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A BACKWARD PROBLEM FOR THE NONLINEAR DIFFUSION EQUATION WITH COUPLING OPERATOR OF LOCAL AND NONLOCAL TYPE AND GAUSSIAN WHITE NOISE ON THE MEASUREMENT

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Abstract. In this paper, we consider the problem of recovering the heat distribution for a nonlinear diffusion equation with local and nonlocal operators with Gaussian white noise. As commonly acknowledged, the problem is severely ill-posed according to Hadamard's definition. Consequently, we propose the Fourier truncation method to regularize the problem. With different assumptions on the exact solution, the estimation of the expectation of the error between the regularized solution and the exact solution is obtained.

1. Introduction

The coupling operator of local and nonlocal type appears in many real-world applications. The couping operator was used to describe the diffusion law of particles that follow the Levy and Brownian processes simultaneously. In practical aspects, the coupling operator was used in the modeling of biological population dynamic where a population with density can possibly alternate both short and long range random walks. This could be motivated, for instance, by a superposition between local exploration of the environment and hunting strategies. Another concrete application of coupling operators of local and nonlocal type can be found in plasma physics. In astrophysical plasmas, the magnetic fields can be used to confine high temperature plasmas. This

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is considered as one of the most promising mechanism to achieve controlled nuclear fusion. That is to say, understanding heat transport in magnetized plasmas in a currently open and challenging problem in plasma physics due to its complex structure. In addition, the heat transport in magnetized plasmas is strongly anisotroplic. Particularly, the parallel heat flux is many orders of magnitude larger than the perpendicular heat flux. The situation is even worse in the low collisionlity plasmas of interest to controlled fusion, the closure relation of the parallel heat flux typically involves non - local operators along the field line, turning the parallel heat transport equation into an integero - differential equation combining of both local and nonlocal operators. Other applications of coupling local and nonlocal diffusion can be found in the model of light - sensitive Belousove - Zhabotinsky reaction in chemistry. (see [5–7, 11, 17]).

In theoretical aspects, the diffusion operators with coupling of local and nonlocal type has been studied in several aspects containing elliptic boundary value problem, the semilinear elliptic equation, the logistic equation, the shape optimization problems for mixed operators, the continuous dependence estimates for viscosity solutions, the nonlocal Allen - Cahn type equation, the nonlocal variant of the classical Cahn - Hillard equation, the wave equations with frational damping and the decay estimate for evolution equations (see [3, 4, 9, 10, 12–14, 16, 22, 27, 31]).

Inspired by the various applications of nonlinear diffusion with coupling operator, let $D = (0, \pi)$, we investigate the problem of determining the temperature distribution $u(x, t)$ for $t \in [0, T)$ which satisfies the following nonlinear diffusion equation

$$u_t(x,t) - \alpha\Delta u(x,t) + \beta(-\Delta)^\gamma u(x,t) = f(x,t,u(x,t)),\ (x,t) \in D \times [0,T], \tag{1.1}$$

and conditions

$$u(0,t) = u(\pi,t) = 0,\ \ t \in [0,T], \tag{1.2}$$

$$u(x,T) = g(x),\ x \in D, \tag{1.3}$$

where the numbers $\alpha, \beta > 0$, $\gamma \in (0,1)$, the final time $T > 0$, the final data $g(x)$ and the nonlinear source $f(x,t,u)$ are given. The fractional Laplacian operator $(-\Delta)^\gamma$ which will be defined in section 2.

The problem (1.1)-(1.3) is widely acknowledged as severely ill-posed, indicating that the solution does not exhibit continuous dependence on the input data. In other words, even minor perturbations in the input data can lead to significant changes in the solution. Therefore, implementing an appropriate regularization process is essential to obtain a stable solution.

The problem (1.1)-(1.3) for the case $\alpha > 0, \beta = 0$ becomes the backward problem for classical nonlinear parabolic equation, which has been extensively investigated in many papers (see [32, 33]). For instance, in [32], Trong and co-authors use an association of the quasi-reversibility method and the quasi-boundary value method to regularize the problem. In [33], Trong and Tuan regularized the problem in the two - dimensional case by using the Fourier truncation method. When $\alpha = 0, \beta > 0$, the problem (1.1) - (1.3) will become the backward problem for the space-fractional diffusion equation has been studied by many mathematicians (see [26, 36–38]). Especially, in [37], Zheng introduced the negative exponential regularization technique to tackle the problem of backward diffusion in the homogeneous case. In [26], Triet, Khieu, Khanh, Hung regularized the problem in the nonhomogeneous case. When $\alpha > 0, \beta > 0$, the problem (1.1) - (1.3) in the nonhomogeneous case has been considered by some authors (see [20, 25]). For example, in [20], Khieu and Hung used the filter method to regularize the problem . In [25], Li and Zhang tackled the problem by applying the Landweber iterative regularization method.

In addition, there is the error in the measurement, so we need to assume the presence of an approximation g_ε. If the error comes from controllable sources, it is assumed to be bounded by a fixed $\varepsilon > 0$ and has been studied much in previous papers. While this model simplifies the assessment of solution errors, it may not fully reflect reality. Recently, researchers have been considering the inverse problem for the heat equation with random input data to better understand and model the diffusion of pollutants in the environment. Environmental factors like temperature, sunlight, wind, rain, and humidity introduce randomness into the problem's data. Consequently, numerical data deviates from exact data due to random noise, with the random noise model offering a more realistic representation. Some scientists have embraced this approach, as evidenced by recent studies (see [28, 35]). However, evaluating the error of the solution becomes more complex because the solution itself is a random variable. One widely-used random process is white noise, owing to its broad applications across various disciplines such as engineering, science, and business. It finds utility in electronic systems, signal processing, econometric models, and acoustics, among other fields.

As far as we are aware, the problem (1.1)-(1.3) in the nonlinear case with Gaussian white noise has not been explored and this is the motivation of our paper. Hence, in this present article, we study the problem (1.1)-(1.3) with the following random model

$$g_\varepsilon(x) = g(x) + \varepsilon\xi(x),$$

where $\varepsilon > 0$ represents the magnitude of the noise and ξ is a Gaussian white noise process. To address the regularization of the problem, we will employ the Fourier truncation method. Considering various conditions on the exact solution, we aim to determine the convergence rate of Hölder or logarithmic type of the expectation of the error between the regularized solution and the exact solution.

The rest of this present paper is organized into four sections. In section 2, we introduces some definitions and find the solution of the problem. In section 3, we prove the ill-posedness of the problem. In section 4, we propose the regularization method and estimate the expectation of the error between the regularized solution and the exact solution. In section 5, we give a numerical example to illustrate the effectiveness of the theory. In section 6, we give a conclusion.

2. Preliminaries and fundamental solution

Throughout this paper, we denote $D = (0, \pi)$.

Definition 2.1. Let us consider

$$L^2(D) = \left\{ v : D \to \mathbb{R} \,\middle|\, v \text{ is Lebesgue measurable and } \int_0^\pi |v(x)|^2 \, dx < \infty \right\},$$

with the inner product

$$\langle v_1, v_2 \rangle = \int_0^\pi v_1(x) v_2(x) \, dx, \text{ for } v_1, v_2 \in L^2(D)$$

and

$$\|v\| = \left(\int_0^\pi |v(x)|^2 \, dx \right)^{1/2}.$$

Lemma 2.2. ([15]) *Let $\{\lambda_n\}_{n\in\mathbb{N}}$ are all the eigenvalues of the operator $-\Delta$, and $\{\phi_n(x)\}_{n\in\mathbb{N}}$ are the corresponding eigenfunctions satisfy*

$$\begin{cases} -\Delta \phi_n(x) = \lambda_n \phi_n(x), x \in D, \\ \phi_n(x) = 0, x \in \partial D, \end{cases}$$

where $\Delta = \dfrac{d^2}{dx^2}$ is the one-dimensional Laplace operator. Then

$$\lambda_n = n^2 \text{ and } \phi_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx). \tag{2.1}$$

Note that $\{\phi_n(x)\}_{n\in\mathbb{N}}$ is an orthonormal basis of $L^2(D)$.

Definition 2.3. ([19]) Let $v \in L^2(D)$. For every $\alpha > 0$, the fractional Laplacian operator is defined as follows

$$(-\Delta)^{\alpha} v(x) = \sum_{n=1}^{\infty} n^{2\alpha} \langle v, \phi_n \rangle \phi_n(x),$$

where $\phi_n(x)$ is given by (2.1).

Definition 2.4. ([19]) For $s > 0$, let us consider

$$H^s(D) = \left\{ v \in L^2(D) : \sum_{n=1}^{\infty} n^{2s} \left|\langle v, \phi_n \rangle\right|^2 < \infty \right\}$$

and

$$\|v\|_{H^s(D)} = \left(\sum_{n=1}^{\infty} n^{2s} \left|\langle v, \phi_n \rangle\right|^2 \right)^{1/2},$$

where $\phi_n(x)$ is given by (2.1).

Notify that $H^s(D)$ is a Hilbert space with the inner product

$$\langle f, g \rangle_{H^s(D)} = \sum_{n=1}^{\infty} n^{2s} \langle f, \phi_n \rangle \langle g, \phi_n \rangle.$$

Definition 2.5. Let us consider

$$\begin{aligned} &C([0,T]; L^2(D)) \\ &= \left\{ v : [0,T] \to L^2(D) \text{ is measurable and } \sup_{0\le t\le T} \|v(.,t)\| < \infty \right\} \end{aligned}$$

and

$$\|v\|_{C([0,T];L^2(D))} = \sup_{0\le t\le T} \|v(.,t)\|.$$

Definition 2.6. Let us consider

$$\begin{aligned} &C([0,T]; H^s(D)) \\ &= \left\{ v : [0,T] \to H^s(D) \text{ is measurable and } \sup_{0\le t\le T} \|v(.,t)\|_{H^s(D)} < \infty \right\} \end{aligned}$$

and

$$\|v\|_{C([0,T];H^s(D))} = \sup_{0\le t\le T} \|v(.,t)\|_{H^s(D)}.$$

Definition 2.7. ([30]) Given a measure probability space Ω. Let us consider the Bochner space

$$L^2(\Omega, L^2(D)) = \Big\{ v : \Omega \to L^2(D) \text{ is measurable and } \mathbb{E}\, \|v\|^2 < \infty \Big\}$$

and

$$\|v\|_{L^2(\Omega, L^2(D))} = \sqrt{\mathbb{E}\, \|v\|^2}.$$

Definition 2.8. Let us consider the normed space

$$V_T = \left\{ v : [0,T] \to L^2(\Omega, L^2(D)) \text{ is measurable and } \sup_{0 \le t \le T} \sqrt{\mathbb{E}\, \|v(.,t)\|^2} < \infty \right\}$$

and

$$\|v\|_{V_T} = \underset{0 \le t \le T}{sup} \sqrt{\mathbb{E}\, \|v(.,t)\|^2}.$$

Definition 2.9. ([8]) Let H be a Hilbert space. We say that ξ is a white noise process if $Cov_\xi = I$ and the random variables are Gaussian: for all functions $g_1, g_2 \in H$, the random variables $\langle \xi, g_j \rangle$ have normal distributions $\mathcal{N}(0, \|g_j\|^2)$ and $Cov\left(\langle \xi, g_1 \rangle, \langle \xi, g_2 \rangle\right) = \langle g_1, g_2 \rangle$.

Lemma 2.10. ([8]) *Let ξ be a white noise process in a Hilbert space H and $\{\phi_n\}$ be an orthonormal basis in H. Define ξ_n by $\xi_n = \langle \xi, \phi_n \rangle$. Then $\{\xi_n\}$ are independent and identically distributed standard Gaussian random variables.*

In what follows, we introduce some assumptions.

(H1): $g \in L^2(D)$.

(H2): $f : [0,\pi] \times [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfying $f(x,y,0) = 0$ and

$$|f(x,t,u) - f(x,t,v)| \le K|u - v|,$$

for $K > 0$ independent of x, t, u, v.

Theorem 2.11. *Suppose that assumptions (H1) - (H2) are fulfilled. If the problem* (1.1)-(1.3) *has a solution in $C([0,T]; L^2(D))$ then the solution is given by*

$$u(x,t) = \sum_{n=1}^{\infty} \left[e^{(\alpha n^2 + \beta n^{2\gamma})(T-t)} g_n - \int_t^T e^{(\alpha n^2 + \beta n^{2\gamma})(s-t)} f_n(u)(s) ds \right] \phi_n(x), \tag{2.2}$$

where

$$g_n = \langle g, \phi_n \rangle,$$
$$f_n(u)(s) = \langle f(., s, u(., s)), \phi_n \rangle.$$

Proof. If $u(x,t)$ represents a solution of the problem (1.1)-(1.3), and it takes the form $u(x,t) = \sum\limits_{n=1}^{\infty} u_n(t)\sin(nx)$, where $u_n(t) = \langle u(.,t), \phi_n\rangle$, then by multiplying both sides of equation (1.1) by $\sin(nx)$ and integrating over the domain D, we obtain:

$$\frac{d}{dt}u_n(t) + (\alpha n^2 + \beta n^{2\gamma})u_n(t) = f_n(u)(t). \tag{2.3}$$

Multiplying both sides of (2.2) by $e^{(\alpha n^2+\beta n^{2\gamma})t}$ and taking the integral from t to T, we obtain

$$\int_t^T \left(e^{(\alpha n^2+\beta n^{2\gamma})s}u_n(s)\right)' ds = \int_t^T e^{(\alpha n^2+\beta n^{2\gamma})s} f_n(u)(s)\, ds.$$

Then, we have

$$e^{(\alpha n^2+\beta n^{2\gamma})T}u_n(T) - e^{(\alpha n^2+\beta n^{2\gamma})t}u_n(t) = \int_t^T e^{(\alpha n^2+\beta n^{2\gamma})s} f_n(u)(s)\, ds.$$

It implies that

$$u_n(t) = e^{(\alpha n^2+\beta n^{2\gamma})(T-t)}g_n - \int_t^T e^{(\alpha n^2+\beta n^{2\gamma})(s-t)} f_n(u)(s)\, ds.$$

So we get

$$u(x,t) = \sum_{n=1}^{\infty}\left[e^{(\alpha n^2+\beta n^{2\gamma})(T-t)}g_n - \int_t^T e^{(\alpha n^2+\beta n^{2\gamma})(s-t)} f_n(u)(s) ds\right]\phi_n(x).$$

This completes the proof. □

In the next section, we will give an example to prove the ill-posedness of the problem (1.1)-(1.3).

3. Example of the ill-posedness of the problem (1.1)-(1.3) with Gaussian white noise

We give an example which shows that the problem (1.1)-(1.3) has a solution and its solution is not stable. Let us consider the problem

$$\begin{cases} u_t(x,t) - 0.1\Delta u(x,t) + 0.2(-\Delta)^{\gamma}u(x,t) = f(x,t,u), \quad (x,t)\in(0,\pi)\times[0,1], \\ u(0,t) = u(\pi,t) = 0\,,\ t\in[0,1], \\ u(x,1) = g(x),\, x\in(0,\pi), \end{cases} \tag{3.1}$$

where $\gamma = 0.7$ and

$$f(x,t,u(x,t)) = -8\gamma t^{8\gamma-1}\sin(t^{8\gamma})\sin(x) + 0.2\cos(t^{8\gamma})\sin(x) + 0.1u(x,t),$$

$$g(x) = \cos(1)\sin(x).$$

The exact solution of the problem (3.1) is

$$u_{ex}(x,t) = \cos(t^{8\gamma})\sin(x).$$

Let us choose the measured data

$$g_n(x) = g(x) + \frac{1}{n^{\frac{3}{4}}}\sum_{p=1}^{n}\langle \xi, \phi_p\rangle \phi_p(x),$$

where $\phi_p(x) = \sqrt{\dfrac{2}{\pi}}\sin(px)$. We get

$$\mathbb{E}\left\|g_n - g\right\|^2 = \frac{1}{n^{\frac{3}{2}}}\mathbb{E}\left(\sum_{p=1}^{n}\xi_p^2\right),$$

where $\xi_p = \langle \xi, \phi_p\rangle$.

It follows from Lemma (2.2) that $\mathbb{E}(\xi_p^2) = 1$. It leads to

$$\mathbb{E}\left\|g_n - g\right\|^2 = \frac{1}{n^{\frac{1}{2}}}. \tag{3.2}$$

The exact solution of the problem (3.1) corresponding to the measured data g_n is

$$u_n(x,t) = \sum_{p=1}^{\infty}\Big[e^{(0.1p^2+0.2p^{2\gamma})(T-t)}g_{n,p} - \int_t^T e^{(0.1p^2+0.2p^{2\gamma})(s-t)}f_p(u_n)(s)ds\Big]\phi_p(x), \tag{3.3}$$

where

$$g_{n,p} = \langle g_n, \phi_p\rangle,$$
$$f_p(u_n)(s) = \langle f(., s, u_n(., s)), \phi_p\rangle.$$

We get

$$\begin{aligned}
&\mathbb{E}\left\|u_n(.,t) - u_{ex}(.,t)\right\|^2 \\
&\quad = \mathbb{E}\Big(\sum_{p=1}^{\infty}\Big[e^{(0.1p^2+0.2p^{2\gamma})(T-t)}(g_{n,p} - g_p) \\
&\qquad - \int_t^T e^{(0.1p^2+0.2p^{2\gamma})(s-t)}(f_p(u_n)(s) - f_p(u_{ex})(s))ds\Big]^2\Big) \\
&\quad \ge \mathbb{E}\Big(\Big[e^{(0.1n^2+0.2n^{2\gamma})(T-t)}(g_{n,n} - g_n) \\
&\qquad - \int_t^T e^{(0.1n^2+0.2n^{2\gamma})(s-t)}(f_n(u_n)(s) - f_n(u_{ex})(s))ds\Big]^2\Big).
\end{aligned}$$

Using the inequality $(a+b)^2 \geq \frac{1}{2}a^2 - b^2$, $a, b \in \mathbb{R}$, we have the estimate

$$\begin{aligned}\mathbb{E}\left\|u_n(.,t) - u_{ex}(.,t)\right\|^2 \geq \frac{1}{2}\mathbb{E}\left[e^{(0.1n^2+0.2n^{2\gamma})(T-t)}(g_{n,n} - g_n)\right]^2 \\ - \mathbb{E}\left[\int_t^T e^{(0.1n^2+0.2n^{2\gamma})(s-t)}(f_n(u_n)(s) - f_n(u_{ex})(s))ds\right]^2 \\ := I_1 - I_2. \end{aligned} \tag{3.4}$$

Firstly, we have

$$\begin{aligned} I_1 &= \frac{1}{2}\mathbb{E}\left[e^{(0.1n^2+0.2n^{2\gamma})(T-t)}(g_{n,n} - g_n)\right]^2 \\ &\geq \frac{1}{2n^{\frac{3}{2}}} e^{2(0.1n^2+0.2n^{2\gamma})(T-t)}\mathbb{E}(\xi_n^2) \\ &\geq \frac{1}{2}\frac{e^{2(0.1n^2+0.2n^{2\gamma})(T-t)}}{n^{\frac{3}{2}}}. \end{aligned} \tag{3.5}$$

Secondly, using Hölder's inequality, we get

$$\begin{aligned} \mathbb{E}I_2 &\leq \mathbb{E}\left(\int_t^T 1^2 ds\right) \\ &\quad \times \left(\int_t^T e^{2(0.1n^2+0.2n^{2\gamma})(s-t)}\langle f(.,s,u_n(.,s)) - f(.,s,u_{ex}(.,s)), \phi_n\rangle^2 ds\right) \\ &\leq \mathbb{E}\left(\int_t^T 1^2 ds\right)\left(\int_t^T e^{2(0.1n^2+0.2n^{2\gamma})(s-t)} 0.1^2 \langle u_n(.,s) - u_{ex}(.,s), \phi_p\rangle^2 ds\right) \\ &\leq 0.1^2 T \int_t^T \mathbb{E}\left(\sum_{p=1}^{\infty}\langle u_n(.,s) - u_{ex}(.,s), \phi_p\rangle^2\right) ds \\ &\leq 0.1^2 T \int_t^T \sup_{t\leq s\leq T} \mathbb{E}\left\|u_n(.,s) - u_{ex}(.,s)\right\|^2 ds \\ &\leq 0.1^2 T^2 \sup_{0\leq t\leq T} \mathbb{E}\left\|u_n(.,t) - u_{ex}(.,t)\right\|^2. \end{aligned} \tag{3.6}$$

Combining (3.4), (3.5) and (3.6) yields

$$\sup_{0\leq t\leq T} \mathbb{E}\left\|u_n(.,t) - u_{ex}(.,t)\right\|^2 \geq \frac{1}{2(1+0.1^2T^2)}\frac{e^{2(0.1n^2+0.2n^{2\gamma})T}}{n^{\frac{3}{2}}}. \tag{3.7}$$

From (3.2), we notice that

$$\mathbb{E}\left\|g_n - g\right\|^2 \to 0, \tag{3.8}$$

when $n \to \infty$.

It implies from (3.7) that

$$\underset{0\le t\le T}{sup}\ \mathbb{E}\left\|u_n(.,t) - u_{ex}(.,t)\right\|^2 \to \infty, \tag{3.9}$$

when $n \to \infty$. From (3.8) and (3.9), we deduce that the problem (1.1)-(1.3) fails the stability condition. Hence, the problem (1.1)-(1.3) is ill-posed.

Next we will give a regularization method for the problem (1.1)-(1.3).

4. Regularization and error estimate

Lemma 4.1. *Let $g_\varepsilon \in L^2(D)$. Suppose that $\lim_{\varepsilon\to 0} N(\varepsilon) = +\infty$ and $\lim_{\varepsilon\to 0} \varepsilon^2 N(\varepsilon) = 0$. Put $g_{N(\varepsilon)}$ such that*

$$g_{N(\varepsilon)}(x) = \sum_{n=1}^{N(\varepsilon)} \langle g_\varepsilon, \phi_n\rangle \phi_n(x).$$

Suppose that $g \in H^s(D)$. Then we have the following estimate

$$\mathbb{E}\left\|g_{N(\varepsilon)} - g\right\|^2 \le \varepsilon^2 N(\varepsilon) + \frac{1}{(N(\varepsilon))^{2s}}\left\|g\right\|^2_{H^s(D)} \tag{4.1}$$

for any $s > 0$.

Proof. We have

$$\begin{aligned}\mathbb{E}\left\|g_{N(\varepsilon)} - g\right\|^2 &= \mathbb{E}\left(\sum_{n=1}^{N(\varepsilon)} \langle g_\varepsilon - g, \phi_n\rangle^2\right) + \mathbb{E}\left(\sum_{n>N(\varepsilon)} \langle g, \phi_n\rangle^2\right)\\ &= \varepsilon^2\mathbb{E}\left(\sum_{n=1}^{N(\varepsilon)} \xi_n^2\right) + \sum_{n>N(\varepsilon)} n^{-2s} n^{2s}\langle g, \phi_n\rangle^2.\end{aligned}$$

Then, we get

$$\mathbb{E}\left\|g_{N(\varepsilon)} - g\right\|^2 \le \varepsilon^2 N(\varepsilon) + \frac{1}{(N(\varepsilon))^{2s}}\left\|g\right\|^2_{H^s(D)}.$$

This completes the proof. $\square$

We know that the terms $e^{(\alpha n^2+\beta n^{2\gamma})(T-t)}$ and $e^{(\alpha n^2+\beta n^{2\gamma})(s-t)}$ (n large) are the instability causes. Hence, to obtain the stability of the solution, we apply the Fourier truncation method to cut-off the high frequency term in the solution and establish a regularized solution as follows

$$u^{\varepsilon}_{N(\varepsilon)}(x,t) = \sum_{n=1}^{B_{N(\varepsilon)}} \Big[e^{(\alpha n^2+\beta n^{2\gamma})(T-t)} (g_{N(\varepsilon)})_n - \int_t^T e^{(\alpha n^2+\beta n^{2\gamma})(s-t)} f_n(u^{\varepsilon}_{N(\varepsilon)})(s) ds \Big] \phi_n(x), \tag{4.2}$$

where $B_{N(\varepsilon)}$ satisfies $\lim\limits_{\varepsilon\to 0} B_{N(\varepsilon)} = +\infty$ and will be chosen later.

Theorem 4.2. *Suppose that assumptions (H1)-(H2) are fulfilled. Then the integral equation* (4.2) *has a unique solution* $u^{\varepsilon}_{N(\varepsilon)} \in V_T$.

Proof. For $u \in V_T$, we put

$$G(u)(x,t) = \sum_{n=1}^{B_{N(\varepsilon)}} \Big[e^{(\alpha n^2+\beta n^{2\gamma})(T-t)} (g_{N(\varepsilon)})_n - \int_t^T e^{(\alpha n^2+\beta n^{2\gamma})(s-t)} f_n(u)(s) ds \Big] \sin(nx).$$

We claim that, for every $u, v \in V_T, t \in [0,T], m \geq 1$, we have

$$\begin{aligned} &\mathbb{E} \left\| G^m(u)(.,t) - G^m(v)(.,t) \right\|^2 \\ &\leq K^{2m} \frac{(T-t)^m}{m!} T^m e^{2m(\alpha+\beta)(B_{N(\varepsilon)})^2 T} \sup_{0\leq t\leq T} \mathbb{E} \left\| u(.,t) - v(.,t) \right\|^2 . \end{aligned} \tag{4.3}$$

In the case of $m = 1$, we get

$$\begin{aligned} &\mathbb{E} \left\| G(u)(.,t) - G(v)(.,t) \right\|^2 \\ &= \mathbb{E} \left(\sum_{n=1}^{B_{N(\varepsilon)}} \Big[\int_t^T e^{(\alpha n^2+\beta n^{2\gamma})(s-t)} (f_n(u)(s) - f_n(v)(s)) ds \Big]^2 \right) \\ &\leq (T-t) e^{2(\alpha+\beta)(B_{N(\varepsilon)})^2 T} \int_t^T \mathbb{E} \left(\sum_{n=1}^{\infty} \Big| f_n(u)(s) - f_n(v)(s) \Big|^2 \right) ds \\ &= (T-t) e^{2(\alpha+\beta)(B_{N(\varepsilon)})^2 T} \int_t^T \mathbb{E} \left\| f(.,s,u(.,s)) - f(.,s,v(.,s)) \right\|^2 ds \\ &\leq K^2 (T-t) e^{2(\alpha+\beta)(B_{N(\varepsilon)})^2 T} \int_t^T \underset{t\leq s\leq T}{sup}\, \mathbb{E} \left\| u(.,s) - v(.,s) \right\|^2 ds \end{aligned}$$

$$\leq K^2T(T-t)e^{2(\alpha+\beta)(B_{N(\varepsilon)})^2T} \underset{0\leq t\leq T}{sup} \mathbb{E}\left\|u(.,t)-v(.,t)\right\|^2.$$

Thus, (4.3) holds for $m = 1$. Suppose that (4.3) holds for $m = k$, we shall prove (4.3) holds for $m = k+1$. In fact, we have

$$\begin{aligned}
&\mathbb{E}\left\|G^{k+1}(u)(.,t)-G^{k+1}(v)(.,t)\right\|^2 \\
&\quad = \mathbb{E}\left(\sum_{n=1}^{B_{N(\varepsilon)}}\Big[\int_t^T e^{(\alpha n^2+\beta n^{2\gamma})(s-t)}(f_n(G^k(u))(s)-f_n(G^k(v))(s))ds\Big]^2\right) \\
&\quad \leq (T-t)e^{2(\alpha+\beta)(B_{N(\varepsilon)})^2T}\int_t^T \mathbb{E}\left(\sum_{n=1}^{\infty}\left|f_n(G^k(u))(s)-f_n(G^k(v))(s)\right|^2\right)ds \\
&\quad = (T-t)e^{2(\alpha+\beta)(B_{N(\varepsilon)})^2T}\int_t^T \mathbb{E}\left\|f(.,s,G^k(u))(.,s)-f(.,s,G^k(v))(.,s)\right\|^2 ds \\
&\quad \leq K^2(T-t)e^{2(\alpha+\beta)(B_{N(\varepsilon)})^2T}\int_t^T \mathbb{E}\left\|G^k(u)(.,s)-G^k(v)(.,s)\right\|^2 ds.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\mathbb{E}\left\|G^{k+1}(u)(.,t)-G^{k+1}(v)(.,t)\right\|^2 \\
&\quad \leq K^2(T-t)e^{2(\alpha+\beta)(B_{N(\varepsilon)})^2T} \\
&\quad\quad \times \int_t^T K^{2k}\frac{(T-s)^k}{k!}T^k e^{2k(\alpha+\beta)(B_{N(\varepsilon)})^2T}\sup_{t\leq s\leq T}\mathbb{E}\left\|u(.,s)-v(.,s)\right\|^2 ds \\
&\quad \leq K^{2(k+1)}(T-t)T^k e^{2(k+1)(\alpha+\beta)(B_{N(\varepsilon)})^2T}\sup_{0\leq t\leq T}\mathbb{E}\left\|u(.,t)-v(.,t)\right\|^2 \\
&\quad\quad \times \int_t^T \frac{(T-s)^k}{k!}ds \\
&\quad \leq K^{2(k+1)}\frac{(T-t)^{k+1}}{(k+1)!}T^{k+1}e^{2(k+1)(\alpha+\beta)(B_{N(\varepsilon)})^2T}\sup_{0\leq t\leq T}\mathbb{E}\left\|u(.,t)-v(.,t)\right\|^2.
\end{aligned}$$

Thus, by the induction principle, we get

$$\begin{aligned}
&\sqrt{\mathbb{E}\left\|G^m(u)(.,t)-G^m(v)(.,t)\right\|^2} \\
&\quad \leq K^m\frac{(T-t)^{m/2}}{(m!)^{1/2}}T^{m/2}e^{m(\alpha+\beta)(B_{N(\varepsilon)})^2T}\sup_{0\leq t\leq T}\sqrt{\mathbb{E}\left\|u(.,t)-v(.,t)\right\|^2}.
\end{aligned}$$

Then we obtain

$$\left\|G^m(u)-G^m(v)\right\|_{V_T} \leq K^m\frac{T^m}{(m!)^{1/2}}e^{m(\alpha+\beta)(B_{N(\varepsilon)})^2T}\left\|u-v\right\|_{V_T}.$$

Since $K^m \frac{T^m}{(m!)^{1/2}} e^{m(\alpha+\beta)(B_{N(\varepsilon)})^2 T} \to 0$ when $m \to \infty$, there exists a positive number m_0 such that

$$K^{m_0} \frac{T^{m_0}}{(m_0!)^{1/2}} e^{m_0(\alpha+\beta)(B_{N(\varepsilon)})^2 T} < 1$$

and G^{m_0} is a contraction. It follows that the equation $G^{m_0}(u) = u$ has a unique solution $u^\varepsilon_{N(\varepsilon)} \in V_T$. In fact, we have $G(G^{m_0}(u^\varepsilon_{N(\varepsilon)}) = G(u^\varepsilon_{N(\varepsilon)})$. Thus $G^{m_0}(G(u^\varepsilon_{N(\varepsilon)}) = G(u^\varepsilon_{N(\varepsilon)})$.

By the uniqueness of the fixed point of G^{m_0}, we have $G(u^\varepsilon_{N(\varepsilon)}) = u^\varepsilon_{N(\varepsilon)}$, so the equation $G(u) = u$ has a unique solution $u^\varepsilon_{N(\varepsilon)} \in V_T$. This completes the proof. $\square$

Next we will give the expectation of the error estimate between the regularized solution and the exact solution under different conditions.

Theorem 4.3. *Suppose that assumptions (H1)-(H2) are fulfilled. Suppose there exists $M_1 > 0$ such that $\|g\|_{H^s(D)} \le M_1$. Let u be the exact solution of the problem* (1.1)-(1.3) *and $u^\varepsilon_{N(\varepsilon)}$ be the regularized solution corresponding to the random data $g_{N(\varepsilon)}$. Suppose there exist $q > 0$ and $Q_1 > 0$ such that*

$$\sum_{n=1}^{\infty} n^{2q} e^{2tn^2} |u_n(t)|^2 \le Q_1, \quad \forall t \in [0, T). \tag{4.4}$$

Then the following estimate holds

$$\mathbb{E}\left\|u^\varepsilon_{N(\varepsilon)}(.,t) - u(.,t)\right\|^2 \le M_2 e^{2K^2T(T-t)} \left(\varepsilon^{\frac{2s(T+t)}{(2s+1)T}} + \left(\ln\left(\frac{1}{\varepsilon}\right)\right)^{-q} \varepsilon^{\frac{2st}{(2s+1)(\alpha+\beta)T}} \right), \tag{4.5}$$

where $t \in [0, T]$ and $M_2 = 4\max\left\{1 + M_1^2, Q_1\left(\frac{s}{(2s+1)(\alpha+\beta)T}\right)^{-q}\right\}$.

Proof. We put

$$v^\varepsilon_{N(\varepsilon)}(x,t) = \sum_{n=1}^{B_{N(\varepsilon)}} \left[e^{(\alpha n^2+\beta n^{2\gamma})(T-t)} g_n - \int_t^T e^{(\alpha n^2+\beta n^{2\gamma})(s-t)} f_n(v^\varepsilon_{N(\varepsilon)})(s) ds \right] \phi_n(x).$$

Firstly, we estimate $\mathbb{E}\left\|u^\varepsilon_{N(\varepsilon)}(.,t) - v^\varepsilon_{N(\varepsilon)}(.,t)\right\|^2$. Using the inequality

$$(a+b)^2 \le 2(a^2+b^2),$$

we get

$$\mathbb{E}\left\|u^\varepsilon_{N(\varepsilon)}(.,t) - v^\varepsilon_{N(\varepsilon)}(.,t)\right\|^2$$

$$\leq 2\mathbb{E}\Big(\sum_{n=1}^{B_{N(\varepsilon)}}\Big|e^{(\alpha n^2+\beta n^{2\gamma})(T-t)}((g_{N(\varepsilon)})_n-g_n)\Big|^2\Big)$$
$$+2\mathbb{E}\Big(\sum_{n=1}^{B_{N(\varepsilon)}}\Big|\int_t^T e^{(\alpha n^2+\beta n^{2\gamma})(s-t)}(f_n(u^{\varepsilon}_{N(\varepsilon)})(s)-f_n(v^{\varepsilon}_{N(\varepsilon)})(s))ds\Big|^2\Big).$$

Using Hölder's inequality, we get

$$\mathbb{E}\left\|u^{\varepsilon}_{N(\varepsilon)}(.,t)-v^{\varepsilon}_{N(\varepsilon)}(.,t)\right\|^2$$
$$\leq 2e^{2(\alpha+\beta)(B_{N(\varepsilon)})^2(T-t)}\mathbb{E}\Big(\sum_{n=1}^{\infty}\Big|(g_{N(\varepsilon)})_n-g_n\Big|^2\Big)$$
$$+2(T-t)e^{-2(\alpha+\beta)(B_{N(\varepsilon)})^2t}$$
$$\times\Big(\int_t^T e^{2(\alpha+\beta)(B_{N(\varepsilon)})^2s}\mathbb{E}\sum_{n=1}^{\infty}\Big|f_n(u^{\varepsilon}_{N(\varepsilon)})(s)-f_n(v^{\varepsilon}_{N(\varepsilon)})(s)\Big|^2ds\Big)$$
$$\leq 2e^{2(\alpha+\beta)(B_{N(\varepsilon)})^2(T-t)}\mathbb{E}\left\|g_{N(\varepsilon)}-g\right\|^2$$
$$+2Te^{-2(\alpha+\beta)(B_{N(\varepsilon)})^2t}$$
$$\times\int_t^T e^{2(\alpha+\beta)(B_{N(\varepsilon)})^2s}\mathbb{E}\left\|f(.,s,u^{\varepsilon}_{N(\varepsilon)}(.,s))-f(.,s,v^{\varepsilon}_{N(\varepsilon)}(.,s))\right\|^2ds.$$

Then we obtain

$$e^{2(\alpha+\beta)(B_{N(\varepsilon)})^2t}\mathbb{E}\left\|u^{\varepsilon}_{N(\varepsilon)}(.,t)-v^{\varepsilon}_{N(\varepsilon)}(.,t)\right\|^2$$
$$\leq 2e^{2(\alpha+\beta)(B_{N(\varepsilon)})^2T}\mathbb{E}\left\|g_{N(\varepsilon)}-g\right\|^2$$
$$+2K^2T\int_t^T e^{2(\alpha+\beta)(B_{N(\varepsilon)})^2s}\mathbb{E}\left\|u^{\varepsilon}_{N(\varepsilon)}(.,s)-v^{\varepsilon}_{N(\varepsilon)}(.,s)\right\|^2ds.$$

Applying Gronwall's inequality, we get

$$e^{2(\alpha+\beta)(B_{N(\varepsilon)})^2t}\mathbb{E}\left\|u^{\varepsilon}_{N(\varepsilon)}(.,t)-v^{\varepsilon}_{N(\varepsilon)}(.,t)\right\|^2$$
$$\leq 2e^{2(\alpha+\beta)(B_{N(\varepsilon)})^2T}e^{2K^2T(T-t)}\mathbb{E}\left\|g_{N(\varepsilon)}-g\right\|^2.$$

Therefore, we have

$$\mathbb{E}\left\|u^{\varepsilon}_{N(\varepsilon)}(.,t)-v^{\varepsilon}_{N(\varepsilon)}(.,t)\right\|^2$$
$$\leq 2e^{2(\alpha+\beta)(B_{N(\varepsilon)})^2(T-t)}e^{2K^2T(T-t)}\mathbb{E}\left\|g_{N(\varepsilon)}-g\right\|^2.$$

From Lemma 4.1, we obtain

$$\mathbb{E}\left\|u^{\varepsilon}_{N(\varepsilon)}(.,t)-v^{\varepsilon}_{N(\varepsilon)}(.,t)\right\|^2$$

$$\leq 2e^{2(\alpha+\beta)(B_{N(\varepsilon)})^2(T-t)}e^{2K^2T(T-t)}\Big(\varepsilon^2 N(\varepsilon) + \frac{1}{(N(\varepsilon))^{2s}}\left\|g\right\|^2_{H^s(D)}\Big)$$

$$\leq 2e^{2(\alpha+\beta)(B_{N(\varepsilon)})^2(T-t)}e^{2K^2T(T-t)}\Big(\varepsilon^2 N(\varepsilon) + \frac{M_1^2}{(N(\varepsilon))^{2s}}\Big). \tag{4.6}$$

Now we estimate $\left\|u(.,t) - v^{\varepsilon}_{N(\varepsilon)}(.,t)\right\|^2$.
Using the inequality $(a+b)^2 \leq 2(a^2+b^2)$, we obtain

$$\begin{aligned}
&\left\|u(.,t) - v^{\varepsilon}_{N(\varepsilon)}(.,t)\right\|^2 \\
&\leq 2\sum_{n > B_{N(\varepsilon)}} n^{-2q}e^{-2tn^2}n^{2q}e^{2tn^2} \\
&\quad \times \Big[e^{(\alpha n^2+\beta n^{2\gamma})(T-t)}g_n - \int_t^T e^{(\alpha n^2+\beta n^{2\gamma})(s-t)}f_n(u)(s)ds\Big]^2 \\
&\quad + 2\sum_{n=1}^{B_{N(\varepsilon)}}\Big[\int_t^T e^{(\alpha n^2+\beta n^{2\gamma})(s-t)}\Big(f_n(v^{\varepsilon}_{N(\varepsilon)})(s) - f_n(u)(s)\Big)ds\Big]^2.
\end{aligned}$$

Using Hölder's inequality, we get

$$\begin{aligned}
&\left\|u(.,t) - v^{\varepsilon}_{N(\varepsilon)}(.,t)\right\|^2 \\
&\leq 2(B_{N(\varepsilon)})^{-2q}e^{-2t(B_{N(\varepsilon)})^2}\sum_{n=1}^{\infty} n^{2\gamma}e^{2tn^2} \\
&\quad \times \Big[e^{(\alpha n^2+\beta n^{2\gamma})(T-t)}g_n - \int_t^T e^{(\alpha n^2+\beta n^{2q})(s-t)}f_n(u)(s)ds\Big]^2 \\
&\quad + 2(T-t)\int_t^T\sum_{n=1}^{\infty} e^{2(\alpha+\beta)(B_{N(\varepsilon)})^2(s-t)}\Big(f_n(v^{\varepsilon}_{N(\varepsilon)})(s) - f_n(u)(s)\Big)^2 ds \\
&\leq 2(B_{N(\varepsilon)})^{-2q}e^{-2t(B_{N(\varepsilon)})^2}Q_1 \\
&\quad + 2(T-t)\int_t^T e^{2(\alpha+\beta)(B_{N(\varepsilon)})^2(s-t)}\left\|f(v^{\varepsilon}_{N(\varepsilon)})(s) - f(u)(s)\right\|^2 ds \\
&\leq 2(B_{N(\varepsilon)})^{-2q}e^{-2t(B_{N(\varepsilon)})^2}Q_1 \\
&\quad + 2K^2Te^{-2(\alpha+\beta)(B_{N(\varepsilon)})^2t}\int_t^T e^{2(\alpha+\beta)(B_{N(\varepsilon)})^2s}\left\|v^{\varepsilon}_{N(\varepsilon)})(.,s) - u(.,s)\right\|^2 ds.
\end{aligned}$$

Then we get

$$\begin{aligned}
&e^{2(\alpha+\beta)(B_{N(\varepsilon)})^2t}\left\|u(.,t) - v^{\varepsilon}_{N(\varepsilon)}(.,t)\right\|^2 \\
&\quad \leq 2(B_{N(\varepsilon)})^{-2q}e^{-2t(B_{N(\varepsilon)})^2}Q_1e^{2(\alpha+\beta)(B_{N(\varepsilon)})^2t}
\end{aligned}$$

$$+ 2K^2T \int_t^T e^{2(\alpha+\beta)(B_{N(\varepsilon)})^2 s} \left\| u(.,s) - v^{\varepsilon}_{N(\varepsilon)})(.,s) \right\|^2 ds.$$

Using Gronwall's inequality, we obtain

$$\begin{aligned} & e^{2(\alpha+\beta)(B_{N(\varepsilon)})^2 t} \left\| u(.,t) - v^{\varepsilon}_{N(\varepsilon)}(.,t) \right\|^2 \\ & \quad \leq 2Q_1 (B_{N(\varepsilon)})^{-2q} e^{-2t(B_{N(\varepsilon)})^2} e^{2(\alpha+\beta)(B_{N(\varepsilon)})^2 t} e^{2K^2T(T-t)}. \end{aligned}$$

It implies that

$$\left\| u(.,t) - v^{\varepsilon}_{N(\varepsilon)}(.,t) \right\|^2 \leq 2Q_1 (B_{N(\varepsilon)})^{-2q} e^{-2t(B_{N(\varepsilon)})^2} e^{2K^2T(T-t)}. \tag{4.7}$$

Combining (4.6) and (4.7) gives

$$\begin{aligned} & \mathbb{E} \left\| u^{\varepsilon}_{N(\varepsilon)}(.,t) - u(.,t) \right\|^2 \\ & \leq 2\mathbb{E} \left\| u^{\varepsilon}_{N(\varepsilon)}(.,t) - v^{\varepsilon}_{N(\varepsilon)}(.,t) \right\|^2 + 2 \left\| u(.,t) - v^{\varepsilon}_{N(\varepsilon)}(.,t) \right\|^2 \\ & \leq 4e^{2K^2T(T-t)} \\ & \quad \times \left[e^{2(\alpha+\beta)(B_{N(\varepsilon)})^2(T-t)} \left(\varepsilon^2 N(\varepsilon) + \frac{M_1^2}{(N(\varepsilon))^{2s}} \right) + Q_1 (B_{N(\varepsilon)})^{-2q} e^{-2t(B_{N(\varepsilon)})^2} \right]. \end{aligned} \tag{4.8}$$

We choose $N(\varepsilon) = \varepsilon^{-\frac{2}{2s+1}}$ and $B_{N(\varepsilon)} = \left(\frac{s}{(2s+1)(\alpha+\beta)T} \ln(\frac{1}{\varepsilon}) \right)^{\frac{1}{2}}$. Then we have

$$\begin{aligned} & \mathbb{E} \left\| u^{\varepsilon}_{N(\varepsilon)}(.,t) - u(.,t) \right\|^2 \\ & \quad \leq 4e^{2K^2T(T-t)} \\ & \quad \times \left[(1+M_1^2) \varepsilon^{\frac{2s(T+t)}{(2s+1)T}} + Q_1 \left(\frac{s}{(2s+1)(\alpha+\beta)T} \ln\left(\frac{1}{\varepsilon}\right) \right)^{-q} \varepsilon^{\frac{2st}{(2s+1)(\alpha+\beta)T}} \right]. \end{aligned}$$

Putting $M_2 = 4 \max \left\{ 1 + M_1^2,\ Q_1 \left(\frac{s}{(2s+1)(\alpha+\beta)T} \right)^{-q} \right\}$, we get the estimate

$$\begin{aligned} & \mathbb{E} \left\| u^{\varepsilon}_{N(\varepsilon)}(.,t) - u(.,t) \right\|^2 \\ & \quad \leq M_2 e^{2K^2T(T-t)} \left(\varepsilon^{\frac{2s(T+t)}{(2s+1)T}} + \left(\ln\left(\frac{1}{\varepsilon}\right) \right)^{-q} \varepsilon^{\frac{2st}{(2s+1)(\alpha+\beta)T}} \right). \end{aligned}$$

This completes the proof. $\square$

Remark 4.4. The error estimate derived from (4.5) at the initial time $t = 0$ is given by

$$\mathbb{E}\left\|u^{\varepsilon}_{N(\varepsilon)}(.,0) - u(.,0)\right\|^2 \leq M_2 e^{2K^2T^2}\left(\varepsilon^{\frac{2s}{2s+1}} + \left(\ln\left(\frac{1}{\varepsilon}\right)\right)^{-q}\right).$$

This error estimate exhibits a logarithmic-type convergence for all $t \in [0,T]$ due to the insufficiently strong condition on the exact solution. Consequently, in the subsequent theorem, with a stronger condition on the exact solution, we aim to establish a superior error estimate between the regularized solution and the exact solution for all $t \in [0,T]$.

Theorem 4.5. *Suppose that assumptions (H1)-(H2) are fulfilled and g be as in Theorem* 4.2. *Let u be the exact solution of the problem* (1.1)-(1.3) *and $u^{\varepsilon}_{N(\varepsilon)}$ be the regularized solution corresponding to the random data $g_{N(\varepsilon)}$. Suppose there exist $r > 0$ and $Q_2 > 0$ such that*

$$\sum_{n=1}^{\infty} e^{2rn^{2\alpha}}|u_n(t)|^2 \leq Q_2, \quad \forall t \in [0,T]. \tag{4.9}$$

Then the following estimate holds

$$\mathbb{E}\left\|u^{\varepsilon}_{N(\varepsilon)}(.,t) - u(.,t)\right\|^2 \leq M_3 e^{2K^2T(T-t)}\left(\varepsilon^{\frac{2s(T+t)}{(2s+1)T}} + \varepsilon^{\frac{2sr}{(2s+1)(\alpha+\beta)T}}\right), \tag{4.10}$$

where $M_3 = 4\max\{1 + M_1^2, Q_2\}$.

Proof. Now we estimate $\left\|u(.,t) - v^{\varepsilon}_{N(\varepsilon)}(.,t)\right\|^2$. Using the inequality $(a+b)^2 \leq 2(a^2+b^2)$, we obtain

$$\begin{aligned}
&\left\|u(.,t) - v^{\varepsilon}_{N(\varepsilon)}(.,t)\right\|^2 \\
&\quad\leq 2\sum_{n > B_{N(\varepsilon)}} e^{-2rn^2}e^{2rn^2}\left[e^{(\alpha n^2+\beta n^{2\gamma})(T-t)}g_n - \int_t^T e^{(\alpha n^2+\beta n^{2\gamma})(s-t)}f_n(u)(s)ds\right]^2 \\
&\quad\quad + 2\sum_{n=1}^{B_{N(\varepsilon)}}\Big[\int_t^T e^{(\alpha n^2+\beta n^{2\gamma})(s-t)}\Big(f_n(v^{\varepsilon}_{N(\varepsilon)})(s) - f_n(u)(s)\Big)ds\Big]^2.
\end{aligned}$$

Using Hölder's inequality, we get

$$\begin{aligned}
&\left\|u(.,t) - v^{\varepsilon}_{N(\varepsilon)}(.,t)\right\|^2 \\
&\quad\leq 2e^{-2r(B_{N(\varepsilon)})^2}\sum_{n=1}^{\infty} e^{2rn^2}
\end{aligned}$$

$$\times\left[e^{(\alpha n^2+\beta n^{2\gamma})(T-t)}g_n-\int_t^T e^{(\alpha n^2+\beta n^{2\gamma})(s-t)}f_n(u)(s)ds\right]^2$$
$$+2(T-t)\int_t^T\sum_{n=1}^{\infty}e^{2(\alpha+\beta)(B_{N(\varepsilon)})^2(s-t)}\Big(f_n(v^{\varepsilon}_{N(\varepsilon)})(s)-f_n(u)(s)\Big)^2ds$$
$$\le 2e^{-2r(B_{N(\varepsilon)})^2}Q_2$$
$$+2(T-t)\int_t^T e^{2(\alpha+\beta)(B_{N(\varepsilon)})^2(s-t)}\left\|f(v^{\varepsilon}_{N(\varepsilon)})(s)-f(u)(s)\right\|^2ds$$
$$\le 2e^{-2r(B_{N(\varepsilon)})^2}Q_2$$
$$+2K^2Te^{-2(\alpha+\beta)(B_{N(\varepsilon)})^2t}\int_t^T e^{2(\alpha+\beta)(B_{N(\varepsilon)})^2s}\left\|v^{\varepsilon}_{N(\varepsilon)})(.,s)-u(.,s)\right\|^2ds.$$

Then we get

$$e^{2(\alpha+\beta)(B_{N(\varepsilon)})^2t}\left\|u(.,t)-v^{\varepsilon}_{N(\varepsilon)}(.,t)\right\|^2$$
$$\le 2e^{-2r(B_{N(\varepsilon)})^2}Q_2e^{2(\alpha+\beta)(B_{N(\varepsilon)})^2t}$$
$$+2K^2T\int_t^T e^{2(\alpha+\beta)(B_{N(\varepsilon)})^2s}\left\|u(.,s)-v^{\varepsilon}_{N(\varepsilon)}(.,s)\right\|^2ds.$$

Using Gronwall's inequality, we obtain

$$e^{2(\alpha+\beta)(B_{N(\varepsilon)})^2t}\left\|u(.,t)-v^{\varepsilon}_{N(\varepsilon)}(.,t)\right\|^2\le 2Q_2e^{-2r(B_{N(\varepsilon)})^2}e^{2(\alpha+\beta)(B_{N(\varepsilon)})^2t}e^{2K^2T(T-t)}.$$

It implies that

$$\left\|u(.,t)-v^{\varepsilon}_{N(\varepsilon)}(.,t)\right\|^2\le 2Q_2e^{-2r(B_{N(\varepsilon)})^2}e^{2K^2T(T-t)}. \tag{4.11}$$

Combining (4.6) and (4.11) gives

$$\mathbb{E}\left\|u^{\varepsilon}_{N(\varepsilon)}(.,t)-u(.,t)\right\|^2$$
$$\le 2\mathbb{E}\left\|u^{\varepsilon}_{N(\varepsilon)}(.,t)-v^{\varepsilon}_{N(\varepsilon)}(.,t)\right\|^2+2\left\|u(.,t)-v^{\varepsilon}_{N(\varepsilon)}(.,t)\right\|^2$$
$$\le 4e^{2K^2T(T-t)}\left[e^{2(\alpha+\beta)(B_{N(\varepsilon)})^2(T-t)}\Big(\varepsilon^2N(\varepsilon)+\frac{M_1^2}{(N(\varepsilon))^{2s}}\Big)+Q_2e^{-2r(B_{N(\varepsilon)})^2}\right]. \tag{4.12}$$

We choose $N(\varepsilon)=\varepsilon^{-\frac{2}{2s+1}}$ and $B_{N(\varepsilon)}=\left(\frac{s}{(2s+1)(\alpha+\beta)T}\ln(\frac{1}{\varepsilon})\right)^{\frac{1}{2}}$. Then we have

$$\mathbb{E}\left\|u^{\varepsilon}_{N(\varepsilon)}(.,t)-u(.,t)\right\|^2\le 4e^{2K^2T(T-t)}\left[(1+M_1^2)\varepsilon^{\frac{2s(T+t)}{(2s+1)T}}+Q_2\varepsilon^{\frac{2sr}{(2s+1)(\alpha+\beta)T}}\right].$$

Putting $M_3 = 4\max\{1+M_1^2, Q_2\}$, we obtain

$$\mathbb{E}\left\|u^{\varepsilon}_{N(\varepsilon)}(.,t)-u(.,t)\right\|^2 \le M_3 e^{2K^2T(T-t)}\left(\varepsilon^{\frac{2s(T+t)}{(2s+1)T}}+\varepsilon^{\frac{2sr}{(2s+1)(\alpha+\beta)T}}\right).$$

This completes the proof. □

Remark 4.6. (1) We notice that the error estimate provided in (4.10) demonstrates Hölder-type convergence for all $t\in[0,T]$. This convergence rate is better than the logarithmic-type rate described in Theorem 4.2. However, it's important to note that the error estimate (4.11) requires a strong condition on the exact solution $u(x,t)$, which can be considered a disadvantage.

(2) In the next theorem, we will give an error estimate of the expectation between the regularized solution and the exact solution in $C([0,T];H^q(D))$.

Theorem 4.7. *Suppose that assumptions (H1)-(H2) are fulfilled and g is as in Theorem* 4.2. *Let u be the exact solution of the problem* (1.1)-(1.3) *and $u^{\varepsilon}_{N(\varepsilon)}$ be the regularized solution corresponding to the random data $g_{N(\varepsilon)}$. If the condition* (4.9) *holds then we get the error estimate*

$$\begin{aligned}\mathbb{E}&\left\|u^{\varepsilon}_{N(\varepsilon)}(.,t)-u(.,t)\right\|^2_{H^q(D)}\\ &\le M_4\left(\ln\left(\frac{1}{\varepsilon}\right)\right)^q\left[e^{2K^2T(T-t)}\left(\varepsilon^{\frac{2s(T+t)}{(2s+1)T}}+\varepsilon^{\frac{2sr}{(2s+1)(\alpha+\beta)T}}\right)+\varepsilon^{\frac{2sr}{(2s+1)(\alpha+\beta)T}}\right]\end{aligned} \tag{4.13}$$

for $t\in[0,T]$, where $M_4 = 4\left(\frac{s}{(2s+1)(\alpha+\beta)T}\right)^q \max\{1+M_1^2,Q_2\}$.

Proof. We put

$$w^{\varepsilon}_{N(\varepsilon)}(x,t)=\sum_{n=1}^{B_{N(\varepsilon)}}\left[e^{(\alpha n^2+\beta n^{2\gamma})(T-t)}g_n-\int_t^T e^{(\alpha n^2+\beta n^{2\gamma})(s-t)}f_n(u)(s)ds\right]\phi_n(x).$$

We deduce that

$$\begin{aligned}\mathbb{E}||u^{\varepsilon}_{N(\varepsilon)}(.,t)-w^{\varepsilon}_{N(\varepsilon)}(.,t)||^2_{H^q(D)} &= \mathbb{E}\left(\sum_{n=1}^{B_{N(\varepsilon)}} n^{2q}\left\langle u^{\varepsilon}_{N(\varepsilon)}(.,t)-u(.,t),\phi_n\right\rangle^2\right)\\ &\le \left|B_{N(\varepsilon)}\right|^{2q}\mathbb{E}\left(\sum_{n=1}^{B_{N(\varepsilon)}}\left\langle u^{\varepsilon}_{N(\varepsilon)}(.,t)-u(.,t),\phi_n\right\rangle^2\right)\\ &\le \left|B_{N(\varepsilon)}\right|^{2q}\mathbb{E}||u^{\varepsilon}_{N(\varepsilon)}(.,t)-u(.,t)||^2.\end{aligned} \tag{4.14}$$

It follows from (4.12) and (4.14) that

$$\begin{aligned}
&\mathbb{E}\left\|u^{\varepsilon}_{N(\varepsilon)}(\cdot,t)-w^{\varepsilon}_{N(\varepsilon)}(\cdot,t)\right\|^2_{H^q(D)}\\
&\quad\le 2\,|B_{N(\varepsilon)}|^{2q}\,e^{2K^2T(T-t)}\\
&\qquad\times\left[e^{2(\alpha+\beta)(B_{N(\varepsilon)})^2(T-t)}\left(\varepsilon^2N(\varepsilon)+\frac{M_1^2}{(N(\varepsilon))^{2s}}\right)+Q_2e^{-2r(B_{N(\varepsilon)})^2}\right]. \qquad (4.15)
\end{aligned}$$

We consider the function

$$G(\omega)=\omega^{2q}e^{-H\omega^2},\quad H>0,\ \ \alpha>0. \qquad (4.16)$$

From the derivative of G is $G'(\omega)=2\omega^{2q-1}e^{-H\omega^2}(q-H\omega^2)$, we know that G is decreasing when $H\omega^2>q$. Since $\lim\limits_{\varepsilon\to 0}B_{N(\varepsilon)}=+\infty$, we see that if ε small enough then $2r\big(B_{N(\varepsilon)}\big)^2>q$. Replacing $H=2r$, $\omega=B_{N(\varepsilon)}$ into (4.16), we obtain for $n>B_{N(\varepsilon)}$

$$G(n)=n^{2q}e^{-2rn^2}\ \ \le\ \ G\big(B_{N(\varepsilon)}\big)=|B_{N(\varepsilon)}|^{2q}e^{-2r|B_{N(\varepsilon)}|^2}.$$

It implies that

$$\begin{aligned}
\left\|u(\cdot,t)-w^{\varepsilon}_{N(\varepsilon)}(\cdot,t)\right\|^2_{H^q(D)}&=\sum_{n>B_{N(\varepsilon)}}n^{2q}\left\langle u(\cdot,t),\phi_n\right\rangle^2\\
&=\sum_{n>B_{N(\varepsilon)}}G(n)\,e^{2rn^2}\left\langle u(\cdot,t),\phi_n\right\rangle^2\\
&\le G\big(B_{N(\varepsilon)}\big)\sum_{n>B_{N(\varepsilon)}}e^{2rn^2}\left\langle u(\cdot,t),\phi_n\right\rangle^2\\
&\le Q_2\,|B_{N(\varepsilon)}|^{2q}\,e^{-2r|B_{N(\varepsilon)}|^2}. \qquad (4.17)
\end{aligned}$$

Combining (4.15) and (4.17) gives

$$\begin{aligned}
&\mathbb{E}\left\|u^{\varepsilon}_{N(\varepsilon)}(\cdot,t)-u(\cdot,t)\right\|^2_{H^q(D)}\\
&\quad\le 2\,\mathbb{E}\left\|u^{\varepsilon}_{N(\varepsilon)}(\cdot,t)-w^{\varepsilon}_{N(\varepsilon)}(\cdot,t)\right\|^2_{H^q(D)}+2\left\|u(\cdot,t)-w^{\varepsilon}_{N(\varepsilon)}(\cdot,t)\right\|^2_{H^q(D)}\\
&\quad\le 4\,|B_{N(\varepsilon)}|^{2q}e^{2K^2T(T-t)}\\
&\qquad\times\left[e^{2(\alpha+\beta)(B_{N(\varepsilon)})^2(T-t)}\left(\varepsilon^2N(\varepsilon)+\frac{M_1^2}{(N(\varepsilon))^{2s}}\right)+Q_2e^{-2r(B_{N(\varepsilon)})^2}\right]\\
&\qquad+4Q_2\,|B_{N(\varepsilon)}|^{2q}\,e^{-2r|B_{N(\varepsilon)}|^2}.
\end{aligned}$$

We choose $N(\varepsilon)=\varepsilon^{-\frac{2}{2s+1}}$ and $B_{N(\varepsilon)}=\left(\frac{s}{(2s+1)(\alpha+\beta)T}\ln(\frac{1}{\varepsilon})\right)^{\frac{1}{2}}$. Then we have

$$
\begin{aligned}
&\mathbb{E}\left\|u^{\varepsilon}_{N(\varepsilon)}(.,t)-u(.,t)\right\|^2_{H^q(D)}\\
&\le 4e^{2K^2T(T-t)}\left(\frac{s}{(2s+1)(\alpha+\beta)T}\ln\left(\frac{1}{\varepsilon}\right)\right)^q\\
&\quad\times\left[(1+M_1^2)\varepsilon^{\frac{2s(T+t)}{(2s+1)T}}+Q_2\varepsilon^{\frac{2sr}{(2s+1)(\alpha+\beta)T}}\right]\\
&\quad+4Q_2\left(\frac{s}{(2s+1)(\alpha+\beta)T}\ln\left(\frac{1}{\varepsilon}\right)\right)^q\varepsilon^{\frac{2sr}{(2s+1)(\alpha+\beta)T}}.
\end{aligned}
$$

Putting $M_4=4\left(\frac{s}{(2s+1)(\alpha+\beta)T}\right)^q\max\{1+M_1^2,Q_2\}$, we obtain

$$
\begin{aligned}
&\mathbb{E}\left\|u^{\varepsilon}_{N(\varepsilon)}(.,t)-u(.,t)\right\|^2_{H^q(D)}\\
&\le M_4\left(\ln\left(\frac{1}{\varepsilon}\right)\right)^q\left[e^{2K^2T(T-t)}\left(\varepsilon^{\frac{2s(T+t)}{(2s+1)T}}+\varepsilon^{\frac{2sr}{(2s+1)(\alpha+\beta)T}}\right)+\varepsilon^{\frac{2sr}{(2s+1)(\alpha+\beta)T}}\right],
\end{aligned}
$$

which completes the proof. □

Remark 4.8. In physics and engineering, the estimation on a Hilbert scale space, for instance $H^q(D)$ is significant. Moreover, getting the error estimate in $H^q(D)$ is more difficult than in $L^2(D)$. So, the result in above theorem is new and remarkable.

5. Numerical example

In this section, we construct an illustrate example for our regularization method. We consider the following problem

$$
\begin{cases}
u_t(x,t)-0.1\Delta u(x,t)+0.2(-\Delta)^{\gamma}u(x,t)=f(x,t,u(x,t)),\ (x,t)\in(0,\pi)\times[0,1],\\
u(0,t)=u(\pi,t)=0\,,\ t\in[0,1],\\
u(x,1)=g(x),\ x\in(0,\pi),
\end{cases}
\tag{5.1}
$$

where $\gamma=0.7$ and

$$
\begin{aligned}
f(x,t,u(x,t))&=-8\gamma t^{8\gamma-1}\sin(t^{8\gamma})\sin(x)+0.2\cos(t^{8\gamma})\sin(x)+0.1u(x,t),\\
g(x)&=\cos(1)\sin(x).
\end{aligned}
$$

The exact solution of the problem (5.1) is

$$u_{exact}(x,t)=\cos(t^{8\gamma})\sin(x).$$

We get the regularization parameters

$$N = [N(\varepsilon)] = [\varepsilon^{\frac{-2}{3}}] \text{ and } B_N = [B_{N(\varepsilon)}] = \left[\left(\tfrac{10}{9} ln\left(\tfrac{1}{\varepsilon}\right)\right)\right]^{\frac{1}{2}}.$$

Consider the random data

$$g_N(x) = \cos(1)\sin(x) + \varepsilon \sum_{n=1}^{N} \langle \xi, \phi_n \rangle \phi_n(x),$$

where $\phi_n(x) = \sqrt{\dfrac{2}{\pi}} \sin(nx)$ and $\langle \xi, \phi_n \rangle$ are Gaussian random variables with mean 0 and variance 1.

In MATLAB, when generating random numbers from a normal distribution, the function commonly employed is $randn$.

From (4.2), we get the regularized solution at the point (x, t)

$$u_N^{\varepsilon}(x,t) = \sum_{n=1}^{B_N} \Big[e^{(0.1n^2+0.2n^{2\gamma})(1-t)} (g_N)_n - \int_t^1 e^{(0.1n^2+0.2n^{2\gamma})(s-t)} f_n(u_N^{\varepsilon})(s) ds \Big] \phi_n(x),$$

where

$$(g_N)_n = \langle g_N, \phi_n \rangle,$$
$$f_n(u_N^{\varepsilon})(s) = \langle f(., s, u_N^{\varepsilon}(., s)), \phi_n \rangle.$$

Next, we divide the time interval [0,1] into 10 subintervals by 11 points

$$t_j = \frac{j-1}{10}, j = 1, 2, ..., 11.$$

Put $\varepsilon = 0.1$, $\varepsilon = 0.01$, $\varepsilon = 0.001$, respectively.

For various values of ε, we aim to compute the expected error between the regularized solution and the exact solution, denoted by

$$\mathbb{E} \left\| u_N^{\varepsilon}(., t_j) - u_{exact}(., t_j) \right\|^2.$$

To achieve this, we generate a statistical sample of size $M = 100$. Specifically, in each of the $k - th$ simulations $(k = 1, 2, ...100)$, let $u_{N,k}^{\varepsilon}(., t_j)$ denote the regularized solution. For a specific value of ε, we use the Picard iteration to compute $u_{N,k}^{\varepsilon}(., t_j)$ as follows

$$\begin{cases} u_{\varepsilon,0}(x, t_j) = 0, \\ u_{\varepsilon,q}(x, t_j) = \sum_{n=1}^{B_N} \Big[e^{(0.1n^2+0.2n^{2\gamma})(1-t_j)} (g_N)_n \\ \qquad - \int_{t_j}^1 e^{(0.1n^2+0.2n^{2\gamma})(s-t_j)} f_n(u_{\varepsilon,q-1}(x, t_j))(s) ds \Big] \phi_n(x), \end{cases}$$

with $q = 1, 2, 3, ...$

The iteration is carried out and terminated at q_0 when

$$\mathbb{E} \left\| u_{\varepsilon,q_0}(., t_j) - u_{\varepsilon,q_0-1}(., t_j) \right\|^2 \leq 10^{-10}.$$

Then we choose u_{ε,q_0} to approximate $u^{\varepsilon}_{N,k}$ and the expectation of the error at the time $t_j, j = 1, 2, ...11$ is calculated by

$$\mathbb{E}\left\|u^{\varepsilon}_N(.,t_j) - u_{exact}(.,t_j)\right\|^2 = \frac{1}{M}\left(\sum_{k=1}^{M}\left\|u^{\varepsilon}_{N,k}(.,t_j) - u_{exact}(.,t_j)\right\|^2\right).$$

The results of our computational method is in the following Table.

Table 1. The expectation of the error between the regularized solution $u^{\varepsilon}_N(.,t)$ and the exact solution $u_{exact}(.,t)$ at different values of time corresponding to $\varepsilon = 0.1, 0.01, 0.001$ and $M = 100$.

	$\mathbb{E}\left\|u^{\varepsilon}_N(.,t) - u_{exact}(.,t)\right\|^2$		
t,ε	$\varepsilon = 0.1$	$\varepsilon = 0.01$	$\varepsilon = 0.001$
$t = 0$	$4.0723e-02$	$3.1283e-03$	$2.5396e-03$
$t = 0.1$	$3.8448e-02$	$2.4596e-03$	$1.9125e-03$
$t = 0.2$	$3.6325e-02$	$1.9015e-03$	$1.3940e-03$
$t = 0.3$	$3.4355e-02$	$1.4435e-03$	$9.6046e-04$
$t = 0.4$	$3.2511e-02$	$1.0773e-03$	$6.3542e-04$
$t = 0.5$	$3.0807e-02$	$7.9376e-04$	$3.8708e-04$
$t = 0.6$	$2.9240e-02$	$5.8766e-04$	$2.1070e-04$
$t = 0.7$	$2.7822e-02$	$4.5178e-04$	$1.0141e-04$
$t = 0.8$	$2.6561e-02$	$3.7617e-04$	$4.7649e-05$
$t = 0.9$	$2.5426e-02$	$3.3496e-04$	$2.4374e-05$
$t = 1$	$2.4192e-02$	$2.8794e-04$	$2.8163e-06$

6. Conclusion

In this study, by Fourier truncation method, we regularized the nonlinear diffusion equation with coupling operator and Gaussian white noise. With some conditions on the exact solution, we obtained the error estimate between the regularized solution and the exact solution. In the future, we will consider the problems with locally Lipschitz condition on the source term.

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