta u|^2.$$

Thus (2.2) easily follows. □

For r_0 as above, we denote

$$\mathcal{N}_j = \{u \in \mathcal{N}_\mu,\ \beta_j(u) < r_0\}, \quad \mathcal{N}_j^+ = \mathcal{N}_j \cap \mathcal{N}_\mu^+, \quad c_j = \inf_{u \in \mathcal{N}_j^+} J_\mu(u).$$

Let $\phi(x) \in \mathcal{C}_0^\infty(B(0, 2\delta_0))$ be a radially symmetric function such that

$$0 \le \phi(x) \le 1 \text{ in } B(0, 2\delta_0),\ \phi(x) = 1 \text{ in } B(0, \delta_0).$$

Set $u_{\varepsilon,j}(x) = \phi(x - a_j) y_\varepsilon(x - a_j)$. The following properties hold.

Lemma 2.4. *Suppose that $N \ge 5$, and $0 \le \mu < \bar{\mu}$. Then, as $\varepsilon \to 0$, we have the following estimates:*

$$\int_\Omega \left(|\Delta u_{\varepsilon,j}|^2 - \mu \frac{|u_{\varepsilon,j}|^2}{|x - a_j|^4} \right) = \mathcal{A}_\mu^{\frac{N}{4}} + O(\varepsilon^{2(b(\mu)-\delta)}),$$
$$\int_\Omega |u_{\varepsilon,j}|^{2^*} = \mathcal{A}_\mu^{\frac{N}{4}} + O(\varepsilon^{2^*(b(\mu)-\delta)}),$$
$$\int_\Omega |x - a_j|^{\alpha-4} |u_{\varepsilon,j}|^2 = O(\varepsilon^{2(b(\mu)-\delta)}).$$

Proof. The proof is similar to [6] and [16]. □

In order to estimate c_j for a fixed $j \in \{1, 2, ..., k\}$, we need the following proposition.

Proposition 2.5. *Assume that $(F0), (V1)$, and $(V2)$ hold, and $0 < \mu < \bar{\mu}$. Then there exists $\lambda_2 > 0$ such that, for all $\lambda \in (0, \lambda_2)$, we have*

$$c_j < \frac{2}{N} \mathcal{A}_\mu^{\frac{N}{4}} - \chi \lambda^2.$$

Proof. For $u_{\varepsilon,j}(x)$, we can assume from Lemma 2.1 that there exists $t_{\varepsilon,j}^+ = t^+(u_{\varepsilon,j}) > 0$ such that $t_{\varepsilon,j}^+ u_{\varepsilon,j} \in \mathcal{N}_j^+$ for any $\lambda \in (0, \lambda_1)$. Since

$$\beta_j(t_{\varepsilon,j}^+ u_{\varepsilon,j}) = \frac{\int_\Omega \psi_j(x) |\Delta u_{\varepsilon,j}|^2}{\int_\Omega |\Delta u_{\varepsilon,j}|^2} \to \frac{\int_\Omega \psi_j(x) |\Delta u_0|^2}{\int_\Omega |\Delta u_0|^2} = 0 \quad as\ \varepsilon \to 0,$$

there exists $\varepsilon_1 > 0$ such that $\beta_j(t_{\varepsilon,j}^+ u_{\varepsilon,j}) < r_0$ for any $\varepsilon \in (0, \varepsilon_1)$, that is $t_{\varepsilon,j}^+ u_{\varepsilon,j} \in \mathcal{N}_j^+$. To complete the proof, it suffices to show that there exists $\lambda_2 > 0$ such that for all $\lambda \in (0, \lambda_2)$ and $\varepsilon \in (0, \varepsilon_1)$, we have

$$\sup_{t > t_0} J_\mu(t u_{\varepsilon,j}) < \frac{2}{N} \mathcal{A}_\mu^{\frac{N}{4}} - \chi \lambda^2.$$

Let $t^+(u_{\varepsilon,j}) > 0$ as above, we get

$$\begin{aligned}
J_\mu(t^+_{\varepsilon,j}u_{\varepsilon,j}) &\le \sup_{t>t_0} J_\mu(tu_{\varepsilon,j})\\
&\le \sup_{t>0}\left\{\frac{t^2}{2}\int_\Omega \left(|\Delta u_{\varepsilon,j}|^2 - \mu V(x)u^2_{\varepsilon,j}\right) - \frac{t^{2^*}}{2^*}\int_\Omega |u_{\varepsilon,j}|^{2^*}\right\}\\
&\quad - t_0\lambda\int_\Omega f u_{\varepsilon,j}\\
&\le \frac{2}{N}\left(\int_\Omega (|\Delta u_{\varepsilon,j}|^2 - \mu V(x)u^2_{\varepsilon,j})\right)^{\frac{N}{4}}\left(\int_\Omega |u_{\varepsilon,j}|^{2^*}\right)^{\frac{4-N}{4}}\\
&\quad - t_0\lambda\int_\Omega f u_{\varepsilon,j}\\
&\le \frac{2}{N}\left(\int_\Omega\left(|\Delta u_{\varepsilon,j}|^2 - \mu\frac{u^2_{\varepsilon,j}}{|x-a_j|^4}\right) + \mu\int_\Omega |x-a_j|^{\beta-4}u^2_{\varepsilon,j}\right)^{\frac{N}{4}}\\
&\quad \times\left(\int_\Omega |u_{\varepsilon,j}|^{2^*}\right)^{\frac{4-N}{4}} - t_0\lambda\int_\Omega f u_{\varepsilon,j}\\
&\le \frac{2}{N}\mathcal{A}_\mu^{\frac{N}{4}} + o(\varepsilon^{2(b(\mu)-\delta)}) - t_0\lambda\int_\Omega f u_{\varepsilon,j}.
\end{aligned}$$

Then, for all $0 < \varepsilon < \varepsilon_1$, we have

$$\sup_{t>t_0} J_\mu(tu_{\varepsilon,j}) \le \frac{2}{N}\mathcal{A}_\mu^{\frac{N}{4}} + o(\epsilon^{2(b(\mu)-\delta)}) - C_4\lambda\varepsilon^{b(\mu)-\delta},$$

where C_4 is a positive constant independent of ε and j.
Let

$$o(\varepsilon^{2(b(\mu)-\delta)}) = k(\varepsilon)\varepsilon^{2(b(\mu)-\delta)},$$

where $k(\varepsilon) \to 0$ as $\varepsilon \to 0$ and ε be such that

$$\varepsilon^{2(b(\mu)-\delta)} = (\frac{2\chi}{C_4})^2\lambda^2, \quad k(\varepsilon) < \chi(\frac{C_4}{2\chi})^2.$$

If λ_2 is chosen small enough and $\lambda \in (0, \lambda_2)$, then $0 < \varepsilon < \varepsilon_1$. Therefore, it follows from $t^+_{\varepsilon,j}u_{\varepsilon,j} \in \mathcal{N}^+_j$ that

$$c_j = \inf_{u\in\mathcal{N}^+_j} J_\mu(u) \le J_\mu(t^+_{\varepsilon,j}u_{\varepsilon,j}) \le \sup_{t>t_0} J_\mu(tu_{\varepsilon,j}) < \frac{2}{N}\mathcal{A}_\mu^{\frac{N}{4}} - \chi\lambda^2.$$

This completes the proof. □

3. Proof of Theorem 1.1

To prove Theorem 1.1, several lemmas are in order first.

Lemma 3.1. *Let $\lambda \in (0, \lambda_1)$, $0 < \mu < \bar{\mu}$, and $u \in \mathcal{N}_j^+$. Then there exists $\rho_u > 0$ and a differential function $s : B(0, \rho_u) \to \mathbb{R}^+$ such that $s(0) = 1$ and for all $w \in H_0^2(\Omega)$, $\|w\| < \rho_u$, we have $s(w)(u - w) \in \mathcal{N}_j^+$, and*

$$\langle s'(0), w\rangle = \frac{2\int (\Delta u \Delta w - \mu V(x)uw) - 2^* \int |u|^{2^*} uw - \lambda \int fw}{\int (|\Delta u|^2 - \mu V(x)u^2) - (2^* - 1)\int |u|^{2^*}}. \tag{3.1}$$

Proof. We follow some ideas from the paper [20] Lemma 2.4. We define a function $F : \mathbb{R} \times H_0^2(\Omega) \to \mathbb{R}$ as follows:

$$F(s, w) = s\left(\int |\Delta(u - w)|^2 - \mu V(x)(u - w)^2\right) - s^{2^*-1}\int |u-w|^{2^*} - \lambda \int f(u-w).$$

Since $u \in \mathcal{N}_j^+$, we have that

$$F(1, 0) = \left(\int |\Delta u|^2 - \mu V(x)(u)^2\right) - \int |u|^{2^*} - \lambda \int fu = 0$$

and

$$\frac{\partial F}{\partial s}(1, 0) = \left(\int |\Delta u|^2 - \mu V(x)(u)^2\right) - (2^* - 1)\int |u|^{2^*} \neq 0.$$

Applying the implicit function theorem at the point $(1, 0)$, we get the result. □

Lemma 3.2. *Assume that $(F0), (V1)$ and $(V2)$ are verified, and $0 < \mu < \bar{\mu}$. Then there exists $\lambda_3 > 0$, and a sequence $\{u_n^j\} \subset \mathcal{N}_j^+$ such that for any $0 < \lambda < \lambda_3$,*

$$J_\mu(u_n^j) \to c_j, \quad J'_\mu(u_n^j) \to 0 \;\; \text{in } H^{-2}(\Omega).$$

Furthermore $\{u_n^j\}$ contains a subsequence satisfying $u_n^j \rightharpoonup u^j$ in $H_0^2(\Omega)$ and $u^j \neq 0$ is a weak solution of $(\mathcal{P}_\mu)$.

Proof. For every $u \in \mathcal{N}_j^+$, we obtain from (V2) that

$$\begin{aligned} J_\mu(u) &= \frac{1}{2}\int_\Omega (|\Delta u|^2 - \mu V(x)u^2) - \frac{1}{2^*}\int_\Omega |u|^{2^*} - \lambda \int_\Omega fu \\ &= \left(\frac{1}{2} - \frac{1}{2^*}\right)\int_\Omega (|\Delta u|^2 - \mu V(x)u^2) - \left(1 - \frac{1}{2^*}\right)\lambda \int_\Omega fu \\ &\geq \left(\frac{1}{2} - \frac{1}{2^*}\right)(1 - C)\int_\Omega |\Delta u|^2 - \lambda\left(1 - \frac{1}{2^*}\right)\|f\|_- \|u\|, \end{aligned}$$

it follows that J_μ is coercive and bounded from below on $\mathcal{N}_j^+$. By Lemma 2.1, we have that $\mathcal{N}_\mu^-$ and $\mathcal{N}_\mu^+ \cup \mathcal{N}_\mu^0$ are two closed sets in $H_0^2(\Omega)$, and so

$$\mathcal{N}_j \cap (\mathcal{N}_\mu^+ \cup \mathcal{N}_\mu^0) = (\mathcal{N}_j \cap \mathcal{N}_\mu^+) \cup \varnothing = \mathcal{N}_j^+.$$

Using Ekelands variational principle we get a minimizing sequence $\{u_n^j\}$ with the following properties:

(i) $J_\mu(u_n^j) < c_j + \frac{1}{n}$.

(ii) $J_\mu(w) \geq J_\mu(u_n^j) - \frac{1}{n}\|w - u_n^j\|, \quad \forall w \in \mathcal{N}_j^+$.

Next, we shall prove that $J'_\mu(u_n^j) \to 0$, as $n \to +\infty$.
For n large enough, let assume that $\|J'_\mu(u_n^j)\| > 0$. By applying Lemma 3.1 with $u = u_n^j$, and $w = \rho\frac{J'_\mu(u_n^j)}{\|J'_\mu(u_n^j)\|}$, $\rho > 0$, we can find some $s_n(\rho) = s\left(\rho\frac{J'_\mu(u_n^j)}{\|J'_\mu(u_n^j)\|}\right)$ such that

$$w_\rho = s_n(\rho)\left(u_n - \rho\frac{J'_\mu(u_n^j)}{\|J'_\mu(u_n^j)\|}\right) \in \mathcal{N}_\mu.$$

From condition (ii), we get

$$\begin{aligned}\frac{1}{n}\|w_\rho - u_n^j\| &\geq J_\mu(u_n^j) - J_\mu(w_\rho)\\ &= (1 - s_n(\rho))\langle J'_\mu(w_\rho), u_n^j\rangle + \rho s_n(\rho)\langle J'_\mu(w_\rho), \frac{J'_\mu(u_n^j)}{\|J'_\mu(u_n^j)\|}\rangle + o(\rho).\end{aligned}$$

Dividing by ρ and passing to the limit as ρ goes to zero we derive that

$$\begin{aligned}\frac{1}{n}(1 + |s'_n(0)|\|u_n\|) &\geq -s'_n(0)\langle J'_\mu(u_n^j), u_n^j\rangle + \|J'_\mu(u_n^j)\|\\ &= \|J'_\mu(u_n^j)\|,\end{aligned}$$

where $s'_n(0) = \langle s'(0), \frac{J'_\mu(u_n^j)}{\|J'_\mu(u_n^j)\|}\rangle$. Thus, we have

$$\|J'_\mu(u_n^j)\| \leq \frac{C_5}{n}(1 + s'_n(0)).$$

At this point, we need to prove that $|s'_n(0)|$ is uniformly bounded with respect to n. By (3.1), we get

$$|s'_n(0)| \leq \frac{C_6}{\int\left(|\Delta u_n^j|^2 - \mu V(x)(u_n^j)^2\right) - (2^* - 1)\int |u_n^j|^{2^*}}.$$

Hence, we must prove that $\int \left(|\Delta u_n^j|^2 - \mu V(x)(u_n^j)^2\right) - (2^* - 1)\int |u_n^j|^{2^*}$ is bounded away from zero. Arguing by contradiction, assume that for a subsequence still called $\{u_n\}$, we get

$$\int \left(|\Delta u_n^j|^2 - \mu V(x)(u_n^j)^2\right) - (2^* - 1)\int |u_n^j|^{2^*} = o(1), \tag{3.2}$$

thus we obtain

$$\int |u_n^j|^{2^*} > C_7.$$

By (3.2), and the fact that $u_n^j \in \mathcal{N}_\mu$, we have

$$\lambda \int f u_n^j = (2^* - 2)\int |u_n^j|^{2^*} + o(1),$$

then there exists $\lambda_3 > 0$, such that $\|J'_\mu(u_n^j)\| \to 0$ for $0 < \lambda < \lambda_3$.

Now, we prove that $u_j \neq 0$. For the ease of notation, we note $u_n = u_n^j$ and $u = u^j$. Arguing by contradiction, we assume that $u = 0$, then

$$\lim_{n\to\infty}\int_\Omega f u_n = \int_\Omega f u = 0,$$

that is

$$\begin{aligned} c_j + o(1) &= J_\mu(u_n) \\ &= \frac{1}{2}\int_\Omega (|\Delta u_n|^2 - \mu V(x)u_n^2) - \frac{1}{2^*}\int_\Omega |u_n|^{2^*} + o(1) \\ &= \frac{2}{N}\int_\Omega (|\Delta u_n|^2 - \mu V(x)u_n^2) + o(1), \end{aligned}$$

and

$$\begin{aligned} \int_\Omega (|\Delta u_n|^2 - \mu V(x)u_n^2) &= \int_\Omega |u_n|^{2^*} + o(1) \\ &\le \left(\frac{\int_\Omega |\Delta u_n|^2}{\mathcal{A}_0}\right)^{2^*/2} + o(1) \\ &\le \left(\frac{\int_\Omega (|\Delta u_n|^2 - \mu V(x)u_n^2)}{(1-C)\mathcal{A}_0}\right)^{2^*/2} + o(1), \end{aligned}$$

together, these imply that

$$Nc_j + 1 \ge \int_\Omega (|\Delta u_n|^2 - \mu V(x)u_n^2) \ge ((1-C)\mathcal{A}_0)^{N/4} > 0$$

holds for n large. It follows from (V2) and (1.3), that

$$(1-C)\|u_n\|^2 \le \int_\Omega (|\Delta u_n|^2 - \mu V(x)u_n^2) \le C_5\|u_n\|^2.$$

That is
$$0 < \nu_1 \leq \|u_n\| \leq \nu_2.$$
Setting
$$\lim_{n\to\infty}\int_\Omega (|\Delta u_n|^2 - \mu V(x)u_n^2) = \lim_{n\to\infty}\int_\Omega |u_n|^{2^*} = l_1 > 0.$$
Define the Levy-type concentration functions by
$$\varrho_n(r) = \sup_{y\in\Omega}\int_{B(y,r)} |u_n|^{2^*},$$
there exists $r_n > 0$ such that $\varrho_n(r_n) = \dfrac{l_1}{2}$. Moreover, there exists $y_n \in \Omega$ such that
$$\int_{B(y_n,r_n)} |u_n|^{2^*} = \varrho_n(r_n) = \frac{l_1}{2}.$$
Let us define the rescaled functions by
$$w_n(\tilde{x}) = r_n^{\frac{N-4}{2}} u_n(r_n\tilde{x} + y_n),$$
thus, $w_n(\tilde{x}) \in H_0^2(\Omega_n)$ and so $\Omega_n = (\Omega - y_n)/r_n$, by extending $w_n(\tilde{x})$ to be zero for $\tilde{x}$ outside Ω_n, we drive that $w_n(\tilde{x}) \in \mathcal{D}^{2,2}(\mathbb{R}^\mathbb{N})$. From
$$\int_{\mathbb{R}^\mathbb{N}} |\Delta w_n(\tilde{x})|^2 d\tilde{x} = \int_\Omega |\Delta u_n(x)|^2 dx,$$
we know that $\{w_n\}$ is bounded in $\mathcal{D}^{2,2}(\mathbb{R}^\mathbb{N})$. Thus, we can assume that
$$w_n(\tilde{x}) \rightharpoonup w_0(\tilde{x}) \ in\ \mathcal{D}^{2,2}(\mathbb{R}^\mathbb{N}), \quad w_n(\tilde{x}) \to w_0(\tilde{x}) \ a.e\ in\ \mathbb{R}^\mathbb{N}.$$
Recalling that $r_n\tilde{x} + y_n \in \Omega$ and Ω is bounded, we can assume, up to a subsequence, that $r_n \to \bar{r} \geq 0$ and $y_n \to \bar{y} \in \overline{\Omega}$. We will divide the discussion into two cases, whenever writing $r_n\tilde{x} + y_n \in$ (or $\notin$)$B(a_j, \delta_0)$, we always mean that there is a natural number N_1 such that for all $n > N_1$, there holds $r_n\tilde{x} + y_n \in$ (or $\notin$)$B(a_j, \delta_0)$.

Case (A): If $r_n\tilde{x} + y_n \notin B(a_j, \delta_0)$, it follows from the definition of $\beta_j(u)$, ψ_j and r_0 that
$$\frac{\delta_0}{3} = r_0 = \beta_j(u_n) = \frac{\int_\Omega \psi_j(x)|\Delta u_n|^2}{\int_\Omega |\Delta u_n|^2} = \frac{\int_{\mathbb{R}^\mathbb{N}} \psi_j(r_n\tilde{x} + y_n)|\Delta w_n(\tilde{x})|^2}{\int_{\mathbb{R}^\mathbb{N}} |\Delta w_n(\tilde{x})|^2} \to \delta_0,$$
which is a contradiction.

Case (B): If $r_n\tilde{x}+y_n \in B(a_j,\delta_0)$, then

$$\begin{aligned}J_\mu(u_n) &= \frac{1}{2}\int_\Omega (|\Delta u_n|^2 - \mu V(x)u_n^2) - \frac{1}{2^*}\int_\Omega |u_n|^{2^*} + o(1)\\ &= \frac{1}{2}\int_{\mathbb{R}^\mathbb{N}} \left(|\Delta w_n(\tilde{x})|^2 - \mu r_n^4 V(r_n\tilde{x}+y_n)w_n^2(\tilde{x})\right) - \frac{1}{2^*}\int_{\mathbb{R}^\mathbb{N}} |w_n(\tilde{x})|^{2^*} + o(1)\\ &= \frac{2}{N}\int_{\mathbb{R}^\mathbb{N}} \left(|\Delta w_n(\tilde{x})|^2 - \mu r_n^4 V(r_n\tilde{x}+y_n)w_n^2(\tilde{x})\right) + o(1).\end{aligned}$$

For any $w \in \mathcal{D}^{2,2}(\mathbb{R}^\mathbb{N})$, we denote

$$\tilde{J}_\mu(w) = \frac{1}{2}\int_{\mathbb{R}^\mathbb{N}} \left(|\Delta w|^2 - \mu r_n^4 V(r_n\tilde{x}+y_n)w^2\right) - \frac{1}{2^*}\int_{\mathbb{R}^\mathbb{N}} |w|^{2^*},$$

thus, we have

$$J_\mu(u_n) = \tilde{J}_\mu(w_n) + o(1).$$

Therefore, it follows from $J_\mu(u_n) \to c_j < \frac{2}{N}\mathcal{A}_\mu^{\frac{N}{4}} - \chi\lambda^2$, that

$$\frac{2}{N}\mathcal{A}_\mu^{\frac{N}{4}} - \chi\lambda^2 > c_j = \lim_{n\to\infty} J_\mu(u_n) = \lim_{n\to\infty} \tilde{J}_\mu(u_n) \geq \frac{2}{N}\mathcal{A}_\mu^{\frac{N}{4}}.$$

Again, we get a contradiction. Hence, we have shown that $u^j \neq 0$. □

Proposition 3.3. *For $j \in \{1,2,...,k\}$, if $r_n\tilde{x}+y_n \in B(a_j,\delta_0)$ and $\{w_n\} \subset \mathcal{D}^{2,2}(\mathbb{R}^\mathbb{N})$ are such that $0 < \nu_3 \leq \|w_n\| \leq \nu_4$, and*

$$\int_{\mathbb{R}^\mathbb{N}} (|\Delta w_n|^2 - \mu V(x)w_n^2) = \int_{\mathbb{R}^\mathbb{N}} |w_n|^{2^*} + o(1),$$

then

$$\tilde{J}_\mu(w_n) \geq \frac{2}{N}\mathcal{A}_\mu^{\frac{N}{4}}.$$

Proof. The proof is divided into several cases.

(i) If $r_n \to \bar{r} > 0$ and $y_n \to \bar{y}$, we set $q = (\bar{y}-a_j)/\bar{r}$, then by using $(V1)$ and the minimization problem (1.1), we have

$$\begin{aligned}\int_{\mathbb{R}^\mathbb{N}} \left(|\Delta w_n|^2 - \frac{\mu w_n^2}{|\tilde{x}+q|^2}\right) &\geq \mathcal{A}_\mu \left(\int_{\mathbb{R}^\mathbb{N}} |w_n|^{2^*}\right)^{\frac{2}{2^*}}\\ &= \mathcal{A}_\mu \left(\int_{\mathbb{R}^\mathbb{N}} \left(|\Delta w_n|^2 - \mu r_n^4 V(r_n\tilde{x}+y_n)w_n^2\right)\right)^{\frac{2}{2^*}} + o(1)\\ &= \mathcal{A}_\mu \left(\int_{\mathbb{R}^\mathbb{N}} \left(|\Delta w_n|^2 - \frac{\mu}{|\tilde{x}+q|^2}w_n^2\right)\right)^{\frac{2}{2^*}} + o(1).\end{aligned}$$

Thus, we obtain

$$\int_{\mathbb{R}^\mathbb{N}} \left(|\Delta w_n|^2 - \frac{\mu}{|\tilde{x}+q|^2}w_n^2\right) \geq \mathcal{A}_\mu^{N/4}.$$

Hence, we have

$$\tilde{J}_\mu(w_n) = \frac{2}{N}\int_{\mathbb{R}^{\mathbb{N}}}\left(|\Delta w_n|^2 - \mu r_n^4 V(r_n\tilde{x}+y_n)w_n^2\right) \geq \frac{2}{N}\mathcal{A}_\mu^{N/4}.$$

(ii) If $r_n \to \bar{r} = 0$ and $y_n \to \bar{y} \neq a_j$, then it follows

$$r_n^4 V(r_n\tilde{x}+y_n) \leq \frac{r_n^4}{|r_n\tilde{x}+y_n-a_j|^4} - r_n^4\,|r_n\tilde{x}+y_n-a_j|^{\alpha-4} \to 0.$$

Thus, we get

$$\tilde{J}_\mu(w_n) = \frac{1}{2}\int_{\mathbb{R}^{\mathbb{N}}}|\Delta w_n|^2 - \frac{1}{2^*}\int_{\mathbb{R}^{\mathbb{N}}}|w_n|^{2^*} + o(1). \tag{3.3}$$

Then, by the Sobolev inequality, we drive

$$\int_{\mathbb{R}^{\mathbb{N}}}|\Delta w_n|^2 \geq \mathcal{A}_0\left(\int_{\mathbb{R}^{\mathbb{N}}}|w_n|^{2^*}\right)^{2/2^*} = \mathcal{A}_0\left(\int_{\mathbb{R}^{\mathbb{N}}}|\Delta w_n|^2\right)^{2/2^*},$$

hence

$$\int_{\mathbb{R}^{\mathbb{N}}}|\Delta w_n|^2 \geq \mathcal{A}_0^{N/4}.$$

It follows from (3.3) that

$$\tilde{J}_\mu(w_n) = \frac{2}{N}\int_{\mathbb{R}^{\mathbb{N}}}|\Delta w_n|^2 + o(1) \geq \frac{2}{N}\mathcal{A}_0^{N/4} \geq \frac{2}{N}\mathcal{A}_\mu^{\frac{N}{4}}.$$

(iii) If $r_n \to \bar{r} = 0,\ \bar{y} = a_j$ and $(y_n - a_j)/r_n \to q' \in \mathbb{R}^{\mathbb{N}}$, then similarly to (i) we get $\tilde{J}_\mu(w_n) \geq \frac{2}{N}\mathcal{A}_\mu^{\frac{N}{4}}$.

(iv) If $r_n \to \bar{r} = 0,\ \bar{y} = a_j$ and $(y_n - a_j)/r_n \to \infty$, then similarly to (ii) we get $\tilde{J}_\mu(w_n) \geq \frac{2}{N}\mathcal{A}_0^{\frac{N}{4}} \geq \frac{2}{N}\mathcal{A}_\mu^{\frac{N}{4}}$. This completes the proof. □

Lemma 3.4. *The weak limit u^j of $\{u_n^j\}$ in Lemma 3.2, is indeed strong.*

Proof. We argue by contradiction, and again we note $u_n = u_n^j$ and $u = u^j$. Suppose that $\|u_n - u\| \to \nu_5 > 0$ and denote $v_n = u_n - u$. Then we have $v_n \rightharpoonup 0$ in $H_0^2(\Omega)$ and $\|v_n\| \to \nu_5 > 0$. Using BrezisLieb Lemma, and the fact that u is a weak solution of $(\mathcal{P}_\mu)$, we drive

$$\int_\Omega\left(|\Delta v_n|^2 - V(x)v_n^2(x)\right) - \int_\Omega |v_n|^{2^*} = o(1).$$

Very similar to the previous proof, we can find $y_n \in \Omega$, $r_n > 0$ such that

$$\int_{B(y_n,r_n)}|u_n^j|^{2^*} = \frac{l_2}{2} > 0.$$

Letting $z_n(\tilde{x}) = r_n^{\frac{N-4}{2}} v_n(r_n\tilde{x}+y_n)$, and extending z_n to be zero for $\tilde{x}$ outside $\Omega_n = r_n^{-1}(\Omega - y_n)$, hence $z_n(\tilde{x}) \in D^{2,2}(\mathbb{R}^{\mathbb{N}})$, and so

$$\int_{\mathbb{R}^{\mathbb{N}}} \left(|\Delta z_n(\tilde{x})|^2 - r_n^4 V(r_n\tilde{x}+y_n) z_n^2(\tilde{x})\right) d\tilde{x} - \int_{\mathbb{R}^{\mathbb{N}}} |z_n(\tilde{x})|^{2^*} d\tilde{x} = o(1). \tag{3.4}$$

We will divide the proof into two cases.

Case (C): While $r_n\tilde{x}+y_n \notin B(a_j,\delta_0)$, then from the definition of $\beta_j(u_n)$, we have

$$r_0 = \beta_j(u_n) = \frac{\int_{\mathbb{R}^{\mathbb{N}}} \psi_j(r_n\tilde{x}+y_n)|\Delta z_n(\tilde{x})|^2}{\int_{\mathbb{R}^{\mathbb{N}}} |\Delta z_n(\tilde{x})|^2} \to \delta_0,\ as\ n\to\infty,$$

which is impossible.

Case (D): While $r_n\tilde{x}+y_n \in B(a_j,\delta_0)$, then

$$\begin{aligned} J_\mu(u_n) &= \frac{1}{2}\int_\Omega (|\Delta v_n|^2 - \mu V(x)v_n^2) - \frac{1}{2^*}\int_\Omega |v_n|^{2^*} + J_\mu(u) + o(1) \\ &= \frac{1}{2}\int_{\mathbb{R}^{\mathbb{N}}} \left(|\Delta z_n(\tilde{x})|^2 - \mu r_n^4 V(r_n\tilde{x}+y_n) z_n^2(\tilde{x})\right) - \frac{1}{2^*}\int_{\mathbb{R}^{\mathbb{N}}} |z_n(\tilde{x})|^{2^*} \\ &\quad + J_\mu(u) + o(1) \\ &= \tilde{J}_\mu(z_n) + J_\mu(u) + o(1). \end{aligned}$$

It follows from (3.4), Proposition 3.3 and Lemma 2.2, that

$$J_\mu(u_n^j) = \tilde{J}_\mu(z_n) + J_\mu(u^j) + o(1) \geq \frac{2}{N}\mathcal{A}_\mu^{\frac{N}{4}} - \chi\lambda^2,$$

which contradict the choice of $\{u_n^j\}$. Hence, the weak limit u^j of $\{u_n^j\}$ is indeed strong. □

Now, we are in a position for the proof of Theorem 1.1.

Proof of Theorem 1.1

For $j \in \{1,...,k\}$, we choose $\lambda^* = \min\{\lambda_1,\lambda_2,\lambda_3\}$. From Lemmas 3.2 and 3.4, we can assume that there exists $\{u_n^j\} \subset \mathcal{N}_j^+$ such that $J_\mu(u_n^j) \to c_j,\ u_n^j \to u^j$ in $H_0^2(\Omega)$, that is, $u^j \neq 0$ is a weak solution of $(\mathcal{P}_\mu)$. For all $i \neq j$, suppose $u^i = u^j$, it follows from Proposition 2.3 that

$$2\int_\Omega |\Delta u^i|^2 \geq 3\left(\int_{\Omega\setminus B(a_j,\delta_0)} |\Delta u^i|^2 + \int_{\Omega\setminus B(a_i,\delta_0)} |\Delta u^i|^2\right) \geq 3\int_\Omega |\Delta u^i|^2,$$

which is a contradiction. Therefore, $u^i \neq u^j$. This implies that problem $(\mathcal{P}_\mu)$ has at least k solutions $u^j \in \mathcal{N}_j^+$.

Remark 3.5. We point out that the techniques developed in the present paper can be further combined with those in a forthcoming work, where we will establish the existence of at least k solutions in $\mathcal{N}_\mu^-$. Hence, the problem $(\mathcal{P}_\mu)$ has at least $2k$ solutions in $H_0^2(\Omega)$.

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