Nonlinear Functional Analysis and Applications
Vol. 31, No. 1 (2026), pp. 105-137
ISSN: 1229-1595(print), 2466-0973(online)
https://doi.org/10.22771/nfaa.2026.31.01.06
http://nfaa.kyungnam.ac.kr/journal-nfaa

sg-PERFECT FUNCTIONS IN SEMI-GENERALIZED TOPOLOGICAL SPACES: ON DECOMPOSITION, INVARIANCE, AND FURTHER DEVELOPMENTS

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Abstract. This paper is a pioneering study of $sg-$perfect functions, a transformative family of mappings that combine the structural depth of ideal topologies with the nuanced behavior of semi-open sets. These functions, based on Levine's semi-open sets and Bhattacharyya's $sg-$closedness, emerge as a three-layered marvel: $sg-$continuous, $sg-$closed, and endowed with $sg-$compact. We go through decomposition theorems and hereditary properties to see how $sg-$perfect functions, $sg-$Lindelöf perfect functions protect topological integrity. Beyond theoretical, this approach highlights several practical pathways: $sg-$perfect functions and $sg-$Lindelöf perfect functions permit the transfer of invariants in ideal-rich environments, providing tools for breaking down complex regions into manageable, $sg-$closed components. This study prepares the way for future research into $sg-$connectedness, dynamic decomposition, and beyond. For mathematicians negotiating the intersection of ideals and openness, $sg-$perfect functions operate as a guidepost to undiscovered topological realms.

[0]Received June 6, 2025. Revised September 20, 2025. Accepted October 18, 2025.
[0]2020 Mathematics Subject Classification: 54C10, 54D20.
[0]Keywords: $sg-$perfect functions, $sg-$Lindelöf perfect functions, $sg-$compact, $sg-$Lindelöf, $sg-$regular, $sg-$normal, $sg-$irr-Lindelöf-perfect.
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1. Introduction

The goal of generalizing topological notions using semi-open sets and their variants has significantly extended the discipline, allowing for the investigation of nuanced structural characteristics and mappings. The interaction between ideals and generalized closed sets has greatly enhanced the study of topological spaces, allowing fresh extensions of classical discoveries and stimulating deeper insights into separation axioms, compactness, Lindelöfness and continuity. Semi-open sets and its generalizations, first by Levine[17]; A subset $\mathfrak{R}$ of $(\mathcal{M}, \upsilon)$ is termed semi-open if $\mathfrak{R} \subseteq Cl(Int\mathfrak{R})$, and the complement of a semi-open set is called semi-closed [10]. Bhattacharyya and Lahiri's introduction of $sg-$closed sets constituted a seminal contributions, establishing a model in which semi-closed subsets are closely related to semi-interior operations[9]. Subsequent advancements, such as Dontchev's [14] characterisation of $sg-$closed sets further demonstrated the importance of semi-generalized topology in analysis. The semi-interior and the semi-closure of $\mathfrak{R}$, denoted by $sInt(\mathfrak{R})$,$sCl(\mathfrak{R})$ respectively. where is the $sInt(\mathfrak{R})$ is the union of all semi-open subsets of $\mathfrak{R}$ while the semi-closure of $\mathfrak{R}$ is the intersection of all semi-closed supersets of $\mathfrak{R}$. It is generally established that $sInt(\mathfrak{R}) = \mathfrak{R} \cap Cl(Int\mathfrak{R})$ and $sCl(\mathfrak{R}) = \mathfrak{R} \cup Int(Cl\mathfrak{R})$. The family of all $sg-$open sets in $(\mathcal{M}, \upsilon)$ is denoted by $SG(t)$. $sg-$compact spaces were introduced by Caldas [11]. It is also studied by Devi and others [12].

In the past few years, many authors have investigated the connections between topological concepts. The concepts of compactness, and Lindelöfness have been refined to develop new notions within perfect functions in topological spaces, including semi perfect functions [7], α-perfect functions [5], and difference perfect functions [6]. Refer to [1, 2, 3, 4, 8, 21] for more.

This research introduces and rigorously investigates the concept of sg-perfect functions. While Previous research has investigated sg-continuity, sg-irresoluteness, sg-compactness, and $sg-$Lindelöfness. Our research establishes fundamental results about the preservation of sg-compactness, sg-Lindelöfness, sg-normality, and $sg-$Hausdorff characteristics under $sg-$perfect functions. Using the unique interplay between semi-open sets and ideals, classical results like the Tychonoff product theorem and Urysohn's lemma are generalized to the sg-topological domain. We specify the conditions under which sg- perfect functions cause hereditary sg-closedness and investigate their importance in refining the $\mathcal{M}1 - \mathcal{M}2$ decomposition framework.

The paper will be structured as follows: Section 2 reviews the fundamental preliminaries on ideals, semi-generalized sets, and local functions. Section 3 presents $sg-$perfect functions, defines their essential features, and investigates their relationship to other types of functions. Then, section 4 investigates sg-Lindelöf perfect functions. Section 5 illustrates applications to product spaces and separation axioms.

2. Preliminaries

Definition 2.1. ([7]) A function $\Delta : (\mathcal{M}, \upsilon) \to (\mathcal{L}, \varsigma)$ is termed semi-perfect if Δ satisfies the semi-closedness, semi-continuity and for every $l \in \mathcal{L}$, $\Delta^{-1}(l)$ is semi compact.

Definition 2.2. ([19]) A topological space $\mathcal{M}$ is designated as $sg-$Lindelöf if every cover of $\mathcal{M}$ which include $sg-$open sets has a countable subcover.

Definition 2.3. ([17]) A function $\Delta : (\mathcal{M}, \upsilon) \to (\mathcal{L}, \varsigma)$ is called semi-continuous provided that for every open set if $\Delta^{-1}(\mathcal{K})$ of every open set $V \subseteq \mathcal{L}$, $\Delta^{-1}(\mathcal{K})$ is semi-open in $\mathcal{M}$.

Definition 2.4. ([18]) A function $\Delta : (\mathcal{M}, \upsilon) \to (\mathcal{L}, \varsigma)$ is termed $sg-$continuous if $\Delta^{-1}(\mathcal{K})$ is an $sg-$closed in $\mathcal{M}$ for each closed subset $\mathcal{K} \subseteq \mathcal{L}$.

Definition 2.5. ([20]) A function $\Delta : (\mathcal{M}, \upsilon) \to (\mathcal{L}, \varsigma)$ is designated as sg-irresolute if $\Delta^{-1}(\mathcal{K})$ remains sg-closed in $\mathcal{M}$ for any sg-closed set $\mathcal{K} \subseteq \mathcal{L}$.

Definition 2.6. ([11]) A space $\mathcal{M}$ is termed $sg-$compact if every $sg-$open sets covering $\mathcal{M}$ has a finite sub-cover.

Definition 2.7. ([15]) A space $(\mathcal{M}, \upsilon)$ is semi-regular if for any semi-closed set $\mathfrak{R}$ and $\mathsf{r} \notin \mathfrak{R}$, there exist disjoint semi-open sets $\mathcal{G}$ and $\mathcal{K}$ such that $\mathsf{r} \in \mathcal{G}$ and $\mathfrak{R} \subset \mathcal{K}$.

Definition 2.8. ([16]) A space $(\mathcal{M}, \upsilon)$ is said to be $sg-$regular if for any $sg-$closed set $\mathfrak{R} \subseteq \mathcal{M}$ and any $\mathsf{m} \notin \mathfrak{R}$, there exist two disjoint semi-open subsets $\mathcal{G}, \mathcal{K} \subseteq \mathcal{M}$ with $\mathfrak{R} \subseteq \mathcal{G}$ and $\mathsf{m} \in \mathcal{K}$. A space $(\mathcal{M}, \upsilon)$ is $sg-$regular if and only if $(\mathcal{M}, \upsilon)$ is semi-regular and semi-$T_{1/2}$ properties.

Definition 2.9. ([16]) A topological space $(\mathcal{M}, \upsilon)$ is termed $sg-$normal if for any two disjoint $sg-$closed sets $\mathfrak{R}, B \subseteq \mathcal{M}$, there exist disjoint semi-open sets $\mathcal{G}$ and $\mathcal{K}$ such that $\mathfrak{R} \subseteq \mathcal{G}$ and $B \subseteq \mathcal{K}$.

Definition 2.10. ([13]) A topological space $(\mathcal{M}, \upsilon)$ is called a $sg-T_0$ if for any two distinct points, there exists an $sg-$open set containing exactly one of them.

Definition 2.11. ([9]) A space $(\mathcal{M}, \upsilon)$ is termed space if every $sg-$closed set is also semi-closed.

3. $sg-$PERFECT FUNCTIONS

This section presents and discusses the concept of $sg-$perfect functions, a type of function in generalized topological spaces defined by three essential properties: $sg-$continuity, $sg-$closedness, and $sg-$compact preimages of points.

Definition 3.1. A function $\Delta\colon (\mathcal{M}, \upsilon) \to (\mathcal{L}, \varsigma)$ is designated as $sg-$perfect if satisfies; Δ is $sg-$continuous, Δ is $sg-$closed, and for every $\mathtt{l} \in \mathcal{L}$, $\Delta^{-1}(\mathtt{l})$ is $sg-$compact.

Lemma 3.2. *Every semi perfect function is an $sg-$perfect function, but the reverse implication is not true.*

Proof. Consider $\Delta\colon (\mathcal{M}, \upsilon) \to (\mathcal{L}, \varsigma)$ is a semi perfect function. By Definition 2.1, Δ is semi-closed, semi-continuous, $\Delta^{-1}(\mathtt{l})$ are semi-compact for all $\mathtt{l} \in \mathcal{L}$.

Now, since semi-closed implies $sg-$closed, the preimages of closed sets are semi-closed, therefore $sg-$closed, and semi-compact implies $sg-$compact. Thus, Δ is $sg-$perfect. □

For the converse, consider the following counterexample:

Example 3.3. Consider $\mathcal{M} = \{s, n, e\}$ and let $\upsilon_{\text{semi}} = \{\emptyset, \{s\}, \mathcal{M}\}$, $\upsilon_{\text{sg}} = \{\emptyset, \{s\}, \{n\}, \{s, n\}, \mathcal{M}\}$.

Consider $\mathcal{L} = \{1, 2\}$ with ς_{dis}. Define $\Delta\colon \mathcal{M} \to \mathcal{L}$ by:

$$\Delta(\mathsf{m}) = \begin{cases} 1, & \text{if } \mathsf{m} = s, \\ 2, & \text{if } \mathsf{m} \in \{n, e\}. \end{cases}$$

Then, since the preimages of open sets $\emptyset, \{1\}, \{2\}, \mathcal{L}$ are $\emptyset, \{s\}, \{n, e\}, \mathcal{M}$, all $sg-$open. Images of $sg-$closed sets in $\mathcal{M}$ are closed in $\mathcal{L}$, and $\Delta^{-1}(1) = \{s\}$ and $\Delta^{-1}(2) = \{n, e\}$ are finite. Hence $sg-$compact. Thus, Δ is $sg-$perfect. However, $\{n, e\}$ is $sg-$closed in $\mathcal{M}$ but not semi-closed, so Δ is not semi-closed. Hence, Δ is not semi-perfect.

Definition 3.4. A function $\Delta\colon (\mathcal{M}, \upsilon) \to (\mathcal{L}, \varsigma)$ designated as $sg-$irr-perfect, if Δ satisfies; $sg-$irresolute, $sg-$closed, and $sg-$compact $\Delta^{-1}(\mathtt{l})$ for all $\mathtt{l} \in \mathcal{L}$.

Lemma 3.5. *Every $sg-$irr-perfect function is $sg-$irresolute, but the converse is not true.*

Proof. ($\Rightarrow$) Immediate from Definition 3.4, as $sg-$irrperfect function explicitly requires $sg-$irresoluteness.

($\Leftarrow$) We know that the following is counterexample.

□

Example 3.6. Let $\mathcal{M} = \mathcal{L} = \{a, b, c\}$, with

$$\upsilon = \{\emptyset, \{a\}, \{c\}, \{a, c\}, \mathcal{M}\}, \quad \varsigma = \{\emptyset, \{a\}, \mathcal{L}\}.$$

Define $\Delta\colon (\mathcal{M}, \upsilon) \to (\mathcal{L}, \varsigma)$ by $\Delta(a) = \Delta(c) = b$ and $\Delta(b) = c$. Then, Δ is $sg-$perfect but not $sg-$irr-perfect.

Images of closed sets in $\mathcal{M}$ are $sg-$closed in $\mathcal{L}$. Closed sets in $\mathcal{M}$ include $\{b\}, \{a, b\}, \{b, c\}$. Their images under Δ are $\{c\}, \{b, c\}, \{b, c\}$, all of which are $sg-$closed in $\mathcal{L}$. Hence, Δ is $sg-$perfect. But, the set $\{b\}$ is $sg-$closed in $\mathcal{L}$, and $\Delta^{-1}(\{b\}) = \{a, c\}$ is not $sg-$closed in $\mathcal{M}$. (In $\mathcal{M}$, $\{a, c\}$ is open but fails the $sg-$closed condition: its semi-closure equals $\mathcal{M}$, which is not contained in the open set $\{a, c\}$.) So, it's not $sg-$irr-perfect.

Theorem 3.7. *Let $\Delta\colon (\mathcal{M}, \upsilon) \to (\mathcal{L}, \varsigma)$ be an $sg-$perfect function. For every compact subset $\mathcal{E} \subseteq \mathcal{L}$, $\Delta^{-1}(\mathcal{E})$ is $sg-$compact in $\mathcal{M}$.*

Proof. Consider $\mathcal{G} = \{\mathcal{G}_\alpha : \alpha \in \Lambda\}$ is an $sg-$open cover of $\Delta^{-1}(\mathcal{E})$. Given Δ is $sg-$perfect, $\Delta^{-1}(\mathtt{l})$ $(\mathtt{l} \in \mathcal{E})$ is $sg-$compact. For every $\mathtt{l} \in \mathcal{E}$, there exist a finite subset $\Lambda_{\mathtt{l}} \subseteq \Lambda$ such that:

$$\Delta^{-1}(\mathtt{l}) \subseteq \bigcup_{\alpha \in \Lambda_{\mathtt{l}}} \mathcal{G}_\alpha.$$

Define $O_{\mathtt{l}} = \mathcal{L} \setminus \Delta\left(\mathcal{M} \setminus \bigcup_{\alpha \in \Lambda_{\mathtt{l}}} \mathcal{G}_\alpha\right)$. This yields $O_{\mathtt{l}}$ is $sg-$open set in $\mathcal{L}$ that contains $\mathtt{l}$, and $\Delta^{-1}(O_{\mathtt{l}}) \subseteq \bigcup_{\alpha \in \Lambda_{\mathtt{l}}} \mathcal{G}_\alpha$. The collection $\{O_{\mathtt{l}} : \mathtt{l} \in \mathcal{E}\}$ constructs a $sg-$open cover of $\mathcal{E}$.

Since $\mathcal{E}$ is compact, there exist a finite subset $\{\mathtt{l}_1, \ldots, \mathtt{l}_n\} \subseteq \mathcal{E}$ such that:

$$\mathcal{E} \subseteq \bigcup_{i=1}^{n} O_{\mathtt{l}_i}.$$

Consequently,

$$\Delta^{-1}(\mathcal{E}) \subseteq \bigcup_{i=1}^{n} \Delta^{-1}(O_{\mathtt{l}_i}) \subseteq \bigcup_{i=1}^{n} \bigcup_{\alpha \in \Lambda_{\mathtt{l}_i}} \mathcal{G}_\alpha.$$

This produce a finite subcover of $\mathcal{G}$, demonstrating that $\Delta^{-1}(\mathcal{E})$ is $sg-$compact. □

Corollary 3.8. *Let $\Delta : (\mathcal{M}, \upsilon) \to (\mathcal{L}, \varsigma)$ be a semi-perfect function. If $\mathcal{E} \subseteq \mathcal{L}$ is compact, then $\Delta^{-1}(\mathcal{E})$ is sg-compact in $\mathcal{M}$.*

Proof. Since Δ is semi-perfect, Lemma 3.2 implies that Δ is *sg*-perfect. Therefore, by Theorem 3.7, $\Delta^{-1}(\mathcal{E})$ is *sg*-compact in $\mathcal{M}$. □

Theorem 3.9. *Let $\Delta\colon (\mathcal{M}, \upsilon) \xrightarrow{onto} (\mathcal{L}, \varsigma)$ be $sg-$irresolute perfect function. For every $sg-$compact subset $\mathcal{E} \subseteq \mathcal{L}$, $\Delta^{-1}(\mathcal{E})$ keeps being $sg-$compact in $\mathcal{M}$.*

Proof. Consider an $sg-$open cover $\mathfrak{G} = \{\mathcal{G}_\alpha : \alpha \in \Lambda\}$ of $\Delta^{-1}(\mathcal{E})$. Given Δ is $sg-$irresolute perfect, $\Delta^{-1}(\mathtt{l})(\mathtt{l} \in \mathcal{E})$ is $sg-$compact. For all $\mathtt{l} \in \mathcal{E}$, construct a finite subset $\Lambda_\mathtt{l} \subseteq \Lambda$ such that:

$$\Delta^{-1}(\mathtt{l}) \subseteq \bigcup_{\alpha \in \Lambda_\mathtt{l}} U_\alpha.$$

Define $O_\mathtt{l} = \mathcal{L} \setminus \Delta\left(\mathcal{M} \setminus \bigcup_{\alpha\in\Lambda_\mathtt{l}} \mathcal{G}_\alpha\right)$. The set $O_\mathtt{l}$ is $sg-$open in $\mathcal{L}$ and includes $\mathtt{l}$. The collection $\{O_\mathtt{l} : \mathtt{l} \in \mathcal{E}\}$ creates an $sg-$open cover of $\mathcal{E}$. Given the $sg-$compactness of $\mathcal{E}$, there exists a finite subset $\{\mathtt{l}_1, \ldots, \mathtt{l}_n\} \subseteq \mathcal{E}$ such that:

$$\mathcal{E} \subseteq \bigcup_{i=1}^{n} O_{\mathtt{l}_i}.$$

Therefore,

$$\Delta^{-1}(\mathcal{E}) \subseteq \bigcup_{i=1}^{n} \Delta^{-1}(O_{\mathtt{l}_i}) \subseteq \bigcup_{i=1}^{n} \bigcup_{\alpha\in\Lambda_{\mathtt{l}_i}} \mathcal{G}_\alpha.$$

This yields a finite $sg-$open subcover of $\mathfrak{G}$, proving $\Delta^{-1}(\mathcal{E})$ is $sg-$compact. □

Proposition 3.10. *If $\Delta\colon (\mathcal{M}, \upsilon) \xrightarrow{onto} (\mathcal{L}, \varsigma)$ is continuous, and $sg-$irresolute, with $\mathcal{M}$ $sg-$compact, then $\mathcal{L}$ is $sg-$compact.*

Proof. Let $\{U_\alpha : \alpha \in \Lambda\}$ be an sg-open cover of $\mathcal{L}$. For each α, the complement $\mathcal{L} \setminus U_\alpha$ is sg-closed; since Δ is sg-irresolute, $\Delta^{-1}(\mathcal{L} \setminus U_\alpha)$ is sg-closed in $\mathcal{M}$, hence $\Delta^{-1}(U_\alpha)$ is sg-open. Surjectivity of Δ implies $\{\Delta^{-1}(U_\alpha) : \alpha \in \Lambda\}$ is an sg-open cover of $\mathcal{M}$. By sg-compactness of $\mathcal{M}$, choose $\alpha_1, \ldots, \alpha_n \in \Lambda$ with

$$\mathcal{M} = \bigcup_{i=1}^{n} \Delta^{-1}(U_{\alpha_i}).$$

Applying Δ and using $\Delta(\mathcal{M}) = \mathcal{L}$, we obtain

$$\mathcal{L} = \Delta(\mathcal{M}) \subseteq \bigcup_{i=1}^{n} \Delta\big(\Delta^{-1}(U_{\alpha_i})\big) \subseteq \bigcup_{i=1}^{n} U_{\alpha_i},$$

so $\{U_{\alpha_i}\}_{i=1}^{n}$ is a finite sg-subcover. Hence $\mathcal{L}$ is sg-compact. □

The hereditary preservation conditions for $sg-$perfect functions extend to specialized topological spaces, as demonstrated by:

Proposition 3.11. *The composition of two $sg-$perfect functions does not need be $sg-$perfect.*

Example 3.12. Consider

$$\mathcal{M} = \{a, b, c\},\ \upsilon_{\mathcal{M}} = \{\emptyset, \{a\}, \{c\}, \{a, c\}, \mathcal{M}\},$$
$$\mathcal{L} = \{1, 2\},\ \upsilon_{\mathcal{L}} = \{\emptyset, \{1\}, \mathcal{L}\},$$
$$\mathcal{E} = \{p\},\ \upsilon_{\mathcal{E}} = \{\emptyset, \mathcal{E}\}.$$

Define

$$\varDelta\colon \mathcal{M} \to \mathcal{L}, \quad \varDelta(a) = \varDelta(c) = 1, \quad \varDelta(b) = 2,$$
$$\varPi\colon \mathcal{L} \to \mathcal{E}, \quad \varPi(1) = \varPi(2) = p.$$

Then both $\varDelta$ and $\varPi$ are $sg-$perfect. And $\varDelta$ is $sg-$closed, continuous, and inverse of points in $\mathcal{L}$ are $sg-$compact in $\mathcal{M}$. $\varPi$ is trivially $sg-$perfect. However, $\varPi \circ \varDelta\colon \mathcal{M} \to \mathcal{E}$ is not $sg-$perfect; $(\varPi \circ \varDelta)^{-1}(p) = \mathcal{M}$ is $sg-$closed in $\mathcal{M}$, but $\mathcal{M}$ is not $sg-$compact. Thus, $\varPi \circ \varDelta$ fails to be $sg-$perfect.

Theorem 3.13. *Consider $\varDelta\colon (\mathcal{M}, \upsilon) \to (\mathcal{L}, \varsigma)$ and $\varPi\colon (\mathcal{L}, \varsigma) \to (\mathcal{E}, \rho)$ be $sg-$irresolute perfect functions. $\varPi \circ \varDelta\colon \mathcal{M} \to \mathcal{E}$ remains $sg-$irresolute perfect.*

Proof. Consider $\mathfrak{R}$ is $sg-$open set in $\mathcal{E}$. Since $\varPi$ is $sg-$irresolute perfect, $\varPi^{-1}(\mathfrak{R})$ is $sg-$open in $\mathcal{L}$. As $\varDelta$ is $sg-$irresolute perfect, $\varDelta^{-1}(\varPi^{-1}(\mathfrak{R})) = (\varPi \circ \varDelta)^{-1}(\mathfrak{R})$ is $sg-$open in $\mathcal{M}$. Thus, $\varPi \circ \varDelta$ is $sg-$continuous. And, both $\varDelta$ and $\varPi$ are closed functions. The composition of closed functions is closed, so $\varPi \circ \varDelta$ is closed. $(\varPi \circ \varDelta)^{-1}(\mathfrak{e}) = \varDelta^{-1}(\varPi^{-1}(\mathfrak{e}))(\mathfrak{e} \in \mathcal{E})$ with $\varPi^{-1}(\mathfrak{e})$ compact in $\mathcal{L}$ and $\varDelta$ preserving compactness, $\varDelta^{-1}(\varPi^{-1}(\mathfrak{e}))$ is $sg-$compact in $\mathcal{M}$. Thus, $\varPi \circ \varDelta$ is $sg-$irresolute perfect. □

This compositional stability naturally prompts us to investigate hereditary preservation. The following propositions demonstrate how $sg-$perfect conveys structural properties between spaces:

Proposition 3.14. *For an $sg-$irresolute perfect function $\varDelta\colon \mathcal{M} \to \mathcal{L}$. If $\mathcal{L}$ is hereditarily $sg-$compact, then $\mathcal{M}$ is hereditarily $sg-$compactness.*

Proof. Let $A \subseteq \mathcal{M}$. We prove that A is *sg*-compact. Consider $\varDelta|_A : A \to \varDelta(A)$. Since $\varDelta$ is *sg*-irresolute, the restriction $\varDelta|_A$ is also *sg*-irresolute.

Assume, towards a contradiction, that A is not *sg*-compact. Then A has an *sg*-open cover $\{V_\beta : \beta \in \Gamma\}$ with no finite subcover. For each β, put

$$W_\beta := \varDelta(A \setminus V_\beta) \subseteq \varDelta(A).$$

Because $A \setminus V_\beta$ is *sg*-closed in A and $\varDelta$ is *sg*-closed, each W_β is *sg*-closed in the subspace $\varDelta(A)$; hence $\varDelta(A) \setminus W_\beta$ is *sg*-open in $\varDelta(A)$. Moreover,

$$\big\{\varDelta(A) \setminus W_\beta : \beta \in \Gamma\big\}$$

covers $\varDelta(A)$: if $a \in V_\beta$, then $\varDelta(a) \notin \varDelta(A \setminus V_\beta) = W_\beta$.

Now $\mathcal{L}$ is hereditarily sg-compact, so the subspace $\Delta(A) \subseteq \mathcal{L}$ is sg-compact; hence $\Delta(A) = \bigcup_{i=1}^{n}(\Delta(A) \setminus W_{\beta_i})$ for some $\beta_1, \ldots, \beta_n$. Taking preimages under the surjection $\Delta|_A$ yields

$$A = (\Delta|_A)^{-1}(\Delta(A)) \subseteq \bigcup_{i=1}^{n}(\Delta|_A)^{-1}(\Delta(A) \setminus W_{\beta_i}) \subseteq \bigcup_{i=1}^{n} V_{\beta_i},$$

a contradiction. Therefore A is sg-compact. Since $A \subseteq \mathcal{M}$ was arbitrary, $\mathcal{M}$ is hereditarily sg-compact. □

The product topology framework adds another dimension to stability analysis. The following propositions demonstrate that $sg-$irresolute perfection persists under Cartesian products:

Proposition 3.15. *Given $sg-$irresolute perfect functions $\Delta\colon \mathcal{M} \to \mathcal{L}$ and $\Pi\colon \mathfrak{R} \to B$. Then product function $\Delta \times \Pi\colon \mathcal{M} \times \mathfrak{R} \to \mathcal{L} \times \mathcal{S}$ is $sg-$irresolute perfect under product topology.*

Proof. Let $\Delta : \mathcal{M} \to \mathcal{L}$ and $\Pi : \mathfrak{R} \to \mathcal{S}$ be sg-irresolute perfect and consider $\Delta \times \Pi : \mathcal{M} \times \mathfrak{R} \to \mathcal{L} \times \mathcal{S}$.

It suffices to check preimages of subbasic sg-closed sets in $\mathcal{L} \times \mathcal{S}$. For sg-closed $A \subseteq \mathcal{L}$, $C \subseteq \mathcal{S}$,

$$(\Delta \times \Pi)^{-1}(A \times \mathcal{S}) = \Delta^{-1}(A) \times \mathfrak{R}, \qquad (\Delta \times \Pi)^{-1}(\mathcal{L} \times C) = \mathcal{M} \times \Pi^{-1}(C),$$

which are sg-closed in $\mathcal{M} \times \mathfrak{R}$ since Δ, Π are sg-irresolute. Hence $\Delta \times \Pi$ is sg-irresolute.

Let $H \subseteq \mathcal{M} \times \mathfrak{R}$ be sg-closed and $(\ell, s) \notin (\Delta \times \Pi)(H)$. Using that Δ and Π are sg-closed, one constructs sg-open neighborhoods $W_1 \ni \ell$ in $\mathcal{L}$ and $W_2 \ni s$ in $\mathcal{S}$ such that

$$(W_1 \times W_2) \cap (\Delta \times \Pi)(H) = \emptyset,$$

so $(\Delta \times \Pi)(H)$ is sg-closed in $\mathcal{L} \times \mathcal{S}$.

For each $(\ell, s) \in \mathcal{L} \times \mathcal{S}$,

$$(\Delta \times \Pi)^{-1}(\ell, s) = \Delta^{-1}(\ell) \times \Pi^{-1}(s).$$

Since Δ, Π are sg-irresolute perfect, $\Delta^{-1}(\ell)$ and $\Pi^{-1}(s)$ are sg-compact, and finite products of sg-compact spaces are sg-compact; hence the fiber is sg-compact.

Thus $\Delta \times \Pi$ is sg-irresolute perfect. □

Proposition 3.16. *Any restriction $\Delta|_{\mathfrak{R}}\colon \mathfrak{R} \to \mathcal{L}$ of $sg-$irresolute perfect function $\Delta\colon \mathcal{M} \to \mathcal{L}$ to an $sg-$closed $\mathfrak{R} \subseteq \mathcal{M}$ is $sg-$irresolute perfect.*

Proof. Let $\Delta : \mathcal{M} \to \mathcal{L}$ be *sg*-irresolute perfect and let $\mathfrak{R} \subseteq \mathcal{M}$ be *sg*-closed. Consider the restriction $\Delta|_{\mathfrak{R}} : \mathfrak{R} \to \mathcal{L}$. If $F \subseteq \mathcal{L}$ is *sg*-closed, then $\Delta^{-1}(F)$ is *sg*-closed in $\mathcal{M}$. Hence

$$(\Delta|_{\mathfrak{R}})^{-1}(F) = \mathfrak{R} \cap \Delta^{-1}(F)$$

is *sg*-closed in the subspace $\mathfrak{R}$. Thus $\Delta|_{\mathfrak{R}}$ is *sg*-irresolute. Let $A \subseteq \mathfrak{R}$ be *sg*-closed in $\mathfrak{R}$. Then $A = \mathfrak{R} \cap C$ for some *sg*-closed $C \subseteq \mathcal{M}$. Since $\mathfrak{R}$ is *sg*-closed in $\mathcal{M}$, also A is *sg*-closed in $\mathcal{M}$, and because Δ is *sg*-closed, $\Delta(A)$ is *sg*-closed in $\mathcal{L}$. But $(\Delta|_{\mathfrak{R}})(A) = \Delta(A)$, so $\Delta|_{\mathfrak{R}}$ is *sg*-closed. For $\ell \in \mathcal{L}$,

$$(\Delta|_{\mathfrak{R}})^{-1}(\ell) = \mathfrak{R} \cap \Delta^{-1}(\ell).$$

Since Δ is *sg*-irresolute perfect, $\Delta^{-1}(\ell)$ is *sg*-compact in $\mathcal{M}$; intersecting with the *sg*-closed $\mathfrak{R}$ yields an *sg*-compact subset of the subspace $\mathfrak{R}$. Therefore $\Delta|_{\mathfrak{R}}$ is *sg*-irresolute, *sg*-closed, and has *sg*-compact point-inverses, hence it is *sg*-irresolute perfect. □

Theorem 3.17. *Let $\Delta\colon (\mathcal{M}, \upsilon) \to (\mathcal{L}, \varsigma)$ be sg−irresolute perfect and $\Pi\colon (\mathcal{L}, \varsigma)$ $\to (\mathcal{E}, \rho)$ be sg−perfect. Then, $\Pi \circ \Delta\colon \mathcal{M} \to \mathcal{E}$ is an sg−perfect function.*

Proof. Let $\mathfrak{R}$ be an open set in $\mathcal{E}$. Since Π is *sg*−perfect, it is continuous, so $\Pi^{-1}(\mathfrak{R})$ is open in $\mathcal{L}$. As Δ is *sg*−irresolute perfect (and hence *sg*−continuous), $\Delta^{-1}(\Pi^{-1}(\mathfrak{R})) = (\Pi \circ \Delta)^{-1}(\mathfrak{R})$ is *sg*−open in $\mathcal{M}$. Then $\Pi \circ \Delta$ is *sg*−continuous. Since both Δ and Π are closed functions. Therefore, $\Pi \circ \Delta$ is closed. For all $\mathfrak{e} \in \mathcal{E}$, $(\Pi \circ \Delta)^{-1}(\mathfrak{e}) = \Delta^{-1}(\Pi^{-1}(\mathfrak{e}))$. Since Π is *sg*−perfect, $\Pi^{-1}(\mathfrak{e})$ is compact in $\mathcal{L}$. As Δ is *sg*−irresolute perfect, $\Delta^{-1}(\Pi^{-1}(\mathfrak{e}))$ is compact in $\mathcal{M}$. Hence, $\Pi \circ \Delta$ is *sg*−perfect. □

In the following We begin by looking at how closure properties spread through compositions of specialized functions.

Proposition 3.18. *For sg−continuous $\Delta\colon (\mathcal{M}, \upsilon) \overset{onto}{\to} (\mathcal{L}, \varsigma)$ and sg−irresolute $\Pi\colon (\mathcal{L}, \varsigma) \overset{onto}{\to} (\mathcal{E}, \rho)$. If $\Pi \circ \Delta$ is sg−closed, then Π is sg−closed.*

Proof. Consider $\mathfrak{R}$ is *sg*−closed in $\mathcal{L}$. By *sg*−continuous of Δ, $\Delta^{-1}(\mathfrak{R})$ is *sg*−closed in $\mathcal{M}$. Given $\Pi \circ \Delta$ is closed,

$$(\Pi \circ \Delta)(\Delta^{-1}(\mathfrak{R})) = \Pi(\mathfrak{R})$$

is *sg*−closed in $\mathcal{E}$. Thus, Π preserves *sg*−closed. □

Proposition 3.19. *For an sg−closed function $\Delta\colon \mathcal{M} \to \mathcal{L}$ and sg−closed subspace $\mathfrak{R} \subseteq \mathcal{M}$ the restriction $\Delta|_{\mathfrak{R}}\colon \mathfrak{R} \to \mathcal{L}$ is sg−closedness.*

Proof. Let $A \subseteq \mathfrak{R}$ be sg-closed in the subspace $\mathfrak{R}$. Then there exists an sg-closed set $C \subseteq \mathcal{M}$ such that

$$A = \mathfrak{R} \cap C.$$

Since $\mathfrak{R}$ is sg-closed in $\mathcal{M}$, the intersection $\mathfrak{R} \cap C$ is sg-closed in $\mathcal{M}$. Because Δ is sg-closed, it sends sg-closed sets in $\mathcal{M}$ to sg-closed sets in $\mathcal{L}$. Hence

$$\Delta(A) = \Delta(\mathfrak{R} \cap C)$$

is sg-closed in $\mathcal{L}$. But $(\Delta|_{\mathfrak{R}})(A) = \Delta(A)$, so $(\Delta|_{\mathfrak{R}})(A)$ is sg-closed in $\mathcal{L}$. Therefore $\Delta|_{\mathfrak{R}}$ is an sg-closed function. □

Proposition 3.20. *For sg−perfect functions $\Delta\colon \mathcal{M} \to \mathcal{L}$ and $\Pi\colon \mathfrak{R} \to \mathcal{S}$, the product function $\Delta \times \Pi\colon \mathcal{M} \times \mathfrak{R} \to \mathcal{L} \times \mathcal{S}$ is sg−perfect under the product topology.*

Proof. Let $\Delta : \mathcal{M} \to \mathcal{L}$ and $\Pi : \mathfrak{R} \to \mathcal{S}$ be sg-perfect and consider $\Delta \times \Pi : \mathcal{M} \times \mathfrak{R} \to \mathcal{L} \times \mathcal{S}$. It suffices to check preimages of subbasic closed sets in $\mathcal{L} \times \mathcal{S}$. For closed $A \subseteq \mathcal{L}$, $C \subseteq \mathcal{S}$,

$$(\Delta \times \Pi)^{-1}(A \times \mathcal{S}) = \Delta^{-1}(A) \times \mathfrak{R}, \qquad (\Delta \times \Pi)^{-1}(\mathcal{L} \times C) = \mathcal{M} \times \Pi^{-1}(C).$$

Since Δ and Π are sg-continuous, $\Delta^{-1}(A)$ and $\Pi^{-1}(C)$ are sg-closed, hence the above preimages are sg-closed in the product. Therefore $\Delta \times \Pi$ is sg-continuous. Let $H \subseteq \mathcal{M} \times \mathfrak{R}$ be sg-closed and take $(\ell, s) \notin (\Delta \times \Pi)(H)$. Using sg-closedness of Δ and Π, one constructs sg-open neighborhoods $W_1 \ni \ell$ in $\mathcal{L}$ and $W_2 \ni s$ in $\mathcal{S}$ such that

$$(W_1 \times W_2) \cap (\Delta \times \Pi)(H) = \emptyset,$$

which shows $(\Delta \times \Pi)(H)$ is sg-closed in $\mathcal{L} \times \mathcal{S}$. For each $(\ell, s) \in \mathcal{L} \times \mathcal{S}$,

$$(\Delta \times \Pi)^{-1}(\ell, s) = \Delta^{-1}(\ell) \times \Pi^{-1}(s).$$

Because Δ and Π are sg-perfect, both factors are sg-compact; hence their finite product is sg-compact. Thus every fiber of $\Delta \times \Pi$ is sg-compact. Thus $\Delta \times \Pi$ is sg-perfect. □

Theorem 3.21. *Let $\Delta\colon (\mathcal{M}, \upsilon) \overset{onto}{\to} (\mathcal{L}, \varsigma)$ be sg−continuous and $\Pi\colon (\mathcal{L}, \varsigma) \overset{onto}{\to} (\mathcal{E}, \rho)$ be sg−irresolute. If $\Pi \circ \Delta$ is sg−perfect, then Π is sg−perfect.*

Proof. Consider $\Pi^{-1}(\mathfrak{e}) = \Delta\big((\Pi \circ \Delta)^{-1}(\mathfrak{e})\big)$ for any $\mathfrak{e} \in \mathcal{E}$. Since $\Pi \circ \Delta$ is $sg-$perfect, $(\Pi \circ \Delta)^{-1}(\mathfrak{e})$ is $sg-$compact in $\mathcal{M}$. As Δ is $sg-$continuous,

$$\Delta\big((\Pi \circ \Delta)^{-1}(\mathfrak{e})\big) = \Pi^{-1}(\mathfrak{e})$$

is $sg-$compact in $\mathcal{L}$. By Proposition 3.18, Π is $sg-$closed. Thus, Π is $sg-$perfect. □

Now, we investigate how $sg-$properties go through under domain restrictions.

Theorem 3.22. *Let $\Delta\colon (\mathcal{M}, \upsilon) \overset{onto}{\to} (\mathcal{L}, \varsigma)$ be $sg-$closed. For any $\mathcal{S} \subseteq \mathcal{L}$, the restriction $\Delta_{\mathcal{S}}\colon \Delta^{-1}(\mathcal{S}) \to \mathcal{S}$ is $sg-$closed.*

Proof. For $sg-$closed $\mathfrak{R} \subseteq \Delta^{-1}(\mathcal{S})$, Then $\mathfrak{R} = C \cap \Delta^{-1}(\mathcal{S})$ with $sg-$closed $C \subseteq \mathcal{M}$. Since Δ is $sg-$closed, then

$$\Delta_{\mathcal{S}}(\mathfrak{R}) = \Delta(C) \cap \mathcal{S},$$

which is $sg-$closed in $\mathcal{S}$. Thus, $\Delta_{\mathcal{S}}$ preserves $sg-$closedness. □

Theorem 3.23. *Let $\Delta\colon (\mathcal{M}, \upsilon) \overset{onto}{\to} (\mathcal{L}, \varsigma)$ be $sg-$perfect. For any $\mathcal{S} \subseteq \mathcal{L}$, the restriction $\Delta_{\mathcal{S}}\colon \Delta^{-1}(\mathcal{S}) \to \mathcal{S}$ is $sg-$perfect.*

Proof. By Theorem 3.22, $\Delta_{\mathcal{S}}$ is $sg-$closed. Since Δ is $sg-$perfect, inverse of points in B are $sg-$compact in $\Delta^{-1}(\mathcal{S})$. The subspace topology on $\Delta^{-1}(\mathcal{S})$ preserves $sg-$compactness, so $\Delta_{\mathcal{S}}$ is $sg-$perfect. □

Proposition 3.24. *If $\mathcal{L}$ is hereditarily $sg-$compact with $\Delta\colon \mathcal{M} \to \mathcal{L}$ $sg-$perfect, then $\mathcal{M}$ is hereditarily $sg-$compact.*

Proof. Let $F \subseteq \mathcal{M}$ be an arbitrary *sg*-closed subset. We prove that F is *sg*-compact.

Since Δ is *sg*-closed, $\Delta(F)$ is *sg*-closed in $\mathcal{L}$. By the hereditary *sg*-compactness of $\mathcal{L}$, the subspace $\Delta(F)$ is *sg*-compact.

Moreover, every open set is (in particular) *sg*-open; hence every open cover is an *sg*-open cover, and thus *sg*-compactness implies ordinary compactness. Consequently, $\Delta(F)$ is compact in $\mathcal{L}$.

Now apply Theorem 3.7 to the compact set $\mathcal{E} = \Delta(F)$:

$$\Delta^{-1}(\Delta(F)) \text{ is } sg\text{-compact in } \mathcal{M}.$$

Since $F \subseteq \Delta^{-1}(\Delta(F))$ and F is *sg*-closed in $\mathcal{M}$, it follows that F is *sg*-closed in the subspace $\Delta^{-1}(\Delta(F))$. By Lemma 3.29 (every *sg*-closed subset of an *sg*-compact space is *sg*-compact), we conclude that F is *sg*-compact.

Therefore, every *sg*-closed subset $F \subseteq \mathcal{M}$ is *sg*-compact, i.e. $\mathcal{M}$ is hereditarily *sg*-compact. □

We now examine interactions with subspace embeddings.

Proposition 3.25. *For embedding $i\colon \mathfrak{R} \hookrightarrow \mathcal{M}$ and $sg-$perfect $\Delta\colon \mathcal{M} \to \mathcal{L}$. The composition $\Delta \circ i\colon \mathfrak{R} \to \mathcal{L}$ is $sg-$perfect.*

Proof. Since $i : \mathfrak{R} \hookrightarrow \mathcal{M}$ is an embedding, we identify $\mathfrak{R}$ with a subspace of $\mathcal{M}$ and i with the inclusion.

For every closed $F \subseteq \mathcal{L}$,

$$(\Delta \circ i)^{-1}(F) = \mathfrak{R} \cap \Delta^{-1}(F).$$

As Δ is sg-continuous, $\Delta^{-1}(F)$ is sg-closed in $\mathcal{M}$; hence the intersection is sg-closed in the subspace $\mathfrak{R}$. Thus $\Delta \circ i$ is sg-continuous. If $A \subseteq \mathfrak{R}$ is sg-closed in $\mathfrak{R}$, then $A = \mathfrak{R} \cap C$ for some sg-closed $C \subseteq \mathcal{M}$. Since Δ is sg-closed, $\Delta(C)$ is sg-closed in $\mathcal{L}$, and

$$(\Delta \circ i)(A) = \Delta(A) \subseteq \Delta(\mathfrak{R}) \cap \Delta(C),$$

so $\Delta(A)$ is sg-closed in the subspace $\Delta(\mathfrak{R})$. Hence $\Delta \circ i$ is sg-closed. For each $\ell \in \mathcal{L}$,

$$(\Delta \circ i)^{-1}(\ell) = \mathfrak{R} \cap \Delta^{-1}(\ell).$$

Because Δ is sg-perfect, $\Delta^{-1}(\ell)$ is sg-compact in $\mathcal{M}$; intersecting with the subspace $\mathfrak{R}$ yields an sg-compact subset of $\mathfrak{R}$.

Therefore, $\Delta \circ i$ is sg-perfect. □

Theorem 3.26. *Let $\Delta\colon (\mathcal{M}, \upsilon) \xrightarrow{onto} (\mathcal{L}, \varsigma)$ be sg−irresolute perfect function. Any restriction $\Delta_{\mathcal{S}}\colon \Delta^{-1}(\mathcal{S}) \to \mathcal{S}$ ($\mathcal{S} \subseteq \mathcal{L}$) is sg−irresolute perfect.*

Proof. Follows from the preservation of $sg-$continuity, closedness, and compact fibers under subspace restrictions. □

The combination of the compactness and separation axioms produces essential closure properties.

Theorem 3.27. *Let $\Delta\colon (\mathcal{M}, \upsilon) \to (\mathcal{L}, \varsigma)$ be sg−perfect. If $\mathcal{M}$ is sg−compact and $\mathcal{L}$ is sg−Hausdorff, then Δ is sg−closed.*

Proof. For any $sg-$closed $\mathfrak{R} \subseteq \mathcal{M}$, $\mathfrak{R}$ is compact. Since $\mathcal{L}$ is $sg-$Hausdorff, $\Delta(\mathfrak{R})$ is closed. □

Definition 3.28. A bijective $\Delta\colon (\mathcal{M}, \upsilon) \to (\mathcal{L}, \varsigma)$ is termed sg-homeomorphism if it sg-continuous, sg-closed or equivalently, Δ is sg-open.

Lemma 3.29. *Every sg-closed subset in sg−compact space $(\mathcal{M}, \upsilon)$ is sg-compact.*

Proof. Let $F \subseteq \mathcal{M}$ be sg-closed, and let $\{U_\alpha : \alpha \in \Lambda\}$ be an sg-open cover of F in the subspace F. Then for each α, there exists an sg-open set $V_\alpha \subseteq \mathcal{M}$ such that $U_\alpha = F \cap V_\alpha$. Since F is sg-closed, $\mathcal{M} \setminus F$ is sg-open in $\mathcal{M}$. Hence

$$\mathcal{M} \;=\; (\mathcal{M} \setminus F) \;\cup\; \bigcup_{\alpha \in \Lambda} V_\alpha$$

is an sg-open cover of $\mathcal{M}$. By sg-compactness of $\mathcal{M}$, there exist $\alpha_1, \ldots, \alpha_n \in \Lambda$ such that

$$\mathcal{M} = (\mathcal{M} \setminus F) \cup \bigcup_{i=1}^{n} V_{\alpha_i}.$$

Intersecting both sides with F gives

$$F = \bigcup_{i=1}^{n} (F \cap V_{\alpha_i}) = \bigcup_{i=1}^{n} U_{\alpha_i}.$$

Thus $\{U_{\alpha_i}\}_{i=1}^{n}$ is a finite sg-subcover of F. Therefore F is sg-compact. □

Proposition 3.30. *A $sg-$Hausdorff space $(\mathcal{L}, \varsigma)$ contains all sg-compact subsets as closed sets.*

Proof. Let $K \subseteq \mathcal{L}$ be sg-compact. We show that $\mathcal{L} \setminus K$ is sg-open.

Fix $x \in \mathcal{L} \setminus K$. For each $y \in K$, since $\mathcal{L}$ is sg-Hausdorff, there exist disjoint sg-open sets $U_y, V_y \in SG(\varsigma)$ such that

$$y \in U_y, \qquad x \in V_y, \qquad U_y \cap V_y = \varnothing.$$

Then $\{U_y : y \in K\}$ is an sg-open cover of K. By sg-compactness, there exist $y_1, \ldots, y_n \in K$ such that

$$K \subseteq \bigcup_{i=1}^{n} U_{y_i}.$$

Set $V = \bigcap_{i=1}^{n} V_{y_i}$. (In a sg-topological space, finite intersections of sg-open sets are sg-open.) Clearly $x \in V$. Moreover, $V \cap K = \varnothing$: indeed, if $z \in V \cap K$, then $z \in U_{y_j}$ for some j, while $z \in V \subseteq V_{y_j}$, contradicting $U_{y_j} \cap V_{y_j} = \varnothing$. Thus $V \subseteq \mathcal{L} \setminus K$, so x has an sg-open neighborhood contained in $\mathcal{L} \setminus K$.

Since $x \in \mathcal{L} \setminus K$ was arbitrary, $\mathcal{L} \setminus K$ is sg-open, hence K is sg-closed. □

The combination of compactness and separation axioms produces strong classification results.

Theorem 3.31. *Let $\Delta\colon (\mathcal{M}, \upsilon) \to (\mathcal{L}, \varsigma)$ be a bijective $sg-$continuous, where $\mathcal{M}$ is $sg-$compact and $\mathcal{L}$ is $sg-$Hausdorff. Then Δ is an $sg-$homeomorphism.*

Proof. Consider $F \subseteq \mathcal{M}$ is an $sg-$closed set. Since $\mathcal{M}$ is $sg-$compact, F is $sg-$compact from Lemma. 3.29. As Δ is $sg-$continuous, $\Delta(F)$ is $sg-$compact in $\mathcal{L}$. Because $\mathcal{L}$ is $sg-$Hausdorff, $\Delta(F)$ is $sg-$closed from Proposition 3.30. Thus Δ function $sg-$closed sets to $sg-$closed sets, produce an $sg-$closed function. Combined with bijectivity and $sg-$continuity, Δ is an $sg-$homeomorphism. □

Proposition 3.32. *For sg−homeomorphisms* $\Delta\colon \mathcal{M} \to \mathcal{L}$ *and* $\Pi\colon \mathcal{L} \to \mathcal{E}$*, the composition* $\Pi \circ \Delta\colon \mathcal{M} \to \mathcal{E}$ *is an sg−homeomorphism, and the inverse* $\Delta^{-1}\colon \mathcal{L} \to \mathcal{M}$ *is an sg−homeomorphism.*

Proof. Recall that an sg-homeomorphism is a bijection that is sg-continuous and sg-closed (equivalently sg-open).

The map $\Pi \circ \Delta$ is bijective. For any closed $F \subseteq \mathcal{E}$,

$$(\Pi \circ \Delta)^{-1}(F) = \Delta^{-1}(\Pi^{-1}(F)),$$

and $\Pi^{-1}(F)$ is sg-closed in $\mathcal{L}$ while $\Delta^{-1}(\cdot)$ preserves sg-closed sets; hence $(\Pi \circ \Delta)^{-1}(F)$ is sg-closed in $\mathcal{M}$. Thus $\Pi \circ \Delta$ is sg-continuous. If $A \subseteq \mathcal{M}$ is sg-closed, then $\Delta(A)$ is sg-closed in $\mathcal{L}$ and then $\Pi(\Delta(A))$ is sg-closed in $\mathcal{E}$; hence $\Pi \circ \Delta$ is sg-closed. Therefore $\Pi \circ \Delta$ is an sg-homeomorphism.

Since Δ is bijective, Δ^{-1} exists and is bijective. For any closed $C \subseteq \mathcal{M}$,

$$(\Delta^{-1})^{-1}(C) = \Delta(C),$$

which is sg-closed in $\mathcal{L}$ because Δ is sg-closed; hence Δ^{-1} is sg-continuous. If $B \subseteq \mathcal{L}$ is sg-closed, then

$$\Delta^{-1}(B) = (\Delta^{-1})(B)$$

is sg-closed in $\mathcal{M}$ since Δ is sg-continuous. Thus Δ^{-1} is sg-closed, and so Δ^{-1} is an sg-homeomorphism. □

4. sg−Lindelöf perfect functions

This section investigates sg−Lindelöf perfect functions, We show that every semi-Lindelöf perfect function is also sg−Lindelöf perfect, but the converse is not true, as proved by a counterexample using finite preimage structures.

Definition 4.1. A function $\Delta\colon (\mathcal{M}, \upsilon) \to (\mathcal{L}, \varsigma)$ is termed sg−Lindelöf perfect if Δ satisfies the sg−continuity, sg−closedness, and sg−Lindelöf $\Delta^{-1}(\mathtt{l})$ for every $\mathtt{l} \in \mathcal{L}$.

Lemma 4.2. *Every semi-Lindelöf perfect function inherently qualify as sg-Lindelöf perfect function, but the converse is not true.*

Proof. Consider a semi perfect $\Delta\colon (\mathcal{M}, \upsilon) \to (\mathcal{L}, \varsigma)$. By Definition 2.1, Δ is semi-closed, semi-continuous, semi-Lindelöf $\Delta^{-1}(l)$ for all $l \in \mathcal{L}$. Now, since semi-closed guarantees sg−closed, Δ being semi-closed implies it preserves sg−closed. Also the inverse of sg−closed sets are semi-closed, hence sg−closed. And semi-Lindelöf implies sg−Lindelöf. Thus, Δ is sg−Lindelöf perfect. For the converse, consider the following counterexample. □

Example 4.3. Consider $\mathcal{M} = \{a, b, c\}$ and let $\upsilon_{\text{semi}} = \{\emptyset, \{a\}, \mathcal{M}\}$, $\upsilon_{\text{sg}} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \mathcal{M}\}$. Let $\mathcal{L} = \{1, 2\}$ with ς_{dis}. Define $\Delta\colon \mathcal{M} \to \mathcal{L}$ by:

$$\Delta(\mathsf{m}) = \begin{cases} 1, & \text{if } \mathsf{m} = a, \\ 2, & \text{if } \mathsf{m} \in \{b, c\}. \end{cases}$$

Since $sg-$closed inverse preserved, images of $sg-$closed sets in $\mathcal{M}$ are closed in $\mathcal{L}$, $\Delta^{-1}(1) = \{a\}$ and $\Delta^{-1}(2) = \{b, c\}$ are finite, hence $sg-$Lindelöf. Hence $sg-$Lindelöf perfect. Furthermore, $\{b, c\}$ is $sg-$closed but not semi-closed in υ_{semi}. Thus, Δ is not semi-closed. Therefore, it's not semi-Lindelöf perfect.

Definition 4.4. A function $\Delta\colon (\mathcal{M}, \upsilon) \to (\mathcal{L}, \varsigma)$ is designated as $sg-$irr-Lindelöf perfect, if Δ satisfies; $sg-$irresolute, $sg-$closed, and $sg-$Lindelöf $\Delta^{-1}(\mathtt{l})$ for all $\mathtt{l} \in \mathcal{L}$.

Lemma 4.5. *Every $sg-$irr-Lindelöf-perfect function is $sg-$irresolute, but the converse is not true.*

Proof. ($\Rightarrow$) Direct from Definition 4.4, as $sg-$irr-Lindelöf perfect function explicitly requires $sg-$irresoluteness.
($\Leftarrow$) we give the following example for the convers. □

Example 4.6. Let $\mathcal{M} = \{1, 2, 3\}$ with:

$$\upsilon_{\text{sg}} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \mathcal{M}\},$$

and $\mathcal{L} = \{a, b\}$ with indiscrete $sg-$topology:

$$\varsigma_{\text{sg}} = \{\emptyset, \mathcal{L}\}.$$

Define $\Delta\colon \mathcal{M} \to \mathcal{L}$ by:

$$\Delta(\mathsf{m}) = \begin{cases} a, & \text{if } \mathsf{m} = 1, \\ b, & \text{otherwise.} \end{cases}$$

Since the inverse of the only $sg-$closed set $\emptyset$ and $\mathcal{L}$ are $sg-$closed, Δ is $sg-$irresolute. Also $\Delta(\{3\}) = \{b\}$, which is not $sg-$closed in $\mathcal{L}$. $\Delta^{-1}(b) = \{2, 3\}$ is $sg-$Lindelöf, but Δ fails $sg-$closed. Thus, Δ is $sg-$irresolute but not $sg-$irr-Lindelöf perfect.

Theorem 4.7. *Let $\Delta\colon (\mathcal{M}, \upsilon) \to (\mathcal{L}, \varsigma)$ be an $sg-$perfect function. For any Lindelöf subspace $\mathcal{E} \subseteq \mathcal{L}$, $\Delta^{-1}(\mathcal{E})$ is $sg-$Lindelöf in $\mathcal{M}$.*

Proof. Let $\mathfrak{G} = \{\mathcal{G}_\alpha : \alpha \in \Lambda\}$ convey an $sg-$open cover of $\Delta^{-1}(\mathcal{E})$. Through the $sg-$Lindelöf perfect of Δ, $\Delta^{-1}(\mathtt{l}) \forall\, \mathtt{l} \in \mathcal{E}$ is $sg-$Lindelöf. For any $\mathtt{l} \in \mathcal{E}$, there exists a countable subset $\Lambda_{\mathtt{l}} \subseteq \Lambda$ such that:

$$\Delta^{-1}(\mathtt{l}) \subseteq \bigcup_{\alpha \in \Lambda_{\mathtt{l}}} \mathcal{G}_\alpha.$$

Construct $O_{\mathtt{l}} = \mathcal{L} \setminus \Delta\left(\mathcal{M} \setminus \bigcup_{\alpha \in \Lambda_{\mathtt{l}}} \mathcal{G}_\alpha\right)$. This yields $O_{\mathtt{l}}$ is an $sg-$open neighborhood of $\mathtt{l}$ in $\mathcal{L}$, and $\Delta^{-1}(O_{\mathtt{l}}) \subseteq \bigcup_{\alpha \in \Lambda_{\mathtt{l}}} \mathcal{G}_\alpha$. The collection $\{O_{\mathtt{l}} : \mathtt{l} \in \mathcal{E}\}$ establishes an $sg-$open cover of $\mathcal{E}$.

Since $\mathcal{E}$ is Lindelöf, there exists a countable subset $\{\mathtt{l}_1, \dots, \mathtt{l}_n\} \subseteq \mathcal{E}$ such that:

$$\mathcal{E} \subseteq \bigcup_{i=1}^{n} O_{\mathtt{l}_i}.$$

Consequently,

$$\Delta^{-1}(\mathcal{E}) \subseteq \bigcup_{i=1}^{n} \Delta^{-1}(O_{\mathtt{l}_i}) \subseteq \bigcup_{i=1}^{n} \bigcup_{\alpha \in \Lambda_{\mathtt{l}_i}} \mathcal{G}_\alpha.$$

This generates a countable subcover of $\mathfrak{G}$, indicating that $\Delta^{-1}(\mathcal{E})$ is sg-Lindelöf. □

Theorem 4.8. *Let $\Delta\colon (\mathcal{M}, \upsilon) \xrightarrow{onto} (\mathcal{L}, \varsigma)$ be an sg−irresolutely Lindelöf perfect function. For each sg−Lindelöf subset $\mathcal{E} \subseteq \mathcal{L}$, $\Delta^{-1}(\mathcal{E})$ is sg−Lindelöf in $\mathcal{M}$.*

Proof. Consider an $sg-$open cover $\mathfrak{G} = \{\mathcal{G}_\alpha : \alpha \in \Lambda\}$ of $\Delta^{-1}(\mathcal{E})$. Through the $sg-$irresolute Lindelöf perfect of Δ, $\Delta^{-1}(\mathtt{l})(\mathtt{l} \in \mathcal{E})$ is $sg-$Lindelöf. For every $\mathtt{l} \in \mathcal{E}$, select a countable subset $\Lambda_{\mathtt{l}} \subseteq \Lambda$ satisfying:

$$\Delta^{-1}(\mathtt{l}) \subseteq \bigcup_{\alpha \in \Lambda_{\mathtt{l}}} \mathcal{G}_\alpha.$$

Construct $O_{\mathtt{l}} = \mathcal{L} \setminus \Delta\left(\mathcal{M} \setminus \bigcup_{\alpha \in \Lambda_{\mathtt{l}}} \mathcal{G}_\alpha\right)$. This yields: $O_{\mathtt{l}}$ is $sg-$open of $\mathtt{l}$ in $\mathcal{L}$. By the $sg-$Lindelöfness of $\mathcal{E}$, there exists a finite subset $\{\mathtt{l}_1, \dots, \mathtt{l}_n\} \subseteq \mathcal{E}$ such that:

$$\mathcal{E} \subseteq \bigcup_{i=1}^{n} O_{\mathtt{l}_i}.$$

Consequently

$$\Delta^{-1}(\mathcal{E}) \subseteq \bigcup_{i=1}^{n} \Delta^{-1}(O_{\mathtt{l}_i}) \subseteq \bigcup_{i=1}^{n} \bigcup_{\alpha \in \Lambda_{\mathtt{l}_i}} \mathcal{G}_\alpha.$$

This yields a countable sg-open subcover of $\mathfrak{G}$, then $\Delta^{-1}(\mathcal{E})$ is sg-Lindelöf. □

Proposition 4.9. *If* $\varDelta\colon (\mathcal{M}, \upsilon) \xrightarrow{onto} (\mathcal{L}, \varsigma)$ *is continuous, and sg−irresolute, with* $\mathcal{M}$ *sg−Lindelöf, then* $\mathcal{L}$ *is sg−Lindelöf.*

Proof. Let $\{U_\alpha : \alpha \in \Lambda\}$ be any *sg*-open cover of $\mathcal{L}$. Then $\{\varDelta^{-1}(U_\alpha) : \alpha \in \Lambda\}$ is a family of subsets of $\mathcal{M}$ whose union is $\mathcal{M}$ (since $\varDelta$ is onto).

Because $\varDelta$ is *sg*-irresolute, the preimage of each *sg*-open set is *sg*-open; hence each $\varDelta^{-1}(U_\alpha)$ is *sg*-open in $\mathcal{M}$. Thus $\{\varDelta^{-1}(U_\alpha)\}$ is an *sg*-open cover of $\mathcal{M}$. Since $\mathcal{M}$ is *sg*-Lindelöf, there exists a countable subfamily $\{\alpha_n : n \in \mathbb{N}\} \subseteq \Lambda$ such that

$$\mathcal{M} = \bigcup_{n\in\mathbb{N}} \varDelta^{-1}(U_{\alpha_n}).$$

Applying $\varDelta$ and using surjectivity, we obtain

$$\mathcal{L} = \varDelta(\mathcal{M}) \subseteq \bigcup_{n\in\mathbb{N}} \varDelta\big(\varDelta^{-1}(U_{\alpha_n})\big) \subseteq \bigcup_{n\in\mathbb{N}} U_{\alpha_n}.$$

Hence $\{U_{\alpha_n} : n \in \mathbb{N}\}$ is a countable *sg*-subcover of $\mathcal{L}$. Therefore $\mathcal{L}$ is *sg*-Lindelöf. □

Proposition 4.10. *The composition of two sg−Lindelöf perfect functions need not be sg−Lindelöf perfect.*

Proof. We give the following example for the proof. □

Example 4.11. Let $\mathcal{M} = \{a, b, c\}$, $\mathcal{L} = \{1, 2\}$, and $\mathcal{E} = \{p\}$ with

$$\upsilon_{\mathcal{M}} = \{\emptyset, \{a\}, \{c\}, \{a, c\}, \mathcal{M}\}, \quad \upsilon_{\mathcal{L}} = \{\emptyset, \{1\}, \mathcal{L}\}, \quad \upsilon_{\mathcal{E}} = \{\emptyset, \mathcal{E}\}.$$

Define:

$$\varDelta\colon \mathcal{M} \to \mathcal{L}, \quad \varDelta(a) = \varDelta(c) = 1, \quad \varDelta(b) = 2,$$
$$\varPi\colon \mathcal{L} \to \mathcal{E}, \quad \varPi(1) = \varPi(2) = p.$$

Then $\varDelta$ is *sg*−closed, continuous, and inverse of points in $\mathcal{L}$ is *sg*−Lindelöf in $\mathcal{M}$. So, $\varDelta$ is *sg*−Lindelöf perfect, and also, $\varPi$ is obviously *sg*−Lindelöf perfect.

However, $\varPi \circ \varDelta\colon \mathcal{M} \to \mathcal{E}$ is not *sg*−Lindelöf perfect, because $(\varPi \circ \varDelta)^{-1}(p) = \mathcal{M}$ is *sg*−closed in $\mathcal{M}$, but $\mathcal{M}$ is not *sg*−Lindelöf in this topology. Thus, $\varPi \circ \varDelta$ fails to be *sg*−Lindelöf perfect.

Theorem 4.12. *Let* $\varDelta\colon (\mathcal{M}, \upsilon) \to (\mathcal{L}, \varsigma)$ *and* $\varPi\colon (\mathcal{L}, \varsigma) \to (\mathcal{E}, \rho)$ *be sg−irresolute Lindelöf perfect functions. Then, Their composition* $\varPi \circ \varDelta\colon \mathcal{M} \to \mathcal{E}$ *remains sg−irresolute Lindelöf perfect.*

Proof. Let $\mathfrak{R}$ be an $sg-$open set in $\mathcal{E}$. Since Π is $sg-$irresolute Lindelöf perfect, $\Pi^{-1}(\mathfrak{R})$ is $sg-$open in $\mathcal{L}$. As Δ is $sg-$irresolute Lindelöf perfect, $\Delta^{-1}(\Pi^{-1}(\mathfrak{R})) = (\Pi \circ \Delta)^{-1}(\mathfrak{R})$ is $sg-$open in $\mathcal{M}$. Thus, $\Pi \circ \Delta$ is $sg-$continuous. Since Δ and Π are closed functions, the composition of closed functions is also, closed, so $\Pi \circ \Delta$ is closed. And, $(\Pi \circ \Delta)^{-1}(\mathfrak{e}) = \Delta^{-1}(\Pi^{-1}(\mathfrak{e}))(\mathfrak{e} \in \mathcal{E})$ with $\Pi^{-1}(\mathfrak{e})$ Lindelöf in $\mathcal{L}$ and Δ preserving Lindelöf, $\Delta^{-1}(\Pi^{-1}(\mathfrak{e}))$ is $sg-$Lindelöf in $\mathcal{M}$. Thus, $\Pi \circ \Delta$ is $sg-$irresolute Lindelöf perfect. □

The following propositions demonstrate how $sg-$Lindelöf perfection conveys structural properties between spaces:

Proposition 4.13. *When $\Delta\colon \mathcal{M} \to \mathcal{L}$ is an $sg-$irresolute Lindelöf perfect function and $\mathcal{L}$ is hereditarily $sg-$Lindelöf, $\mathcal{M}$ necessarily inherits hereditary sg-Lindelöf.*

Proof. Let $F \subseteq \mathcal{M}$ be an arbitrary sg-closed subset. We prove that F is sg-Lindelöf.

Since Δ is sg-closed, $\Delta(F)$ is sg-closed in $\mathcal{L}$. By hereditary sg-Lindelöfness of $\mathcal{L}$, the subspace $\Delta(F)$ is sg-Lindelöf. In particular, every open cover of $\Delta(F)$ is an sg-open cover; hence $\Delta(F)$ is Lindelöf.

Now apply Theorem 4.8 to the sg-Lindelöf subset $\mathcal{E} = \Delta(F) \subseteq \mathcal{L}$: since Δ is sg-irresolute Lindelöf perfect, we obtain

$$\Delta^{-1}(\Delta(F)) \text{ is } sg\text{-Lindelöf in } \mathcal{M}.$$

Because $F \subseteq \Delta^{-1}(\Delta(F))$ and F is sg-closed in $\mathcal{M}$, it follows that F is sg-closed in the subspace $\Delta^{-1}(\Delta(F))$. By Lemma 4.24 (every sg-closed subset of an sg-Lindelöf space is sg-Lindelöf), we conclude that F is sg-Lindelöf.

Hence every sg-closed subset of $\mathcal{M}$ is sg-Lindelöf; that is, $\mathcal{M}$ is hereditarily sg-Lindelöf. □

The following propositions demonstrate that $sg-$irresolute Lindelöf perfection persists under Cartesian products:

Proposition 4.14. *Let $\Delta\colon \mathcal{M} \to \mathcal{L}$ and $\Pi\colon \mathfrak{R} \to \mathcal{S}$ be $sg-$irresolute Lindelöf perfect functions. Then the product function $\Delta \times \Pi\colon \mathcal{M} \times \mathfrak{R} \to \mathcal{L} \times \mathcal{S}$ is $sg-$irresolute Lindelöf perfect under the product topology.*

Proof. It suffices to check preimages of basic sg-open rectangles $U \times V$ in $\mathcal{L} \times \mathcal{S}$, where $U \in SG(\varsigma)$ and $V \in SG(\rho)$. Indeed,

$$(\Delta \times \Pi)^{-1}(U \times V) = \Delta^{-1}(U) \times \Pi^{-1}(V),$$

and since Δ, Π are sg-irresolute, $\Delta^{-1}(U)$ and $\Pi^{-1}(V)$ are sg-open; hence their product is sg-open in $\mathcal{M} \times \mathfrak{R}$. Thus $\Delta \times \Pi$ is sg-irresolute.

Both Δ and Π are sg-closed by hypothesis (Lindelöf perfectness). Consequently, the product map $\Delta \times \Pi$ is sg-closed in the product topology: for any

sg-closed $C \subseteq \mathcal{M} \times \mathfrak{R}$, the complement of $(\mathit{\Delta} \times \mathit{\Pi})(C)$ is a union of basic sg-open rectangles obtained by separating points $(\ell, s) \notin (\mathit{\Delta} \times \mathit{\Pi})(C)$ from C via the sg-closedness of $\mathit{\Delta}$ and $\mathit{\Pi}$ (the standard product-closedness argument, identical in form to the classical case).

For any $(\ell, s) \in \mathcal{L} \times \mathcal{S}$,

$$(\mathit{\Delta} \times \mathit{\Pi})^{-1}(\ell, s) = \mathit{\Delta}^{-1}(\ell) \times \mathit{\Pi}^{-1}(s).$$

Since $\mathit{\Delta}$ and $\mathit{\Pi}$ are sg-irresolute Lindelöf perfect, the fibers $\mathit{\Delta}^{-1}(\ell)$ and $\mathit{\Pi}^{-1}(s)$ are sg-Lindelöf. A finite product of sg-Lindelöf spaces is sg-Lindelöf (in the product topology), hence $(\mathit{\Delta} \times \mathit{\Pi})^{-1}(\ell, s)$ is sg-Lindelöf.

Thus, $\mathit{\Delta} \times \mathit{\Pi}$ is sg-irresolute Lindelöf perfect. □

Proposition 4.15. *Any restriction $\mathit{\Delta}|_{\mathfrak{R}} \colon \mathfrak{R} \to \mathcal{L}$ of sg−irresolute Lindelöf perfect function $\mathit{\Delta} \colon \mathcal{M} \to \mathcal{L}$ to an sg−closed $\mathfrak{R} \subseteq \mathcal{M}$, is sg−irresolute Lindelöf perfect.*

Proof. For any sg-closed $B \subseteq \mathcal{L}$,

$$(\mathit{\Delta}|_{\mathfrak{R}})^{-1}(B) = \mathfrak{R} \cap \mathit{\Delta}^{-1}(B).$$

Since $\mathit{\Delta}$ is sg-irresolute, $\mathit{\Delta}^{-1}(B)$ is sg-closed in $\mathcal{M}$; hence the intersection is sg-closed in the subspace $\mathfrak{R}$. Thus $\mathit{\Delta}|_{\mathfrak{R}}$ is sg-irresolute.

If $A \subseteq \mathfrak{R}$ is sg-closed in $\mathfrak{R}$, then $A = \mathfrak{R} \cap C$ for some sg-closed $C \subseteq \mathcal{M}$. Since $\mathit{\Delta}$ is sg-closed, $\mathit{\Delta}(C)$ is sg-closed in $\mathcal{L}$, and

$$(\mathit{\Delta}|_{\mathfrak{R}})(A) = \mathit{\Delta}(A) \subseteq \mathit{\Delta}(C),$$

so $(\mathit{\Delta}|_{\mathfrak{R}})(A)$ is sg-closed (in particular, in the subspace $\mathit{\Delta}(\mathfrak{R})$). Hence $\mathit{\Delta}|_{\mathfrak{R}}$ is sg-closed.

For $\ell \in \mathcal{L}$,

$$(\mathit{\Delta}|_{\mathfrak{R}})^{-1}(\ell) = \mathfrak{R} \cap \mathit{\Delta}^{-1}(\ell).$$

Because $\mathit{\Delta}$ is Lindelöf perfect, $\mathit{\Delta}^{-1}(\ell)$ is sg-Lindelöf in $\mathcal{M}$. Since $\mathfrak{R}$ is sg-closed, the intersection is sg-closed in the sg-Lindelöf space $\mathit{\Delta}^{-1}(\ell)$; by Lemma 4.24, it is sg-Lindelöf. Therefore every fiber of $\mathit{\Delta}|_{\mathfrak{R}}$ is sg-Lindelöf.

Thus $\mathit{\Delta}|_{\mathfrak{R}}$ is sg-irresolute, sg-closed, and has sg-Lindelöf point-preimages; hence it is sg-irresolute Lindelöf perfect. □

Theorem 4.16. *Consider the copmosition $\mathit{\Pi} \circ \mathit{\Delta} \colon \mathcal{M} \to \mathcal{E}$ of sg−irresolute Lindelöf perfect function $\mathit{\Delta} \colon (\mathcal{M}, \upsilon) \to (\mathcal{L}, \varsigma)$ and sg−Lindelöf perfect function $\mathit{\Pi} \colon (\mathcal{L}, \varsigma) \to (\mathcal{E}, \rho)$. Then the composition is sg−Lindelöf perfect.*

Proof. Let $\mathfrak{R}$ be an open set in $\mathcal{E}$. Since Π is $sg-$Lindelöf perfect, it is continuous, so $\Pi^{-1}(\mathfrak{R})$ is open in $\mathcal{L}$. As Δ is $sg-$irresolute Lindelöf perfect, hence $sg-$continuous, $\Delta^{-1}(\Pi^{-1}(\mathfrak{R})) = (\Pi \circ \Delta)^{-1}(\mathfrak{R})$ is $sg-$open in $\mathcal{M}$. Thus, $\Pi \circ \Delta$ is $sg-$continuous. Since, Δ and Π are closed functions, the composition of closed functions is closed, so $\Pi \circ \Delta$ is closed.

Since Π is $sg-$Lindelöf perfect, $\Pi^{-1}(\mathfrak{e})$ is Lindelöf in $\mathcal{L}$. As Δ is $sg-$irresolute Lindelöf perfect, $\Delta^{-1}(\Pi^{-1}(\mathfrak{e}))$ is Lindelöf in $\mathcal{M}$. Since, $(\Pi \circ \Delta)^{-1}(\mathfrak{e}) = \Delta^{-1}(\Pi^{-1}(\mathfrak{e}))(\mathfrak{e} \in \mathcal{E})$, $\Pi \circ \Delta$ is $sg-$Lindelöf perfect. □

Proposition 4.17. *Given $\Delta\colon \mathcal{M} \to \mathcal{L}$ and $\Pi\colon \mathfrak{R} \to \mathcal{S}$ is $sg-$Lindelöf perfect functions. $\Delta \times \Pi\colon \mathcal{M} \times A \to \mathcal{L} \times \mathcal{S}$ is $sg-$Lindelöf perfect under the product topology.*

Proof. For a basic open rectangle $U \times V \subseteq \mathcal{L} \times \mathcal{S}$,

$$(\Delta \times \Pi)^{-1}(U \times V) = \Delta^{-1}(U) \times \Pi^{-1}(V),$$

which is open (hence sg-open) in $\mathcal{M} \times \mathfrak{R}$ since Δ, Π are continuous. Thus $\Delta \times \Pi$ is sg-continuous.

Since Δ and Π are sg-closed, the standard product argument yields that $\Delta \times \Pi$ is sg-closed: points outside the image of an sg-closed set admit separating basic sg-open rectangles obtained from the sg-closedness of the factors.

For $(\ell, s) \in \mathcal{L} \times \mathcal{S}$,

$$(\Delta \times \Pi)^{-1}(\ell, s) = \Delta^{-1}(\ell) \times \Pi^{-1}(s).$$

Each factor is sg-Lindelöf by hypothesis, and a finite product of sg-Lindelöf spaces is sg-Lindelöf in the product topology. Hence every point fiber of $\Delta \times \Pi$ is sg-Lindelöf.

Therefore $\Delta \times \Pi$ is sg-continuous, sg-closed, and has sg-Lindelöf point preimages; i.e., it is sg-Lindelöf perfect. □

Next, we discover an important inheritance characteristic using surjective compositions.

Theorem 4.18. *Let $\Delta\colon (\mathcal{M}, \upsilon) \overset{onto}{\to} (\mathcal{L}, \varsigma)$ be $sg-$continuous and $\Pi\colon (\mathcal{L}, \varsigma) \overset{onto}{\to} (\mathcal{E}, \rho)$ be sg-irresolute. If $\Pi \circ \Delta$ exhibits $sg-$Lindelöf perfect, then Π inherits $sg-$perfect.*

Proof. For each $\mathfrak{e} \in \mathcal{E}$, $\Pi^{-1}(\mathfrak{e}) = \Delta\big((\Pi \circ \Delta)^{-1}(\mathfrak{e})\big)$. Since $\Pi \circ \Delta$ is $sg-$Lindelöf perfect, $(\Pi \circ \Delta)^{-1}(\mathfrak{e})$ is $sg-$Lindelöf in $\mathcal{M}$. Through Δ is $sg-$continuous, $\Delta\big((g \circ \Delta)^{-1}(\mathfrak{e})\big) = \Pi^{-1}(\mathfrak{e})$ is $sg-$Lindelöf in $\mathcal{L}$. By Proposition 3.18, Π is $sg-$closed. Thus, Π is $sg-$Lindelöf perfect. □

Theorem 4.19. *Let* $\Delta\colon (\mathcal{M}, \upsilon) \overset{onto}{\to} (\mathcal{L}, \varsigma)$ *be* $sg-$*Lindelöf perfect. For any* $\mathcal{S} \subseteq \mathcal{L}$*, the restriction* $\Delta_{\mathcal{S}}\colon \Delta^{-1}(\mathcal{S}) \to \mathcal{S}$ *is* $sg-$*Lindelöf perfect.*

Proof. By Theorem 3.22, $\Delta_{\mathcal{S}}$ is $sg-$closed. Since Δ is $sg-$Lindelöf perfect, preimages of points in $\mathcal{S}$ are $sg-$Lindelöf in $\Delta^{-1}(\mathcal{S})$. The subspace topology on $\Delta^{-1}(\mathcal{S})$ preserves $sg-$Lindelöf, so $\Delta_{\mathcal{S}}$ is $sg-$Lindelöf perfect. □

Proposition 4.20. *If* $\mathcal{L}$ *is hereditarily sg-Lindelöf and* $\Delta\colon \mathcal{M} \to \mathcal{L}$ *is sg-Lindelöf perfect, then* $\mathcal{M}$ *is hereditarily sg-Lindelöf.*

Proof. Let $F \subseteq \mathcal{M}$ be an arbitrary sg-closed set. We show that F is sg-Lindelöf.

Since Δ is sg-closed, $\Delta(F)$ is sg-closed in $\mathcal{L}$. By the hereditary sg-Lindelöfness of $\mathcal{L}$, the subspace $\Delta(F)$ is sg-Lindelöf; in particular, $\Delta(F)$ is Lindelöf.

Now apply Theorem 4.8 to the Lindelöf subset $\mathcal{E} = \Delta(F) \subseteq \mathcal{L}$: because Δ is sg-Lindelöf perfect, we obtain that

$$\Delta^{-1}(\mathcal{E}) = \Delta^{-1}(\Delta(F)) \quad \text{is } sg\text{-Lindelöf in } \mathcal{M}.$$

Clearly $F \subseteq \Delta^{-1}(\Delta(F))$. Moreover, F is sg-closed in $\mathcal{M}$, hence it is sg-closed in the subspace $\Delta^{-1}(\Delta(F))$. By Lemma 4.24 (every sg-closed subset of an sg-Lindelöf space is sg-Lindelöf), it follows that F is sg-Lindelöf.

Since F was arbitrary, every sg-closed subset of $\mathcal{M}$ is sg-Lindelöf; i.e. $\mathcal{M}$ is hereditarily sg-Lindelöf. □

Proposition 4.21. *Let* $i\colon A \hookrightarrow \mathcal{M}$ *be an embedding and* $\Delta\colon \mathcal{M} \to \mathcal{L}$ *be* $sg-$*Lindelöf perfect. Then* $\Delta \circ i\colon A \to \mathcal{L}$ *is* $sg-$*Lindelöf perfect.*

Proof. Since i is an embedding, it is injective and continuous, and the topology on A is the subspace topology induced from $\mathcal{M}$.

Because Δ is sg-Lindelöf perfect, it is continuous; hence the composition $\Delta \circ i$ is continuous, and therefore sg-continuous.

Let $F \subseteq A$ be sg-closed in A. Since A carries the subspace topology, there exists a sg-closed set $C \subseteq \mathcal{M}$ such that $F = A \cap C$. Then

$$(\Delta \circ i)(F) = \Delta(i(F)) = \Delta(F) \subseteq \Delta(C).$$

As Δ is sg-closed, $\Delta(C)$ is sg-closed in $\mathcal{L}$; hence $(\Delta \circ i)(F)$ is sg-closed (in particular as a subset of $\mathcal{L}$, or equivalently as a closed set in the subspace $\Delta(A)$). Thus $\Delta \circ i$ is sg-closed in the standard restricted-map sense.

For any $\ell \in \mathcal{L}$,

$$(\Delta \circ i)^{-1}(\ell) = \{a \in A : \Delta(i(a)) = \ell\} = A \cap \Delta^{-1}(\ell).$$

Since Δ is sg-Lindelöf perfect, $\Delta^{-1}(\ell)$ is sg-Lindelöf in $\mathcal{M}$. The set $A \cap \Delta^{-1}(\ell)$ is a subspace of $\Delta^{-1}(\ell)$; hence it is Lindelöf, and therefore sg-Lindelöf.

Consequently, $\Delta \circ i$ is sg-continuous, sg-closed, and has sg-Lindelöf point-preimages; hence it is sg-Lindelöf perfect. □

Theorem 4.22. *Let $\Delta\colon (\mathcal{M}, \upsilon) \overset{onto}{\to} (\mathcal{L}, \varsigma)$ be an $sg-$irresolute Lindelöf perfect function. Then, for any $\mathcal{S} \subseteq \mathcal{L}$, the restriction $\Delta_{\mathcal{S}}\colon \Delta^{-1}(\mathcal{S}) \to \mathcal{S}$ is $sg-$irresolute Lindelöf perfect.*

Proof. In a similar way to Theorem 3.26, and follows from the preservation of sg-continuity, closedness, and Lindelöf inverse under subspace restrictions. □

Theorem 4.23. *For an $sg-$Lindelöf perfect function $\Delta\colon (\mathcal{M}, \upsilon) \to (\mathcal{L}, \varsigma)$, if $\mathcal{M}$ is $sg-$Lindelöf and $\mathcal{L}$ is $sg-$Hausdorff, then Δ is $sg-$closed.*

Proof. For any $sg-$closed $\mathfrak{R} \subseteq \mathcal{M}$, $\mathfrak{R}$ is Lindelöf. Since $\mathcal{L}$ is $sg-$Hausdorff, $\Delta(\mathfrak{R})$ is closed. □

Lemma 4.24. *In sg-Lindelöf space $(\mathcal{M}, \upsilon)$, every sg-closed subset is sg-Lindelöf.*

Proof. Let $F \subseteq \mathcal{M}$ be sg-closed, and let $\{U_\alpha : \alpha \in \Lambda\}$ be an sg-open cover of F (in the subspace topology). For each α, choose an sg-open set $V_\alpha \subseteq \mathcal{M}$ such that $U_\alpha = F \cap V_\alpha$. Since F is sg-closed, $\mathcal{M} \setminus F$ is sg-open, and hence

$$\{V_\alpha : \alpha \in \Lambda\} \ \cup \ \{\mathcal{M} \setminus F\}$$

is an sg-open cover of $\mathcal{M}$. Because $\mathcal{M}$ is sg-Lindelöf, there exists a countable subfamily $\{V_{\alpha_n} : n \in \mathbb{N}\}$ such that

$$\mathcal{M} = \Big(\bigcup_{n \in \mathbb{N}} V_{\alpha_n} \Big) \ \cup \ (\mathcal{M} \setminus F).$$

Intersecting with F yields

$$F = F \cap \mathcal{M} \subseteq \bigcup_{n \in \mathbb{N}} (F \cap V_{\alpha_n}) = \bigcup_{n \in \mathbb{N}} U_{\alpha_n}.$$

Thus $\{U_{\alpha_n} : n \in \mathbb{N}\}$ is a countable sg-subcover of F, proving that F is sg-Lindelöf. □

Proposition 4.25. *sg-Hausdorff spaces $(\mathcal{L}, \varsigma)$ contain all sg-Lindelöf subsets as closed sets.*

Proof. Let $A \subseteq \mathcal{L}$ be sg-Lindelöf and fix $y \in \mathcal{L} \setminus A$. For each $x \in A$, since $\mathcal{L}$ is sg-Hausdorff, choose disjoint sg-open sets U_x, V_x such that $x \in U_x$ and $y \in V_x$ (hence $U_x \cap V_x = \varnothing$). Then $\{U_x : x \in A\}$ is an sg-open cover of A. By the sg-Lindelöf property, there exists a countable subcover $\{U_{x_n} : n \in \mathbb{N}\}$ of A.

Set

$$V := \bigcap_{n\in\mathbb{N}} V_{x_n}.$$

By the standing assumption, V is sg-open; clearly $y \in V$. Moreover, for each n, we have $V \cap U_{x_n} = \varnothing$, hence

$$V \cap A \subseteq V \cap \bigcup_{n\in\mathbb{N}} U_{x_n} = \varnothing.$$

Thus, for every $y \notin A$ there exists an sg-open neighborhood V of y disjoint from A, i.e. $\mathcal{L} \setminus A$ is sg-open. Therefore A is sg-closed. □

Theorem 4.26. *For a bijective sg−continuous function* $\Delta\colon (\mathcal{M}, \upsilon) \to (\mathcal{L}, \varsigma)$*, if* $\mathcal{M}$ *is sg−Lindelöf and* $\mathcal{L}$ *is sg−Hausdorff, then* Δ *is an sg−homeomorphism.*

Proof. Let $F \subseteq \mathcal{M}$ be an any $sg-$closed set. Since $\mathcal{M}$ is $sg-$Lindelöf, F is $sg-$Lindelöf from Lemma 4.24. As Δ is $sg-$continuous, $\Delta(F)$ is $sg-$Lindelöf in $\mathcal{L}$. Because $\mathcal{L}$ is $sg-$Hausdorff, $\Delta(F)$ is $sg-$closed from Proposition 4.25. Thus Δ is a function from $sg-$closed sets to $sg-$closed sets, leading in a $sg-$closed function. By incorporating bijective and $sg-$continuous, Δ is an $sg-$homeomorphism. □

5. **Structural properties and separation axioms for sg-perfect functions**

This section investigates sg-strong functions and their critical significance in maintaining structural features in generalized topological spaces.

Definition 5.1. A function $\Delta\colon (\mathcal{M}, \upsilon) \to (\mathcal{L}, \varsigma)$ is called $sg-$strong if for every $sg-$open cover $\mathfrak{G} = \{\mathcal{G}_\alpha\}_{\alpha\in\Lambda}$ of $\mathcal{M}$, there exists an $sg-$open cover $\mathfrak{K} = \{\mathcal{K}_\gamma\}_{\gamma\in\Gamma}$ of $\mathcal{L}$ such that, for all $\mathcal{K}_\gamma \in \mathfrak{K}$, there exists finite $\Lambda_1 \subset \Lambda$,

$$\Delta^{-1}(\mathcal{K}_\gamma) \subseteq \bigcup_{\alpha\in\Lambda_1} \mathcal{G}_\alpha.$$

Theorem 5.2. *Let* $\Delta\colon (\mathcal{M}, \upsilon) \xrightarrow{onto} (\mathcal{L}, \varsigma)$ *be sg−strong. If* $\mathcal{L}$ *is sg−compact, then* $\mathcal{M}$ *is sg−compact.*

Proof. Let $\mathfrak{G} = \{\mathcal{G}_\alpha\}_{\alpha\in\Lambda}$ be an $sg-$open cover of $\mathcal{M}$. By the $sg-$strong property, there exists an $sg-$open cover $\mathfrak{K} = \{\mathcal{K}_\gamma\}_{\gamma\in\Gamma}$ of $\mathcal{L}$ with, for all $\gamma \in \Gamma$, there exists finite, $\Lambda_\gamma \subset \Lambda$,

$$\Delta^{-1}(\mathcal{K}_\gamma) \subseteq \bigcup_{\alpha\in\Lambda_\gamma} \mathcal{G}_\alpha.$$

Since $\mathcal{L}$ is $sg-$compact, there exists finite $\Gamma_1 \subset \Gamma$ with, for $\mathcal{L} = \bigcup_{\gamma\in\Gamma_1} \mathcal{K}_\gamma$,

$$\mathcal{M} = \bigcup_{\gamma\in\Gamma_1} \Delta^{-1}(\mathcal{K}_\gamma).$$

Each $\Delta^{-1}(\mathcal{K}_\gamma)$ is covered by finite $\{\mathcal{G}_\alpha\}_{\alpha\in\Lambda_\gamma}$, so,

$$\mathcal{M} = \bigcup_{\gamma\in\Gamma_1} \bigcup_{\alpha\in\Lambda_\gamma} \mathcal{G}_\alpha.$$

This delivers a finite subcover of $\mathfrak{G}$. Thus $\mathcal{M}$ is $sg-$compact. □

Proposition 5.3. *If $\Delta\colon \mathcal{M} \to \mathcal{L}$ is $sg-$strong and $\mathcal{L}$ is $sg-$compact, then $\mathcal{M} \times \mathcal{E}$ is $sg-$compact for any $sg-$compact space $\mathcal{E}$.*

Proof. Since $\Delta : \mathcal{M} \to \mathcal{L}$ is sg-strong and $\mathcal{L}$ is sg-compact, Theorem 5.2 implies that $\mathcal{M}$ is sg-compact.

Now let $\{W_\alpha : \alpha \in \Lambda\}$ be an sg-open cover of $\mathcal{M}\times\mathcal{E}$. Fix $m \in \mathcal{M}$. For each $e \in \mathcal{E}$ choose $\alpha(e)$ with $(m, e) \in W_{\alpha(e)}$, and pick a basic product neighborhood $U_e \times V_e$ such that

$$(m, e) \in U_e \times V_e \subseteq W_{\alpha(e)}$$

with U_e sg-open in $\mathcal{M}$ and V_e sg-open in $\mathcal{E}$. Then $\{V_e : e \in \mathcal{E}\}$ is an sg-open cover of $\mathcal{E}$; by sg-compactness of $\mathcal{E}$ choose $e_1, \ldots, e_n$ with $\mathcal{E} \subseteq \bigcup_{i=1}^n V_{e_i}$. Put $U_m = \bigcap_{i=1}^n U_{e_i}$ (still sg-open). Then

$$U_m \times \mathcal{E} \subseteq \bigcup_{i=1}^{n} W_{\alpha(e_i)}.$$

Thus $\{U_m : m \in \mathcal{M}\}$ is an sg-open cover of $\mathcal{M}$. By sg-compactness of $\mathcal{M}$, pick $m_1, \ldots, m_k$ with $\mathcal{M} \subseteq \bigcup_{j=1}^k U_{m_j}$. Consequently,

$$\mathcal{M} \times \mathcal{E} = \bigcup_{j=1}^{k} (U_{m_j} \times \mathcal{E}) \subseteq \bigcup_{j=1}^{k} \bigcup_{i=1}^{n(j)} W_{\alpha(e_i^{(j)})},$$

a finite subcover of the original cover. Hence $\mathcal{M} \times \mathcal{E}$ is sg-compact. □

Lemma 5.4. *Every $sg-$strong function preserves locally finite covers through preimages.*

Proof. Fix $m \in \mathcal{M}$ and put $\ell = \Delta(m)$. Since $\{A_\gamma\}_{\gamma\in\Gamma}$ is locally finite in $\mathcal{L}$, there exists an sg-open neighborhood $V \in SG(\varsigma)$ of ℓ such that $V \cap A_\gamma \neq \emptyset$ for only finitely many indices γ.

Because sg-strong maps are taken to be sg-continuous with respect to sg-open neighborhoods, the preimage $U = \Delta^{-1}(V)$ is an sg-open neighborhood of m in $\mathcal{M}$. Moreover, for each γ,

$$U \cap \Delta^{-1}(A_\gamma) = \Delta^{-1}(V) \cap \Delta^{-1}(A_\gamma) = \Delta^{-1}(V \cap A_\gamma).$$

Hence $U \cap \Delta^{-1}(A_\gamma) \neq \emptyset$ implies $V \cap A_\gamma \neq \emptyset$, and therefore this can occur for only finitely many γ. This proves that the family $\{\Delta^{-1}(A_\gamma)\}$ is locally finite in $\mathcal{M}$. □

Theorem 5.5. *Let $\Delta\colon (\mathcal{M}, \upsilon) \xrightarrow{onto} (\mathcal{L}, \varsigma)$ be sg−strong. If $\mathcal{L}$ is sg−Lindelöf, then $\mathcal{M}$ is sg−Lindelöf.*

Proof. Let $\mathfrak{G} = \{\mathcal{G}_\alpha\}_{\alpha\in\Lambda}$ be an $sg-$open cover of $\mathcal{M}$. By the $sg-$strong property, there exists an $sg-$open cover $\mathfrak{K} = \{\mathcal{K}_\gamma\}_{\gamma\in\Gamma}$ of $\mathcal{L}$ with, for all $\gamma \in \Gamma$, there exists countable $\Lambda_\gamma \subset \Lambda$, such that

$$\Delta^{-1}(\mathcal{K}_\gamma) \subseteq \bigcup_{\alpha\in\Lambda_\gamma} \mathcal{G}_\alpha.$$

Since $\mathcal{L}$ is $sg-$Lindelöf, there exists countable $\Gamma_1 \subset \Gamma$ with $\mathcal{L} = \bigcup_{\gamma\in\Gamma_1} \mathcal{K}_\gamma$ such that

$$\mathcal{M} = \bigcup_{\gamma\in\Gamma_1} \Delta^{-1}(\mathcal{K}_\gamma).$$

Each $\Delta^{-1}(\mathcal{K}_\gamma)$ is covered by countable $\{\mathcal{G}_\alpha\}_{\alpha\in\Lambda_\gamma}$, so,

$$\mathcal{M} = \bigcup_{\gamma\in\Gamma_1} \bigcup_{\alpha\in\Lambda_\gamma} \mathcal{G}_\alpha$$

providing a countable subcover of $\mathfrak{G}$. Thus $\mathcal{M}$ is $sg-$Lindelöf. □

Proposition 5.6. *For sg−strong $\Delta\colon \mathcal{M} \to \mathcal{L}$ and sg−Lindelöf $\mathcal{L}$, $\mathcal{M} \times \mathcal{E}$ is sg−Lindelöf for all sg−Lindelöf $\mathcal{E}$.*

The combination of perfection and separation axioms yields substantial structural results.

Theorem 5.7. *Let $\Delta\colon (\mathcal{M}, \upsilon) \xrightarrow{onto} (\mathcal{L}, \varsigma)$ be sg−perfect. If $\mathcal{M}$ is sg−Hausdorff, then $\mathcal{L}$ is sg−Hausdorff.*

Proof. For $\mathtt{l}_1 \neq \mathtt{l}_2$ in $\mathcal{L}$. Since Δ is $sg-$perfect, $\Delta^{-1}(\mathtt{l}_1)$ and $\Delta^{-1}(\mathtt{l}_2)$ are disjoint $sg-$compact sets. $sg-$Hausdorff of $\mathcal{M}$ gives disjoint $sg-$open neighborhoods:

$$\mathcal{G} \supseteq \Delta^{-1}(\mathtt{l}_1),\ V \supseteq \Delta^{-1}(\mathtt{l}_2),\ \mathcal{G} \cap \mathcal{K} = \emptyset.$$

As Δ is $sg-$closed, construct $sg-$open sets in $\mathcal{L}$:

$$W_1 = \mathcal{L} \setminus \Delta(\mathcal{M} \setminus \mathcal{G}),\ W_2 = \mathcal{L} \setminus \Delta(\mathcal{M} \setminus \mathcal{K}),$$

$\mathtt{l}_i \in W_i$ and $W_1 \cap W_2 = \emptyset$. Thus $\mathcal{L}$ is $sg-$Hausdorff. □

Proposition 5.8. *A finite product of sg−Hausdorff spaces is sg−Hausdorff.*

Proof. It suffices to consider two factors. Let $(\mathcal{M}_1, \upsilon_1)$ and $(\mathcal{M}_2, \upsilon_2)$ be *sg*-Hausdorff and take $(\mathsf{m}_1, \mathsf{m}_2) \neq (\mathsf{n}_1, \mathsf{n}_2)$ in $\mathcal{M}_1 \times \mathcal{M}_2$. Then either $\mathsf{m}_1 \neq \mathsf{n}_1$ or $\mathsf{m}_2 \neq \mathsf{n}_2$. If $\mathsf{m}_1 \neq \mathsf{n}_1$, choose disjoint $\mathcal{G}_1, \mathcal{K}_1 \in SG(\upsilon_1)$ with $\mathsf{m}_1 \in \mathcal{G}_1$, $\mathsf{n}_1 \in \mathcal{K}_1$; then $\mathcal{G}_1 \times \mathcal{M}_2$ and $\mathcal{K}_1 \times \mathcal{M}_2$ are disjoint *sg*-open neighborhoods of the two points. Otherwise choose disjoint $\mathcal{G}_2, \mathcal{K}_2 \in SG(\upsilon_2)$ separating m_2 and n_2, and use $\mathcal{M}_1 \times \mathcal{G}_2$ and $\mathcal{M}_1 \times \mathcal{K}_2$. Thus $\mathcal{M}_1 \times \mathcal{M}_2$ is *sg*-Hausdorff, and the finite case follows by induction. □

Proposition 5.9. *Every sg−Hausdorff space is Hausdorff, but the converse is not true.*

Proof. Let $(\mathcal{M}, \upsilon)$ be a Hausdorff space and let $\mathsf{m}_1 \neq \mathsf{m}_2$ in $\mathcal{M}$. Then there exist disjoint open sets $\mathcal{G}, \mathcal{K} \in \upsilon$ such that $\mathsf{m}_1 \in \mathcal{G}$ and $\mathsf{m}_2 \in \mathcal{K}$. Since every open set is semi-open, and every semi-open set is *sg*-open, we have $\mathcal{G}, \mathcal{K} \in SG(\upsilon)$. Hence $(\mathcal{M}, \upsilon)$ is *sg*-Hausdorff.

To show that the converse fails, consider $\mathcal{M} = \{a, b, c\}$ with

$$\upsilon = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \mathcal{M}\}.$$

Then $(\mathcal{M}, \upsilon)$ is not Hausdorff: the only open neighborhood of c is $\mathcal{M}$, so c cannot be separated from a (nor from b) by disjoint open sets.

On the other hand, $(\mathcal{M}, \upsilon)$ is *sg*-Hausdorff. Indeed, $\{a, c\}$ is semi-open, since $\mathrm{Int}(\{a, c\}) = \{a\}$ and $\mathrm{Cl}(\{a\}) = \{a, c\}$, so $\{a, c\} \subseteq \mathrm{Cl}(\mathrm{Int}(\{a, c\}))$; similarly $\{b, c\}$ is semi-open. Hence $\{a, c\}, \{b, c\} \in SG(\upsilon)$. Therefore, the following disjoint *sg*-open separations exist:

$$a \text{ and } b: \ \{a\} \cap \{b\} = \emptyset, \quad a \text{ and } c: \ \{a\} \cap \{b, c\} = \emptyset, \quad b \text{ and } c: \ \{b\} \cap \{a, c\} = \emptyset.$$

Thus $(\mathcal{M}, \upsilon)$ is *sg*-Hausdorff but not Hausdorff. □

Remark 5.10. For *sg*−perfect $\varDelta \colon \mathcal{M} \to \mathcal{L}$ with $\mathcal{L}$ is *sg*−Hausdorff, then $\mathcal{M}$ is *sg*−Hausdorff.

Remark 5.11. In *sg*−topology:

$$sg{-}\mathrm{T}_2 \subsetneq sg{-}\mathrm{T}_1 \subsetneq sg{-}\mathrm{T}_0$$

with *sg*−Hausdorff (*sg*−T_2) being the strongest.

The interaction between Lindelöf perfect and separation axioms yields fundamental structural preservation.

Theorem 5.12. *Let* $\varDelta \colon (\mathcal{M}, \upsilon) \xrightarrow{onto} (\mathcal{L}, \varsigma)$ *be sg−Lindelöf perfect. If* $\mathcal{M}$ *is sg−Hausdorff, then* $\mathcal{L}$ *is sg−Hausdorff.*

Proof. Let $\mathtt{l}_1 \neq \mathtt{l}_2$ in $\mathcal{L}$. Since Δ is $sg-$Lindelöf perfect, $\Delta^{-1}(\mathtt{l}_1)$ and $\Delta^{-1}(\mathtt{l}_2)$ are disjoint $sg-$Lindelöf subsets of $\mathcal{M}$. $sg-$Hausdorff of $\mathcal{M}$ gives disjoint $sg-$open neighborhoods:

$$\mathcal{G} \supseteq \Delta^{-1}(\mathtt{l}_1),\ \mathcal{K} \supseteq \Delta^{-1}(\mathtt{l}_2),\ \mathcal{G} \cap \mathcal{K} = \emptyset.$$

As Δ is $sg-$closed, construct $sg-$open sets in $\mathcal{L}$ such that

$$W_1 = \mathcal{L} \setminus \Delta(\mathcal{M} \setminus \mathcal{G}),\ W_2 = \mathcal{L} \setminus \Delta(\mathcal{M} \setminus \mathcal{K}),$$

$\mathtt{l}_i \in W_i$ and $W_1 \cap W_2 = \emptyset$. Thus $\mathcal{L}$ is $sg-$Hausdorff. □

Remark 5.13. For $sg-$Lindelöf perfect $\Delta\colon \mathcal{M} \to \mathcal{L}$ with $\mathcal{L}$ is $sg-$Hausdorff, then $\mathcal{M}$ is $sg-$Hausdorff.

Definition 5.14. A topological space $(\mathcal{M}, \upsilon)$ is $sg-$regular if for every point $\mathsf{m} \in \mathcal{M}$ and $sg-$closed set $F \subseteq \mathcal{M}$ with $\mathsf{m} \notin F$, there exist disjoint $sg-$open sets $\mathcal{G}, \mathcal{K} \in SG(\upsilon)$ such that

$$\mathsf{m} \in \mathcal{G},\ F \subseteq \mathcal{K},\ \mathcal{G} \cap \mathcal{K} = \emptyset.$$

Lemma 5.15. *Let (X, υ) be an $sg-$regular space. Then for any $sg-$compact $\mathfrak{R} \subseteq X$ and $sg-$open neighborhood $\mathcal{G}$ of $\mathfrak{R}$, there exists an $sg-$open W such that*

$$\mathfrak{R} \subseteq W \subseteq s\ Cl\ (W) \subseteq \mathcal{G}.$$

Proof. For each $\mathsf{r} \in \mathfrak{R}$, $sg-$regular gives $sg-$open $\mathcal{K}(\mathsf{r})$ with

$$\mathsf{r} \in \mathcal{K}(\mathsf{r}) \subseteq sCl\mathcal{K}(\mathsf{r}) \subseteq \mathcal{G}.$$

The collection $\{\mathcal{K}(\mathsf{r})\}_{\mathsf{r}\in\mathfrak{R}}$ forms an $sg-$open cover of $\mathfrak{R}$, now by $sg-$compact, extract finite subcover $\{\mathcal{K}(\mathsf{r}_1), \ldots, \mathcal{K}(\mathsf{r}_n)\}$. Let $W = \bigcup_{i=1}^{n} \mathcal{K}(\mathsf{r}_i)$. Then,

$$sClW = sCl\bigcup_{i=1}^{n} \mathcal{K}(\mathsf{r}_i) = \bigcup_{i=1}^{n} sCl\mathcal{K}(\mathsf{r}_i) \subseteq \mathcal{G}.$$

□

Lemma 5.16. *Let (X, υ) be an $sg-$regular space. Then for $sg-$Lindelöf $\mathfrak{R} \subseteq \mathcal{M}$ and $sg-$open neighborhood $\mathcal{G}$ of $\mathfrak{R}$, there exists $sg-$open W such that*

$$\mathfrak{R} \subseteq W \subseteq s\ Cl\ (W) \subseteq \mathcal{G}.$$

Proof. For each $\mathsf{r} \in \mathfrak{R}$, $sg-$regular gives $sg-$open $\mathcal{K}(\mathsf{r})$ with

$$\mathsf{r} \in \mathcal{K}(\mathsf{r}) \subseteq sCl\mathcal{K}(\mathsf{r}) \subseteq \mathcal{G}.$$

The collection $\{\mathcal{K}(\mathsf{r})\}_{\mathsf{r}\in\mathfrak{R}}$ forms an $sg-$open cover of $\mathfrak{R}$, now by $sg-$Lindelöf, extract countable subcover $\{\mathcal{K}(\mathsf{r}_1), \ldots, \mathcal{K}(\mathsf{r}_n)\}$. Let $W = \bigcup_{i=1}^{n} \mathcal{K}(\mathsf{r}_i)$. Then,

$$sClW = sCl\bigcup_{i=1}^{n} \mathcal{K}(\mathsf{r}_i) = \bigcup_{i=1}^{n} sCl\mathcal{K}(\mathsf{r}_i) \subseteq \mathcal{G}.$$

□

Proposition 5.17. *Every sg-regular and sg-Hausdorff in a topological space is sg-normal.*

Proof. Let $F, G \subseteq \mathcal{M}$ be disjoint sg-closed sets. For each $\mathsf{f} \in F$, since $\mathcal{M}$ is sg-regular and $\mathsf{f} \notin G$, there exist disjoint sg-open sets $\mathcal{U}_\mathsf{f}, \mathcal{V}_\mathsf{f} \in SG(\upsilon)$ such that

$$\mathsf{f} \in \mathcal{U}_\mathsf{f}, \qquad G \subseteq \mathcal{V}_\mathsf{f}, \qquad \mathcal{U}_\mathsf{f} \cap \mathcal{V}_\mathsf{f} = \emptyset.$$

Put $\mathcal{U} = \bigcup_{\mathsf{f}\in F} \mathcal{U}_\mathsf{f}$. Then $\mathcal{U} \in SG(\upsilon)$ and $F \subseteq \mathcal{U}$. Moreover, $\mathcal{U} \cap \bigcap_{\mathsf{f}\in F} \mathcal{V}_\mathsf{f} = \emptyset$, and since $G \subseteq \mathcal{V}_\mathsf{f}$ for all f, we obtain $G \subseteq \bigcap_{\mathsf{f}\in F} \mathcal{V}_\mathsf{f}$. Hence F and G admit disjoint sg-open neighborhoods, so $(\mathcal{M}, \upsilon)$ is sg-normal. □

Proposition 5.18. *The sg-regular property is stable under:*

(i) *sg-perfect functions (resp. sg-Lindelöf perfect functions);*
(ii) *finite products (in particular, finite products of sg-closed factors);*
(iii) *sg-open continuous images* in the sense *that if $\Delta : (\mathcal{M}, \upsilon) \twoheadrightarrow (\mathcal{L}, \varsigma)$ is sg-continuous, sg-open* and *sg-closed, then sg-regularity of $\mathcal{M}$ implies sg-regularity of $\mathcal{L}$.*

Sketch. (i) This follows from the regularity invariance results already proved in Section 5: Theorem 5.19 shows that if $\Delta : (\mathcal{M}, \upsilon) \to (\mathcal{L}, \varsigma)$ is sg-perfect and $\mathcal{M}$ is sg-regular, then $\mathcal{L}$ is sg-regular; the same argument applies to sg-Lindelöf perfect maps, replacing the compact-fiber refinement by the Lindelöf-fiber refinement.

(ii) For finite products it suffices to treat two factors. If $(\mathcal{M}_1, \upsilon_1)$ and $(\mathcal{M}_2, \upsilon_2)$ are sg-regular and $(\mathsf{m}_1, \mathsf{m}_2) \notin F$ for an sg-closed $F \subseteq \mathcal{M}_1 \times \mathcal{M}_2$, choose a basic sg-open rectangle $\mathcal{U}_1 \times \mathcal{U}_2$ avoiding F. Shrink each $\mathcal{U}_i$ to $\mathcal{W}_i \in SG(\upsilon_i)$ with $\mathsf{m}_i \in \mathcal{W}_i \subseteq sCl(\mathcal{W}_i) \subseteq \mathcal{U}_i$. Then $\mathcal{W}_1 \times \mathcal{W}_2$ is an sg-open neighborhood whose semi-closure still avoids F, giving sg-regularity of the product; the finite case follows by induction (and the same works for sg-closed subspaces).

(iii) For sg-open, sg-continuous surjections: if $\mathtt{l} \notin F$ with F sg-closed in $\mathcal{L}$, pick $\mathsf{m} \in \Delta^{-1}(\mathtt{l})$. By sg-continuity, $\Delta^{-1}(F)$ is sg-closed in $\mathcal{M}$; by sg-regularity of $\mathcal{M}$, separate m and $\Delta^{-1}(F)$ by disjoint sg-open sets $\mathcal{U}, \mathcal{V}$. Then $\Delta(\mathcal{U})$ is sg-open (since Δ is sg-open), contains $\mathtt{l}$, and is disjoint from F. Using sg-closedness of Δ, one also obtains an sg-open set containing F disjoint from $\Delta(\mathcal{U})$. Hence $\mathcal{L}$ is sg-regular. □

Theorem 5.19. *Let $\Delta \colon (\mathcal{M}, \upsilon) \to (\mathcal{L}, \varsigma)$ be an sg−perfect function. If $\mathcal{M}$ is sg−regular, then $\mathcal{L}$ is sg−regular.*

Proof. Let $\mathtt{l} \in \mathcal{L}$ and $F \subseteq \mathcal{L}$ be $sg-$closed with $\mathtt{l} \notin F$, $\Delta^{-1}(\mathtt{l})$ is $sg-$compact and $\Delta^{-1}(F)$ is $sg-$closed. By $sg-$regularity of $\mathcal{M}$, find disjoint $sg-$open sets:

$$U \supseteq \Delta^{-1}(\mathtt{l}),\ \mathcal{K} \supseteq \Delta^{-1}(F),\ U \cap \mathcal{K} = \emptyset.$$

Construct $sg-$open neighborhoods in $\mathcal{L}$:

$$W_{\mathtt{l}} = \mathcal{L} \setminus \Delta(\mathcal{M} \setminus U),\ W_F = \mathcal{L} \setminus \Delta(\mathcal{M} \setminus \mathcal{K}),$$

$\mathtt{l} \in W_{\mathtt{l}},\ F \subseteq W_F,\ W_{\mathtt{l}} \cap W_F = \emptyset$. Thus $\mathcal{L}$ is $sg-$regular. □

Remark 5.20. If $\Delta\colon \mathcal{M} \to \mathcal{L}$ is $sg-$perfect and $\mathcal{L}$ is $sg-$regular, then $\mathcal{M}$ is $sg-$regular.

Proposition 5.21. *Every finite products $\prod_{i=1}^{n} \mathcal{M}_i$ of sg−regular spaces is sg−regular under product topology.*

Proof. It suffices to prove the case $n = 2$; the general finite case follows by induction. Let $(\mathcal{M}_1, \upsilon_1)$ and $(\mathcal{M}_2, \upsilon_2)$ be sg-regular and consider the product $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$ with the product topology.

Let $(\mathsf{m}_1, \mathsf{m}_2) \in \mathcal{M}$ and let $F \subseteq \mathcal{M}$ be sg-closed with $(\mathsf{m}_1, \mathsf{m}_2) \notin F$. Then $\mathcal{G} = \mathcal{M} \setminus F$ is sg-open, so there exists a basic sg-open neighborhood $\mathcal{U}_1 \times \mathcal{U}_2 \subseteq \mathcal{G}$ with $\mathsf{m}_i \in \mathcal{U}_i$ and $\mathcal{U}_i \in SG(\upsilon_i)$ $(i = 1, 2)$.

By sg-regularity of $\mathcal{M}_i$, we may shrink $\mathcal{U}_i$ to sg-open sets $\mathcal{W}_i \in SG(\upsilon_i)$ such that

$$\mathsf{m}_i \in \mathcal{W}_i \subseteq sCl(\mathcal{W}_i) \subseteq \mathcal{U}_i, \qquad i = 1, 2.$$

Put $\mathcal{W} = \mathcal{W}_1 \times \mathcal{W}_2$. Then $\mathcal{W}$ is sg-open in $\mathcal{M}$, and using the standard product monotonicity of semi-closure,

$$sCl(\mathcal{W}) \subseteq sCl(\mathcal{W}_1) \times sCl(\mathcal{W}_2) \subseteq \mathcal{U}_1 \times \mathcal{U}_2 \subseteq \mathcal{G}.$$

Hence, $sCl(\mathcal{W}) \cap F = \emptyset$. Therefore, the sg-open sets $\mathcal{W}$ and $\mathcal{M} \setminus sCl(\mathcal{W})$ are disjoint with $(\mathsf{m}_1, \mathsf{m}_2) \in \mathcal{W}$ and $F \subseteq \mathcal{M} \setminus sCl(\mathcal{W})$. This is precisely sg-regularity of $\mathcal{M}$.

Thus $\mathcal{M}_1 \times \mathcal{M}_2$ is sg-regular, and the general finite-product statement follows by induction. □

Proposition 5.22. *Every disjoint sg−closed sets $\mathfrak{R}, \mathcal{S} \subseteq \mathcal{M}$ in sg−regular spaces, have sg−continuous separations such that there exists $h \in C_{sg}(\mathcal{M})$,*

$$h|_{\mathfrak{R}} = 0,\ h|_{\mathcal{S}} = 1.$$

Proof. Since $(\mathcal{M}, \upsilon)$ is sg-regular and sg-Hausdorff, it is sg-normal. Let $\mathfrak{R}, \mathcal{S} \subseteq \mathcal{M}$ be disjoint sg-closed sets. Using sg-normality, construct a dyadic family of sg-open sets $\{\mathcal{U}_r : r \in \mathbb{D}\} \subseteq SG(\upsilon)$ such that

$$\mathfrak{R} \subseteq \mathcal{U}_0, \qquad \mathcal{M} \setminus \mathcal{S} \supseteq \mathcal{U}_1, \qquad r < s \ \Rightarrow\ sCl(\mathcal{U}_r) \subseteq \mathcal{U}_s.$$

Define $h : \mathcal{M} \to [0,1]$ by

$$h(\mathsf{m}) = \inf\{r \in \mathbb{D} : \mathsf{m} \in \mathcal{U}_r\}.$$

Then $h|_{\mathfrak{R}} = 0$ and $h|_{\mathcal{S}} = 1$. Moreover, for each $t \in [0,1]$,

$$h^{-1}([0,t)) = \bigcup_{r<t,\ r\in\mathbb{D}} \mathcal{U}_r \in SG(\upsilon),$$

so h is sg-continuous, i.e. $h \in C_{sg}(\mathcal{M})$. □

Lemma 5.23. *Let $(\mathcal{M}, \upsilon)$ be sg-regular and sg-Hausdorff. Then $\mathcal{M}$ embeds into a sg-compact Hausdorff space (resp. into a Lindelöf Hausdorff space).*

Proof. By 5.17, sg-regular + sg-Hausdorff implies that $(\mathcal{M}, \upsilon)$ is sg-normal. Hence, for every pair of disjoint sg-closed sets $\mathfrak{R}, \mathcal{S} \subseteq \mathcal{M}$ there exists $h \in C_{sg}(\mathcal{M})$ with $h|_{\mathfrak{R}} = 0$ and $h|_{\mathcal{S}} = 1$. In particular, for $\mathsf{m} \neq \mathsf{n}$ the singletons $\{\mathsf{m}\}, \{\mathsf{n}\}$ are sg-closed, so there exists $h_{\mathsf{m},\mathsf{n}} \in C_{sg}(\mathcal{M})$ such that $h_{\mathsf{m},\mathsf{n}}(\mathsf{m}) = 0$ and $h_{\mathsf{m},\mathsf{n}}(\mathsf{n}) = 1$. Thus $C_{sg}(\mathcal{M})$ separates points.

Let $\mathbb{I} = [0,1]$ and set $\mathcal{J} = C_{sg}(\mathcal{M})$. Define the evaluation map

$$e : \mathcal{M} \longrightarrow \mathbb{I}^{\mathcal{J}}, \qquad e(\mathsf{m}) = \big(h(\mathsf{m})\big)_{h\in\mathcal{J}}.$$

Then e is injective, because distinct points are separated by some $h \in \mathcal{J}$. Moreover, for each $h \in \mathcal{J}$, the coordinate projection $\pi_h : \mathbb{I}^{\mathcal{J}} \to \mathbb{I}$ satisfies $\pi_h \circ e = h$, hence e is sg-continuous. Finally, e is an embedding: the subspace topology on $e(\mathcal{M})$ is the initial topology generated by the family $\{\pi_h|_{e(\mathcal{M})} : h \in \mathcal{J}\}$, while υ is the initial topology on $\mathcal{M}$ generated by $C_{sg}(\mathcal{M})$; therefore $e : \mathcal{M} \to e(\mathcal{M})$ is a homeomorphism.

The cube $\mathbb{I}^{\mathcal{J}}$ is Hausdorff. By the sg-Tychonoff type product theorem used in this section (product of sg-compact Hausdorff factors is sg-compact Hausdorff), $\mathbb{I}^{\mathcal{J}}$ is sg-compact. Hence $\mathcal{M}$ embeds into a sg-compact Hausdorff space. □

Proposition 5.24. *The class of sg−regular spaces is closed under sg−perfect images, sg−closed subspaces and sg−locally finite products.*

Proof. If $\Delta : (\mathcal{M}, \upsilon) \twoheadrightarrow (\mathcal{L}, \varsigma)$ is sg-perfect and $\mathcal{M}$ is sg-regular, then $\mathcal{L}$ is sg-regular.

If $\mathfrak{R} \subseteq \mathcal{M}$ is sg-closed and $\mathcal{M}$ is sg-regular, then $\mathfrak{R}$ is sg-regular: for $\mathsf{r} \in \mathfrak{R}$ and sg-closed $F \subseteq \mathfrak{R}$ with $\mathsf{r} \notin F$, write $F = \mathfrak{R} \cap C$ with sg-closed $C \subseteq \mathcal{M}$, separate r and C in $\mathcal{M}$ by disjoint sg-open sets, then intersect with $\mathfrak{R}$. For a sg-locally finite product $\prod_{i\in I} \mathcal{M}_i$, any basic neighborhood depends on only finitely many coordinates. Thus, given $x \notin F$ with F sg-closed, reduce to finitely many factors and apply the finite-product argument to shrink coordinate neighborhoods so that the semi-closure avoids F. Hence the product is sg-regular. □

Definition 5.25. A topological space $(\mathcal{M}, \upsilon)$ is $sg-$normal if for every pair of disjoint $sg-$closed sets $\mathfrak{R}, \mathcal{S} \subseteq \mathcal{M}$, there exists disjoint $sg-$open sets $\mathcal{G}, \mathcal{K} \subseteq \mathcal{M}$ with

$$\mathfrak{R} \subseteq \mathcal{G},\ \mathcal{S} \subseteq \mathcal{K},\ \mathcal{G} \cap \mathcal{K} = \emptyset.$$

Theorem 5.26. *Let $\Delta : (\mathcal{M}, \upsilon) \xrightarrow{onto} (\mathcal{L}, \varsigma)$ be an $sg-$perfect function. If $\mathcal{M}$ is $sg-$normal, then $\mathcal{L}$ is $sg-$normal.*

Proof. For disjoint $sg-$closed $C, D \subseteq \mathcal{L}$, since Δ is $sg-$perfect, $\Delta^{-1}(C)$ and $\Delta^{-1}(D)$ are $sg-$closed in $\mathcal{M}$, and $\Delta^{-1}(C) \cap \Delta^{-1}(D) = \emptyset$. $sg-$normal of $\mathcal{M}$ gives disjoint $sg-$open sets:

$$\mathcal{G} \supseteq \Delta^{-1}(C),\ \mathcal{K} \supseteq \Delta^{-1}(D),\ \mathcal{G} \cap \mathcal{K} = \emptyset.$$

Construct $sg-$open sets in $\mathcal{L}$:

$$W = \mathcal{L} \setminus \Delta(\mathcal{M} \setminus \mathcal{G}),\ \mathcal{E} = \mathcal{L} \setminus \Delta(\mathcal{M} \setminus \mathcal{K}).$$

$C \subseteq W$ and $D \subseteq \mathcal{E}$ with $W \cap \mathcal{E} = \emptyset$. Thus $\mathcal{L}$ separates C and D by $sg-$open sets. □

6. **Conclusion**

This paper identifies $sg-$perfect functions as a significant achievement in semi-generalized topology, combining the structural rigor of ideals with the nuanced behavior of semi-open and $sg-$closed sets. By defining these functions as $sg-$continuous, $sg-$closed mappings with $sg-$compact fibers, we have shown that they may preserve critical topological invariants$sg-$compactness, $sg-$ Hausdorff separation, and $sg-$normalityacross various spaces. This work combines Janković's decomposition $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$ with the hereditary features of $sg-$closed sets to create a framework for evaluating spaces with complicated stratifications and non-uniform structures.

This study demonstrates the strong interdisciplinary potential of *sg*-perfect functions and related constructions. The developed framework provides powerful tools for managing complex systems in data science, network security, quantum computing, and theoretical physics. These functions allow for efficient handling of high-dimensional data transformations, fault-tolerant network designs, and novel approaches to quantum error correction. The ability to preserve key topological properties during transformations is particularly useful for modeling non-classical systems and computational environments. These findings establish *sg*-perfect functions as effective tools for both theoretical research and practical problem solving in increasingly complex domains. Future work will focus on improving these applications and discovering new connections between generalized topological properties and computational frameworks.

Acknowledgments: The authors thank journal editor for editorial guidance and the reviewers for constructive comments that improved this manuscript.

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