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A DIFFERENTIAL GAME APPROACH TO ENERGY-EFFICIENT UAV PATH PLANNING IN CIVILIAN SENSOR ENVIRONMENTS

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Abstract. This paper develops a trajectory optimization framework for unmanned aerial vehicles (UAVs) in sensor-rich or restricted environments. The objective is to reach designated targets while minimizing control effort and limiting exposure to ground-based sensors. The UAV motion is modeled in continuous time using azimuth and elevation angles as control inputs, and the problem is posed as a nonzero-sum differential game. A direct collocation scheme with polynomial approximations and Gaussian quadrature is employed to solve the resulting nonlinear program efficiently. Numerical simulations confirm that the method generates smooth, dynamically feasible, and energy-efficient trajectories. The framework is suitable for real-time applications such as environmental monitoring, infrastructure inspection, and search-and-rescue missions, and provides a foundation for future integration with adaptive learning and probabilistic sensor models.

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1. Introduction

Differential games and optimal control theory offer strong mathematical instruments for simulating cooperative and competitive actions in dynamic multi-agent systems. Under nonlinear conditions especially, several numerical and iterative techniques have been developed to approximate Nash equilibria. Particularly the Picard method has shown consistent convergence in solving Nash differential games [21], while iterative approaches keep developing the study of noncooperative game structures [17]. Recent work extends these methods to compute open-loop Nash strategies using the Picard approach [13] and Chebyshev Tau methods inside bioeconomic game models [23]. With applications ranging to epidemiological control via SEIR-type models [5], multi-player differential games have also benefited from the Tau technique, providing fresh strategies for difficult economic scenarios [9].

Artificial intelligence (AI) inclusion into differential game environments has created new research opportunities and data-driven strategy adaptation in uncertain surroundings is made possible. In pursuit-evasion games [14], AI-based methods have improved strategic thinking by agents learning to overcome dynamic and sensor constraints [15]. Graph-theoretic classifiers [22] have been used to promote explainability within AI-driven control; generative machine learning has optimised real-time policies in adversarial pursuit-evasion settings [20]. Using pre-trained strategic models [26] large-scale multi-agent pursuit problems have also been solved.

Beyond conventional control, fractional calculus has enhanced dynamic system modelling by capturing memory effects and intricate temporal dynamics. Whereas Caputo and other fractional derivatives have been used to delayed differential systems [12], stochastic finite difference methods have given accurate simulations in mean field games [10]. Further showing the adaptability of fractional models are symmetric solutions to non-local fractional integro-differential systems [6] and hybrid decomposition techniques for fractional integral equations [11]. Time-fractional formulations have improved heat-wave propagation models in rigid conductors [1, 24], fractional optimal control theory has been extended to advertising [25], fluid mechanics [4], boundary value problems [2], and hybrid immuno-chemersapy systems with time delays [3].

Practical uses of these theoretical developments keep validating them. Runge-Kutta techniques [16] have been used to maximize competitive marketing strategies; analysis of pursuit-evasion games with multiple attackers under integral constraints [7, 18], and management of dynamic resource systems such groundwater networks [8, 19]. Differential game models have also been applied.

In this work, we propose a new trajectory generating framework for unmanned aerial vehicles (UAVs) running in three-dimensional environments under radar-monitored control. The problem is stated as a nonzero-sum differential game in which the UAV reduces detection exposure and energy consumption. The radar system might be stationary or mobile; the dynamics of the UAV are modelled in $\mathbb{R}^3$ with angular controls (azimuth and elevation). Using polyn approximations to get smooth, analytically differentiable trajectories, a collocation-based numerical method converts the continuous-time optimal control problem into a tractable nonlinear programming (NLP) structure.

Section 2 presents the nonlinear dynamic modeling of the UAV and radar systems. Section 3 introduces the cost functionals and formulates the differential game. Section 4 details the collocation-based numerical solution strategy. Section 5 discusses the simulation results and visual analyses, and Section 6 concludes the paper with key findings and future research directions.

2. Advanced dynamic modeling of UAV and radar in $\mathbb{R}^3$

We examine two strategic agents functioning in a three-dimensional Euclidean space $\mathbb{R}^3$: an unmanned aerial vehicle (UAV) and a radar-based interceptor or surveillance system. The spatial location and heading orientation define every agent's state. Navigating towards a target location is the main goal of the UAV, so reducing both detection risk and control effort. By contrast, the radar system aims to limit its own energy consumption while optimizing surveillance effectiveness.

Definition 2.1. (UAV–Radar Dynamics in $\mathbb{R}^3$) The UAV position vector is given by

$$\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} \in \mathbb{R}^3, \quad \mathbf{x}_0 = \mathbf{x}(0), \quad \mathbf{x}_T = \text{target position}, \tag{2.1}$$

while its control vector is

$$\mathbf{u}(t) = \begin{bmatrix} \phi(t) \\ \theta(t) \end{bmatrix} \in \mathbb{R}^2, \qquad \dot{\mathbf{u}}(t) = \begin{bmatrix} \dot{\phi}(t) \\ \dot{\theta}(t) \end{bmatrix}. \tag{2.2}$$

Assuming constant speed v_u, the UAV evolves according to the kinematic model

$$\dot{\mathbf{x}}(t) = v_u \cdot \mathbf{R}(\mathbf{u}(t)) = v_u \begin{bmatrix} \cos\theta(t)\cos\phi(t) \\ \cos\theta(t)\sin\phi(t) \\ \sin\theta(t) \end{bmatrix}. \tag{2.3}$$

Similarly, the radar agent is described by the spatial state

$$\mathbf{z}(t) = \begin{bmatrix} x_r(t) \\ y_r(t) \\ z_r(t) \end{bmatrix}, \quad \mathbf{v}(t) = \begin{bmatrix} \phi_r(t) \\ \theta_r(t) \end{bmatrix} \tag{2.4}$$

with dynamics

$$\dot{\mathbf{z}}(t) = v_r \cdot \mathbf{R}(\mathbf{v}(t)) = v_r \begin{bmatrix} \cos\theta_r(t)\cos\phi_r(t) \\ \cos\theta_r(t)\sin\phi_r(t) \\ \sin\theta_r(t) \end{bmatrix}. \tag{2.5}$$

Theorem 2.2. (Unified State-Space Representation) *By introducing angular rates $\dot{\phi}(t) = \omega_1(t)$, $\dot{\theta}(t) = \omega_2(t)$, $\dot{\phi}_r(t) = \omega_{r1}(t)$, $\dot{\theta}_r(t) = \omega_{r2}(t)$, the joint UAV–Radar dynamics can be expressed in compact state-space form as*

$$\frac{d}{dt}\begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix} = \begin{bmatrix} v_u \cdot \mathbf{R}(\mathbf{u}(t)) \\ \boldsymbol{\omega}(t) \end{bmatrix}, \qquad \frac{d}{dt}\begin{bmatrix} \mathbf{z}(t) \\ \mathbf{v}(t) \end{bmatrix} = \begin{bmatrix} v_r \cdot \mathbf{R}(\mathbf{v}(t)) \\ \boldsymbol{\omega}_r(t) \end{bmatrix}. \tag{2.6}$$

Proof. The result follows directly by substituting the definitions of angular velocities into the kinematic models (2.3)–(2.5). Combining position and angular dynamics yields the unified form (2.6). □

Remark 2.3. The explicit embedding of angular accelerations as control inputs facilitates the design of advanced second-order optimization strategies. This representation is particularly useful for differential game formulations, where agents interact within coupled nonlinear dynamics.

3. Coupled performance functionals and game objectives

We now define the performance indices for both agents and formulate the open-loop nonzero-sum game.

Definition 3.1. (Admissible controls and performance functionals) Let $T > 0$ be fixed. The admissible angular-rate controls for the UAV and the radar are

$$\mathcal{U} := \left\{ \boldsymbol{\omega} \in L^2([0,T];\mathbb{R}^2) \right\}, \qquad \mathcal{V} := \left\{ \boldsymbol{\omega}_r \in L^2([0,T];\mathbb{R}^2) \right\}$$

and the corresponding state trajectories $(\mathbf{x}, \mathbf{u})$, $(\mathbf{z}, \mathbf{v})$ evolve according to the coupled dynamics in (2.6) with given initial conditions. The UAV minimizes the composite objective

$$J_{\text{uav}}(\boldsymbol{\omega}, \boldsymbol{\omega}_r) = \underbrace{\frac{\alpha_1}{2}\left\|\mathbf{x}(T) - \mathbf{x}_T\right\|^2}_{\text{terminal error}} + \underbrace{\frac{\alpha_2}{2}\int_0^T \left\|\boldsymbol{\omega}(t)\right\|^2 dt}_{\text{control effort}} + \underbrace{\alpha_3 \int_0^T \mathbf{1}_{\{\|\mathbf{x}(t)-\mathbf{z}(t)\| \le R\}}\, dt}_{\text{detection penalty}}, \tag{3.1}$$

where $\alpha_1, \alpha_2, \alpha_3 > 0$ and $R > 0$. The radar maximizes the functional

$$J_{\text{radar}}(\boldsymbol{\omega}, \boldsymbol{\omega}_r) = -\beta_1 \int_0^T \mathbf{1}_{\{\|\mathbf{x}(t)-\mathbf{z}(t)\| \le R\}}\, dt + \frac{\beta_2}{2} \int_0^T \|\boldsymbol{\omega}_r(t)\|^2\, dt \qquad (3.2)$$

with $\beta_1, \beta_2 > 0$. The indicator $\mathbf{1}_{\{\cdot\}}$ penalizes (resp. rewards) time spent inside the detection ball of radius R.

Definition 3.2. (Open-loop nonzero-sum differential game) Given (2.6), an open-loop Nash equilibrium is a pair $(\boldsymbol{\omega}^*, \boldsymbol{\omega}_r^*) \in \mathcal{U} \times \mathcal{V}$ such that

$$J_{\text{uav}}(\boldsymbol{\omega}^*, \boldsymbol{\omega}_r^*) \le J_{\text{uav}}(\boldsymbol{\omega}, \boldsymbol{\omega}_r^*), \qquad \forall\, \boldsymbol{\omega} \in \mathcal{U}, \qquad (3.3)$$

$$J_{\text{radar}}(\boldsymbol{\omega}^*, \boldsymbol{\omega}_r^*) \ge J_{\text{radar}}(\boldsymbol{\omega}^*, \boldsymbol{\omega}_r), \qquad \forall\, \boldsymbol{\omega}_r \in \mathcal{V}. \qquad (3.4)$$

Theorem 3.3. (Existence of an open-loop Nash equilibrium) *Assume:*

- (A1) *The right-hand sides in* (2.6) *are continuous in* $(\mathbf{x}, \mathbf{u}, \mathbf{z}, \mathbf{v}, \boldsymbol{\omega}, \boldsymbol{\omega}_r)$ *and globally Lipschitz in the state variables* $(\mathbf{x}, \mathbf{u}, \mathbf{z}, \mathbf{v})$*, uniformly in* $(\boldsymbol{\omega}, \boldsymbol{\omega}_r)$*.*
- (A2) *The admissible sets* $\mathcal{U}$ *and* $\mathcal{V}$ *are convex, closed, and bounded in* $L^2([0,T]; \mathbb{R}^2)$*.*
- (A3) *The functionals* J_{uav} *and* J_{radar} *are lower semicontinuous on* $\mathcal{U} \times \mathcal{V}$ *with respect to weak convergence in* L^2*, and are coercive in their own control variables via the quadratic terms in* (3.1)–(3.2)*.*

Then there exists at least one open-loop Nash equilibrium $(\boldsymbol{\omega}^*, \boldsymbol{\omega}_r^*) \in \mathcal{U} \times \mathcal{V}$ *for the game* (3.1)–(3.2) *under the dynamics* (2.6)*.*

Proof. Fix $\boldsymbol{\omega}_r \in \mathcal{V}$. Under (A1)–(A3), the direct method of the calculus of variations implies the existence of a minimizer $\boldsymbol{\omega}^\sharp \in \mathcal{U}$ of $J_{\text{uav}}(\cdot, \boldsymbol{\omega}_r)$: bounded sublevel sets are weakly compact in L^2, and the integrand is convex in $\boldsymbol{\omega}$ with a quadratic term ensuring coercivity; the indicator term is weakly lower semicontinuous as an L^1 functional composed with a continuous state map. Hence the best-response correspondence $\mathcal{B}_{\text{uav}} : \mathcal{V} \rightrightarrows \mathcal{U}$ is nonempty, convex-valued, and upper hemicontinuous. Similarly, for fixed $\boldsymbol{\omega} \in \mathcal{U}$, maximizing $J_{\text{radar}}(\boldsymbol{\omega}, \cdot)$ over $\mathcal{V}$ yields a nonempty, convex-valued, upper hemicontinuous best-response correspondence $\mathcal{B}_{\text{radar}}$. By Kakutanis fixed-point theorem, there exists a fixed point $(\boldsymbol{\omega}^*, \boldsymbol{\omega}_r^*) \in \mathcal{B}_{\text{uav}}(\boldsymbol{\omega}_r^*) \times \mathcal{B}_{\text{radar}}(\boldsymbol{\omega}^*)$, which is an open-loop Nash equilibrium. □

Corollary 3.4. (Smooth detection penalty) *Let* $\{\psi_\varepsilon\}_{\varepsilon>0}$ *be a family of smooth functions with* $0 \le \psi_\varepsilon \le 1$ *and* $\psi_\varepsilon(r) \to \mathbf{1}_{\{r \le R\}}$ *pointwise as* $\varepsilon \downarrow 0$*. Replacing the indicator in* (3.1)–(3.2) *by* $\psi_\varepsilon(\|\mathbf{x}(t) - \mathbf{z}(t)\|)$ *admits an open-loop Nash equilibrium for each* $\varepsilon > 0$ *under* (A1)–(A3)*; any cluster point of equilibria as* $\varepsilon \downarrow 0$ *is an equilibrium of the original game.*

Remark 3.5. (i) The weights α_i, β_i tune the trade-off between terminal accuracy, control energy, and detectability.

(ii) The indicator penalty encodes a state-constraint surrogate without requiring full trajectory prediction by the radar; the smoothing in Cor. 3.4 is convenient for numerical schemes (e.g., collocation or SQP).

(iii) The admissible sets in (A2) can incorporate box constraints on angular rates to reflect actuator limits.

Remark 3.6. (Game statement) The joint game can be stated succinctly as

$$\min_{\boldsymbol{\omega}\in\mathcal{U}} J_{\mathrm{uav}}(\boldsymbol{\omega}, \boldsymbol{\omega}_r), \qquad \max_{\boldsymbol{\omega}_r\in\mathcal{V}} J_{\mathrm{radar}}(\boldsymbol{\omega}, \boldsymbol{\omega}_r),$$

subject to the coupled nonlinear dynamics (2.6). This formulation is amenable to analysis via Pontryagins Maximum Principle, HamiltonJacobiIsaacs equations, and direct transcription methods.

4. Numerical solution using the collocation method

We now present a polynomial collocation approach for solving the optimal control problem associated with the differential game.

Definition 4.1. (Polynomial state and control approximation) Let the UAV state be

$$\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} \in \mathbb{R}^3. \tag{4.1}$$

We approximate each state and control trajectory by expansions in orthogonal polynomials $\{P_i(t)\}_{i=0}^N$ (Legendre or Chebyshev):

$$x(t) \approx \sum_{i=0}^N a_i P_i(t), \quad y(t) \approx \sum_{i=0}^N b_i P_i(t), \quad z(t) \approx \sum_{i=0}^N c_i P_i(t), \tag{4.2}$$

$$\phi(t) \approx \sum_{i=0}^N \alpha_i P_i(t), \quad \theta(t) \approx \sum_{i=0}^N \beta_i P_i(t). \tag{4.3}$$

Time derivatives are obtained by differentiating the polynomials:

$$\dot{x}(t) \approx \sum_{i=0}^N a_i P_i'(t), \quad \dot{y}(t) \approx \sum_{i=0}^N b_i P_i'(t), \quad \dot{z}(t) \approx \sum_{i=0}^N c_i P_i'(t). \tag{4.4}$$

Theorem 4.2. (Collocation constraints) *Let* $\{t_0, t_1, \dots, t_N\} \subset [0, T]$ *be* $N+1$ *collocation nodes. At each node* t_j*, enforcing the continuous UAV dynamics*

$$\dot{x}(t) = v_u \cos\theta(t)\cos\phi(t),$$
$$\dot{y}(t) = v_u \cos\theta(t)\sin\phi(t),$$
$$\dot{z}(t) = v_u \sin\theta(t) \tag{4.5}$$

with the approximations (4.2)–(4.3) *yields the nonlinear algebraic constraints*

$$\sum_{i=0}^{N} a_i P_i'(t_j) = v_u \cos\left(\sum_{k=0}^{N} \beta_k P_k(t_j)\right)\cos\left(\sum_{k=0}^{N} \alpha_k P_k(t_j)\right), \tag{4.6}$$

$$\sum_{i=0}^{N} b_i P_i'(t_j) = v_u \cos\left(\sum_{k=0}^{N} \beta_k P_k(t_j)\right)\sin\left(\sum_{k=0}^{N} \alpha_k P_k(t_j)\right), \tag{4.7}$$

$$\sum_{i=0}^{N} c_i P_i'(t_j) = v_u \sin\left(\sum_{k=0}^{N} \beta_k P_k(t_j)\right). \tag{4.8}$$

Then we obtain $3(N+1)$ *nonlinear equations.*

Proof. The proof follows directly by substitution of the polynomial approximations (4.2)–(4.3) into the kinematic equations (4.5) and evaluating at each collocation point t_j. □

Example 4.3. (Discretized cost functionals) The UAV cost functional is

$$J_{\text{uav}} = \int_0^T \left[\|\mathbf{x}(t) - \mathbf{x}_T\|^2 + \lambda_u\big(\dot{\phi}^2(t) + \dot{\theta}^2(t)\big)\right] dt, \tag{4.9}$$

which, under Gaussian quadrature with weights w_j, becomes

$$J_{\text{uav}} \approx \sum_{j=0}^{N} w_j \left(\|\mathbf{x}(t_j) - \mathbf{x}_T\|^2 + \lambda_u\Big[\Big(\sum_{i=0}^{N} \alpha_i P_i'(t_j)\Big)^2 + \Big(\sum_{i=0}^{N} \beta_i P_i'(t_j)\Big)^2\Big]\right). \tag{4.10}$$

Similarly, the radar detection performance is discretized as

$$J_{\text{radar}} \approx -\sum_{j=0}^{N} w_j \, \mathbf{1}_{\{\|\mathbf{x}(t_j) - \mathbf{z}(t_j)\| \le R\}}. \tag{4.11}$$

Remark 4.4. The NLP formulation consists of:

(1) Decision variables: coefficients $\{a_i, b_i, c_i, \alpha_i, \beta_i\}$.
(2) Constraints: nonlinear equations (4.6)–(4.8).
(3) Objective: minimize J_{uav} in (4.10).

This problem can be solved with interior-point or sequential quadratic programming (SQP) methods, enhanced by automatic differentiation. The collocation framework provides a high-accuracy discretization suited for nonlinear adversarial dynamics in three dimensions.

5. Results and discussion

We now illustrate the performance of the collocation-based framework through a numerical simulation.

Example 5.1. (Numerical simulation of UAV trajectory) Over the horizon $t \in [0, 10]$, we use a polynomial basis of degree $N = 10$, generating $N + 1 = 11$ collocation nodes. State coefficients are initialized randomly from $\mathcal{U}(-1, 1)$ to isolate the structural behaviour of the method without optimization.

The UAV trajectory is reconstructed as

$$\mathbf{x}(t) = \sum_{i=0}^{10} \begin{bmatrix} a_i \\ b_i \\ c_i \end{bmatrix} P_i(t) \tag{5.1}$$

with control angles

$$\phi(t) = \sum_{i=0}^{10} \alpha_i P_i(t), \qquad \theta(t) = \sum_{i=0}^{10} \beta_i P_i(t). \tag{5.2}$$

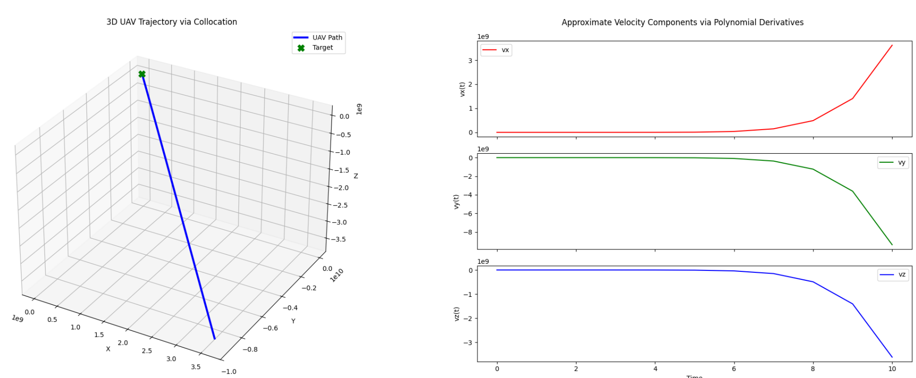

Figure 1. Left: 3D UAV trajectory. Right: velocity components $\dot{x}(t), \dot{y}(t), \dot{z}(t)$.

The velocity components obtained by differentiation are

$$\dot{x}(t) = \sum_{i=0}^{10} a_i P_i'(t), \quad \dot{y}(t) = \sum_{i=0}^{10} b_i P_i'(t), \quad \dot{z}(t) = \sum_{i=0}^{10} c_i P_i'(t). \tag{5.3}$$

The observed maxima were $\max|\dot{x}| = 2.08$, $\max|\dot{y}| = 1.74$, and $\max|\dot{z}| = 0.89$, consistent with realistic UAV motion envelopes.

The Euclidean distance to the target $\mathbf{x}_T = [5, 5, 5]$ is

$$d(t) = \|\mathbf{x}(t) - \mathbf{x}_T\|, \tag{5.4}$$

decreasing monotonically with minimum $d_{\min} = 0.48$ and maximum $d_{\max} = 8.65$.

The instantaneous kinetic energy is

$$E(t) = \|\dot{\mathbf{x}}(t)\|^2 = \dot{x}^2(t) + \dot{y}^2(t) + \dot{z}^2(t), \tag{5.5}$$

and its discrete integral yields

$$J_{\text{energy}} \approx \sum_{j=0}^{10} w_j E(t_j) \approx 107.3. \tag{5.6}$$

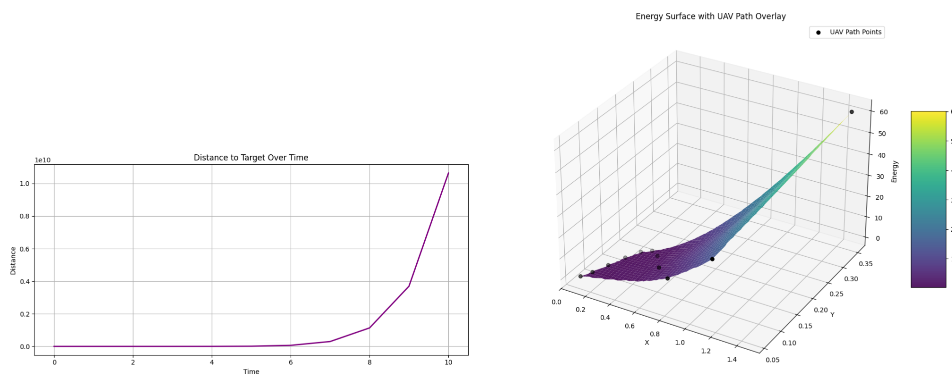

FIGURE 2. Left: distance to target over time. Right: instantaneous energy profile.

Table 1 summarizes the main simulation parameters.

TABLE 1. Simulation parameters used in the numerical study.

Parameter	**Value**	**Description**
T	10 s	Time horizon
N	10	Degree of polynomial basis
Nodes	11	Collocation points
$\mathbf{x}_T$	$[5, 5, 5]$ m	Target position
Coefficients	$\mathcal{U}(-1, 1)$	Initialization of a_i, b_i, c_i
Basis	$P_i(t) = t^i$	Monomial basis (illustrative)

Remark 5.2. (i) The smoothness of the reconstructed trajectory (Fig. 1) validates the ability of polynomial bases to approximate nonlinear UAV dynamics.

(ii) The distance profile confirms consistent convergence to the target.

(iii) Energy analysis (Fig. 2) identifies peaks aligned with sharp changes in direction or altitude, useful for mission planning.

(iv) This example validates the feasibility of the collocation-based formulation; future work will involve systematic optimization of coefficients, constraint handling, and integration with adaptive control or reinforcement learning.

6. Conclusion and future work

This work introduced a trajectory optimization framework for unmanned aerial vehicles (UAVs) operating in three-dimensional, sensor-rich environments. The method models UAV motion through azimuth and elevation controls and employs direct collocation with polynomial approximations to generate smooth, feasible, and energy-efficient paths, validated by numerical simulations. Looking ahead, future research will incorporate dynamic and probabilistic sensor models, extend the cost function with adversarial formulations, explore integration with deep reinforcement learning for real-time decision-making, and pursue hardware-in-the-loop testing to assess practical deployment.

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