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SOME RESULTS OF FIXED POINT THEOREMS IN CONVEX METRIC SPACES

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Abstract. In this paper we study some fixed point theorems for self-mappings satisfying certain contraction principles on a convex complete metric space. In addition, we also improve and extend some very recently results in [9].

1. Introduction and Preliminary

In 1970, Takahashi [11] introduced the notion of convexity in metric spaces and studied some fixed point theorems for nonexpansive mappings in such spaces. A convex metric space is a generalized space. For example, every normed space and cone Banach space is a convex metric space and convex complete metric space, respectively. Subsequently, many mathematicians in [2]-[7], [10, 12] and recently, Moosaei [9] studied fixed point theorems in convex metric spaces.

Our results improve and extend some of Moosaei's results in [9] and Karapinar's results in [8] from a cone Banach space to a convex complete metric space. For instance, Karapinar proved that

Theorem 1.1. ([8, Theorem 2.4]) *Let C be a closed and convex subset of a cone Banach space X with the norm $\|x\|_p = d(x, 0)$, and $T : C \to C$ be a mapping which satisfies the condition*

$$\exists\, q \in [2, 4), \quad \forall\, x, y \in C, \quad d(x, Tx) + d(y, Ty) \leq qd(x, y).$$

Then T has at least one fixed point.

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Letting $x = y$ in the above inequality, it is easy to see that T is an identity mapping. In this paper, results in [8, 9] is improved and extended to a convex complete metric space.

Theorem 1.2. ([8, Theorem 2.6]) *Let C be a closed and convex subset of a cone Banach space X with the norm $\|x\|_p = d(x, 0)$, and $T : C \to C$ be a mapping which satisfies the condition*

$$\exists\, r \in [2, 5), \quad \forall\, x, y \in C, \quad d(Tx, Ty) + d(x, Tx) + d(y, Ty) \leq rd(x, y).$$

Then T has at least one fixed point.

Definition 1.3. ([1]) Let (X, d) be a metric space and $I = [0, 1]$. A mapping $W : X \times X \times I \to X$ is said to be a convex structure on X if for each $(x, y, \lambda) \in X \times X \times I$ and $u \in X$,

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y).$$

A metric space (X, d) together with a convex structure W is called a convex metric space, which is denoted by (X, d, W).

Example 1.4. Let $(X, d, \|.\|)$ be a normed space. The mapping $W : X \times X \times I \to X$ defined by $W(x, y, \lambda) = \lambda x + (1 - \lambda)$y for each $x, y \in X$, $\lambda \in I$ is a convex structure on X.

Definition 1.5. ([1]) Let (X, d, W) be a convex metric space. A nonempty subset C of X is said to be convex if $W(x, y, \lambda) \in C$ whenever $(x, y, \lambda) \in C \times C \times I$.

Lemma 1.6. ([9]) *Let (X, d, W) be a convex metric space, then the following statements hold:*

(i) $d(x, y) = d(x, W(x, y, \lambda)) + d(y, W(x, y, \lambda))$ *for all* $(x, y, \lambda) \in X \times X \times I$.
(ii) $d(x, W(x, y, \lambda)) = (1 - \lambda)d(x, y)$ *for all* $x, y \in X$.
(iii) $d(y, W(x, y, \lambda)) = \lambda d(x, y)$ *for all* $x, y \in X$.

Proof. To prove (i) see [9, Lemma 3.1].
By definition, we have

$$d(x, W(x, y, \lambda)) \leq (1 - \lambda)d(x, y)$$

and on the other hand

$$\begin{aligned}(1 - \lambda)d(x, y) &= d(x, y) - \lambda d(x, y)\\ &= [d(x, W(x, y, \lambda)) + d(y, W(x, y, \lambda))] - \lambda d(x, y)\end{aligned}$$

but

$$d(y, W(x, y, \lambda)) \leq \lambda d(x, y).$$

Therefore

$$(1-\lambda)d(x,y) \le d(x, W(x,y,\lambda)).$$

Thus $d(x, W(x,y,\lambda)) = (1-\lambda)d(x,y)$ for all $x, y \in X$. This completes proof of (ii).

For (iii), by (i) and (ii), we have

$$\begin{aligned} d(x,y) &= d(x, W(x,y,\lambda)) + d(y, W(x,y,\lambda)) \\ &= (1-\lambda)d(x,y) + d(y, W(x,y,\lambda)). \end{aligned}$$

So $d(y, W(x,y,\lambda)) = \lambda d(x,y)$ for all $x, y \in X$. □

2. Main Results

Theorem 2.1. *Let C be a nonempty closed convex subset of a convex complete metric space (X,d,W) and T be a self-mapping of C. If there exist a, b, c, e, f, and k such that*

$$\frac{b+e-|f|(1-\lambda)-|c|\lambda}{1-\lambda} \le k < \frac{a+b+c+e+f-|c|\lambda-|f|(1-\lambda)}{1-\lambda} \tag{2.1}$$

$$ad(x,Tx) + bd(y,Ty) + cd(Tx,Ty) + ed(x,Ty) + fd(y,Tx) \le kd(x,y) \tag{2.2}$$

for all $x, y \in C$, then T has at least one fixed point.

Proof. Fix $\lambda \in (0,1)$. Suppose $x_0 \in C$ is arbitrary. We define a sequence $\{x_n\}_{n=1}$ in the following way:

$$x_n = W(x_{n-1}, T(x_{n-1}, \lambda)), \quad n = 1, 2, 3, \cdots.$$

As C is convex, $x_n \in C$ for all $n \in \mathbb{N}$. By Lemma 1.6 and above relation, we have

$$d(x_{n+1}, x_n) = (1-\lambda)d(x_n, Tx_n), \tag{2.3}$$

$$d(x_n, Tx_{n-1}) = \lambda d(x_{n-1}, Tx_{n-1}) = \frac{\lambda}{1-\lambda} d(x_n, x_{n-1}). \tag{2.4}$$

By relation (2.3)

$$\frac{1}{1-\lambda} d(x_{n+1}, x_n) = d(x_n, Tx_n) \tag{2.5}$$

$$\le d(x_n, Tx_{n-1}) + d(Tx_{n-1}, Tx_n) \tag{2.6}$$

and

$$\frac{c}{1-\lambda} d(x_{n+1}, x_n) - \frac{|c|\lambda}{1-\lambda} d(x_n, x_{n-1}) \le cd(Tx_{n-1}, Tx_n). \tag{2.7}$$

And also by relation (2.3) and triangle inequality we have

$$\frac{1}{1-\lambda} d(x_{n+1}, x_n) = d(x_n, Tx_n) \tag{2.8}$$

$$\le d(x_n, x_{n-1}) + d(x_{n-1}, Tx_n) \tag{2.9}$$

and

$$\frac{f}{1-\lambda}d(x_{n+1},x_n) - |f|d(x_n,x_{n-1}) \leq fd(Tx_{n-1},Tx_n) \tag{2.10}$$

for all $n \in \mathbb{N}$. Now, by substituting x with x_n and y with x_{n-1} in (2.2), we get

$$\begin{aligned} &ad(x_n,Tx_n) + bd(x_{n-1},Tx_{n-1}) + cd(Tx_n,Tx_{n-1}) \\ &+ ed(x_n,Tx_{n-1}) + fd(x_{n-1},Tx_{n-1}) \\ &\leq kd(x_n,x_{n-1}) \end{aligned}$$

so by the relations (2.3),(2.4),(2.7) and (2.10), we obtain

$$\begin{aligned} &\left(\frac{a+c+f}{1-\lambda}\right)d(x_{n+1},x_n) + \left(\frac{b-|c|\lambda+e\lambda}{1-\lambda} - |f|\right)d(x_{n-1},x_n) \\ &\leq kd(x_n,x_{n-1}). \end{aligned}$$

Thus

$$d(x_n,x_{n+1}) \leq \left(\frac{k(1-\lambda)+|f|(1-\lambda)-b-e+|c|\lambda}{a+c+f}\right)d(x_n,x_{n-1})$$

for all $n \in \mathbb{N}$. By the relation (2.1) $\frac{k(1-\lambda)+|f|(1-\lambda)-b-e+|c|\lambda}{a+c+f} \in [0,1)$ and hence, $\{x_n\} \subseteq C$ is a contraction sequence. Therefore, it is a Cauchy sequence. Since C is a closed subset of a complete space, so $\lim_{n\to\infty} x_n = x^*$ for some $x^* \in C$. Now by relation (2.3)

$$\frac{1}{1-\lambda}d(x_{n+1},x_n) = d(x_n,Tx_n) \leq d(x_n,x^*) + d(x^*,Tx_n)$$

we obtain $\lim_{n\to\infty} d(x_n,Tx_n) = 0$ and by

$$d(x^*,Tx_n) \leq d(x^*,x_n) + d(x_n,Tx_n)$$

we get $\lim_{n\to\infty} Tx_n = x^*$.

Now, by substituting x with x^* and y with x_n in relation (2.2), we obtain

$$\begin{aligned} &ad(x^*,Tx^*) + bd(x_n,Tx_n) + cd(Tx^*,Tx_n) + ed(x^*,Tx_n) + fd(x_n,Tx^*) \\ &\leq kd(x^*,x_n). \end{aligned}$$

So

$$(a+c+f)d(x^*,Tx^*) \leq 0.$$

But by relation (2.1) $a+c+f \geq 0$ thus $Tx^* = x^*$. □

The following corollary improves and extends [9, Theorem 3.2].

Corollary 2.2. *Let C be a nonempty closed convex subset of a convex complete metric space (X,d,W) and T be a self-mapping of C. If there exist a,b,c and k such that*

$$2b - |c| \leq k < 2(a+b+c) - |c|, \tag{2.11}$$

$$ad(x, Tx) + bd(y, Ty) + cd(Tx, Ty) \leq kd(x, y) \tag{2.12}$$

for all $x, y \in C$*, then* T *has at least one fixed point.*

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