Nonlinear Functional Analysis and Applications
Vol. 31, No. 1 (2026), pp. 173-180
ISSN: 1229-1595(print), 2466-0973(online)
https://doi.org/10.22771/nfaa.2026.31.01.10
http://nfaa.kyungnam.ac.kr/journal-nfaa

SOME PROPERTIES OF THE SET OF CODISK-RECURRENT VECTORS

Mayada N. Mohammedali[1] and Zeana Zaki Jamil[2]

[1]Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq
Department of Mathematics and Computer Applications, College of Applied Sciences, University of Technology, Baghdad, Iraq
e-mail: `mayada.ali2303@sc.uobaghdad.edu.iq`

[2]Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq
e-mail: `zina.z@sc.uobaghdad.edu.iq`

Abstract. Extending the foundational results presented in [16], this work investigates the topological structure and dynamical behavior of the set of codisk-recurrent vectors under a continuous linear operator on infinite dimensional Hilbert space induced by polynomials, powers, and scalar multiplications.

1. Introduction

An area that has attracted considerable interest from researchers in recent decades is the dynamics of continuous linear operators. One of the first important turning points in the development of this theory is Kitai's PhD thesis [13], which is often regarded as a foundational contribution. For more information on the dynamics of linear operators, see [3, 5, 8, 9, 12, 15]. There are many applications for linear operators on infinite-dimensional spaces, which may exhibit rich dynamical features that highlight deep relationships among dynamical systems, differential geometry, and functional analysis.

[0]Received August 23, 2025. Revised December 7, 2025. Accepted December 13, 2025.
[0]2020 Mathematics Subject Classification: 47A16.
[0]Keywords: Hypercyclic operators, hypercyclic vectors, codisk-cyclic operators, codisk-cyclic vectors, recurrent operators, recurrent vectors.
[0]Corresponding author: M. N. Mohammedali(`mayada.ali2303@sc.uobaghdad.edu.iq`).

Through this work, let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space, and denote $\mathbf{B}(\mathcal{H}) := \{\mathcal{T} : \mathcal{H} \longrightarrow \mathcal{H} : \mathcal{T}\text{is continuous linear operator}\}$. A vector $\mathfrak{h} \in \mathcal{H}$ with $\overline{span(orbt(\mathcal{T}, \mathfrak{h}))} := \overline{span\{\mathcal{T}^n\mathfrak{h} : n \geq 0\}} = \mathcal{H}$ is called a cyclic vector for $\mathcal{T}$. While a vector $\mathfrak{h} \in \mathcal{H}$ is referred to a hypercyclic if $\overline{orbt(\mathcal{T}, \mathfrak{h})} := \overline{\{\mathcal{T}^n\mathfrak{h} : n \geq 0\}} = \mathcal{H}$. Accordingly, $\mathcal{T}$ is hypercyclic whenever such a vector exists. In [17], Jamil divided the cone orbit $COrb(\mathcal{T}, \mathfrak{h}) := \{\lambda\mathcal{T}^n\mathfrak{h} : \lambda \in \mathbb{C}, n \geq 0\}$ into two parts: operators that are diskcyclic if $|\lambda| \leq 1$, and operators that are codiskcyclic if $|\lambda| \geq 1$. See [1, 2, 11, 14], for background information on these structures.

Recurrence is an additional property that has emerged in recent years in linear dynamics; it is especially related to our subject here. A vector $\mathfrak{h} \in \mathcal{H}$ is recurrent for $\mathcal{T}$ if it is in the closure of forward orbit itself, this means, if $\mathfrak{h} \in \overline{Orb(\mathcal{T}, \mathcal{T}\mathfrak{h})} := \overline{\{\mathcal{T}^n\mathfrak{h} : n \geq 1\}}$. In addition, the recurrent vector set for $\mathcal{T}$, denoted as $Rec(\mathcal{T})$, is dense in $\mathcal{H}$ if and only if $\mathcal{T}$ is recurrent. The recent study in [7] emphasized recurrence as a topic of significance within the context of linear dynamics. Although many subsequent studies [4, 10] and [6] have examined the topic from various viewpoints, the notion of codisk-recurrence has been proposed in [16] within a topological dynamical framework, along with a description of the set of codisk-recurrent vectors. This work presents a novel contribution by exploring the structure of this set and its dynamical features induced by various modifications of the operator.

The following is the organization of this paper: Section 2 is dedicated to exploring the structure of the set of codisk-recurrent vectors, which is topologically characterized by demonstrating that it is G_δ-set. It is also established that the codisk-recurrent vectors set is invariant under the polynomial actions of a continuous linear operator that admits a codisk-recurrent vector. In Section 3, we examine how the codisk-recurrent vectors set behaves dynamically under two main operations: raising the operator to positive integer powers and scalar multiplication. The set of codisk-recurrent vectors is demonstrated to be preserved under operator powers. Additionally, when the scalar is on the unit circle in the context of scalar multiplication, the set is preserved. However, when the inequality $|\alpha_2| \leq |\alpha_1|$ holds, every codisk-recurrent vector of $\alpha_1\mathcal{T}$ is codisk-recurrent vector of $\alpha_2\mathcal{T}$. Finally, Section 4 presents the main results of the article.

Notations. We denote the codisk in the complex plane as

$$\mathbb{B}^c := \{\lambda \in \mathbb{C} : |\lambda| \geq 1\}$$

and the unit circle as

$$\mathbb{S}(0, 1) := \{\lambda \in \mathbb{C} : |\lambda| = 1\}.$$

The set of non-negative integers is represented by $\mathbb{N}$.

2. Preliminaries

The structural aspects of the codisk-recurrent vectors set associated with a linear continuous operator on infinite-dimensional Hilbert space were examined in this section. The notion of a codisk-recurrent vector set was presented in [16] as follows:

Definition 2.1. ([16]) A vector $\mathfrak{h}$ in $\mathcal{H}$ is termed codisk-recurrent for $\mathcal{T}$ in $\mathbf{B}(\mathcal{H})$, if there exists a strictly increasing $\{n_k\} \subset \mathbb{N}$ and $\{\lambda_k\} \subset \mathbb{B}^c$ such that $\lambda_k \mathcal{T}^{n_k}\mathfrak{h}$ converges to $\mathfrak{h}$ as k approaches to infinity.

$\mathbb{D}^C Rec(\mathcal{T})$, which represents the set of all codisk-recurrent vectors for $\mathcal{T}$.

Remark 2.2. From the Definition 2.1, $0 \in \mathbb{D}^C Rec(\mathcal{T})$ for any $\mathcal{T} \in \mathbf{B}(\mathcal{H})$, that is, $\mathbb{D}^C Rec(\mathcal{T}) \neq \emptyset$.

The following proposition demonstrates that $\mathbb{D}^C Rec(\mathcal{T})$ is invariant under the action of any linear operator that commutes with it.

Theorem 2.3. *If $\mathcal{T}, \hat{\mathcal{T}} \in \mathbf{B}(\mathcal{H})$ with the property, $\mathcal{T}\hat{\mathcal{T}} = \hat{\mathcal{T}}\mathcal{T}$, then $\hat{\mathcal{T}}(\mathbb{D}^C Rec(\mathcal{T})) \subseteq \mathbb{D}^C Rec(\mathcal{T})$.*

Proof. Let $\mathfrak{h}$ be a vector of $\mathbb{D}^C Rec(\mathcal{T})$. Then there is a strictly increasing $\{n_k\} \subset \mathbb{N}$ and $\{\lambda_k\} \subset \mathbb{B}^c$ with $\lambda_k \mathcal{T}^{n_k}\mathfrak{h} \longrightarrow \mathfrak{h}$ as $k \longrightarrow \infty$. Since $\hat{\mathcal{T}}$ is continuous, then $\hat{\mathcal{T}}(\lambda_k \mathcal{T}^{n_k}\mathfrak{h}) \longrightarrow \hat{\mathcal{T}}(\mathfrak{h})$ as $k \longrightarrow \infty$. Given that $\mathcal{T}\hat{\mathcal{T}} = \hat{\mathcal{T}}\mathcal{T}$, $\lambda_k \mathcal{T}^{n_k}\hat{\mathcal{T}}\mathfrak{h} \longrightarrow \hat{\mathcal{T}}\mathfrak{h}$ as $k \longrightarrow \infty$. Thus $\hat{\mathcal{T}}\mathfrak{h} \in \mathbb{D}^C Rec(\mathcal{T})$. □

This result shows that a codisk-recurrent vector is not merely an element satisfying a convergence condition, but rather generates a subspace of recurrent vectors that remains invariant under all polynomials in $\mathcal{T}$, thanks to Theorem 2.3.

Corollary 2.4. *The set $\Lambda := \{p(\mathcal{T})\mathfrak{h} : p \text{ is a polynomial}\}$, where $\mathfrak{h} \in \mathbb{D}^C Rec(\mathcal{T})$, is a subset of $\mathbb{D}^C Rec(\mathcal{T})$.*

The subsequent outcome demonstrates that the set is a G_δ-set, highlighting a fundamental characteristic of its topological structure.

Theorem 2.5. *Let $\mathcal{T} \in \mathbf{B}(\mathcal{H})$. Then*

$$\mathbb{D}^C Rec(\mathcal{T}) = \bigcap_{r=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcup_{\lambda \in \mathbb{B}^c} \left\{ \mathfrak{h} \in \mathcal{H} : \|\lambda \mathcal{T}^n \mathfrak{h} - \mathfrak{h}\| < \frac{1}{r} \right\}$$

is a G_δ-set.

Proof. We will first prove that for

$$\mathbb{D}^C Rec(\mathcal{T}) = \bigcap_{r=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcup_{\lambda \in \mathbb{B}^c} \left\{ \mathfrak{h} \in \mathcal{H} : \|\lambda \mathcal{T}^n \mathfrak{h} - \mathfrak{h}\| < \frac{1}{r} \right\}$$

$-\ell \in \mathbb{D}^C Rec(\mathcal{T})$, if and only if there exists a strictly increasing $\{n_k\} \subset \mathbb{N}$ and $\{\lambda_k\} \subset \mathbb{B}^c$ with $\lambda_k \mathcal{T}^{n_k}\ell \longrightarrow \ell$ as $k \longrightarrow \infty$ if and only if for all $\varepsilon > 0$, there is $N \in \mathbb{N}$ with $\|\lambda_k \mathcal{T}^{n_k}\ell - \ell\| < \varepsilon$ for all $k \geq N$ if and only if, for all $r \in \mathbb{N}$, there are $n_k \in \mathbb{N}$ and $\lambda_k \in \mathbb{B}^c$ such that $\|\lambda_k \mathcal{T}^{n_k}\ell - \ell\| < \frac{1}{r}$ for all $k \geq N$ if and only if

$$\ell \in \bigcap_{r=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcup_{\lambda \in \mathbb{B}^c} \left\{ \mathfrak{h} \in \mathcal{H} : \|\lambda \mathcal{T}^n \mathfrak{h} - \mathfrak{h}\| < \frac{1}{r} \right\}.$$

We now prove the second claim:

For each $r \geq 1$,we will define the following set:

$$U_r := \bigcup_{n=1}^{\infty} \bigcup_{\lambda \in \mathbb{B}^c} \left\{ \mathfrak{h} \in \mathcal{H} : \|\lambda \mathcal{T}^n \mathfrak{h} - \mathfrak{h}\| < \frac{1}{r} \right\}.$$

Let $n \geq 1$ and $\lambda \in \mathbb{B}^c$. Define a map:

$$\vartheta_{n,\lambda} : \mathcal{H} \longrightarrow \mathcal{H}$$

by

$$\vartheta_{n,\lambda}(\mathfrak{h}) = (\lambda \mathcal{T}^n - I)(\mathfrak{h}).$$

Since $\mathcal{T}^n$ is continuous, it follows that $\vartheta_{n,\lambda}$ is also continuous. According to $V_{n,\lambda} = \left\{ \mathfrak{h} \in \mathcal{H} : \|\lambda \mathcal{T}^n \mathfrak{h} - \mathfrak{h}\| < \frac{1}{r} \right\} = \vartheta_{n,\lambda}^{-1}(\mathbb{B}(0, \frac{1}{r})$, $V_{n,\lambda}$ is an open set. This indicates that U_r is open. Therefore, by (Baire's Theorem), $\mathbb{D}^C Rec(\mathcal{T}) = \cap_{r=1}^{\infty} U_r$ is a G_δ-set. □

3. Algebraic Properties

In this section, the dynamic behavior of the codisk-recurrent vectors set is examined concerning two fundamental operations: the raising of the operator to a positive integer and scalar multiplication powers.

The following theorem demonstrates that $\mathbb{D}^C Rec(\mathcal{T})$ is preserved under all positive powers of the operator.

Theorem 3.1. *An operator $\mathcal{T} \in \mathbf{B}(\mathcal{H})$ is codisk-recurrent if and only if $\mathcal{T}^p$; $p \geq 1$ is codisk-recurrent. Additionally, $\mathcal{T}$ and $\mathcal{T}^p$ possess identical $\mathcal{T}$-codisk-recurrent vectors.*

Proof. It is sufficient to prove that $\mathbb{D}^C Rec(\mathcal{T}) \subseteq \mathbb{D}^C Rec(\mathcal{T}^p)$ to show that $\mathbb{D}^C Rec(\mathcal{T}) = \mathbb{D}^C Rec(\mathcal{T}^p)$. Let $\mathfrak{h} \in \mathbb{D}^C Rec(\mathcal{T})$. Then, there is a strictly increasing $\{n_k\} \subset \mathbb{N}$ and a sequence $\{\lambda_k\} \subset \mathbb{B}^c$ such that

$$\lambda_k \mathcal{T}^{n_k} \mathfrak{h} \longrightarrow \mathfrak{h} \quad \text{as} \quad k \longrightarrow \infty. \tag{3.1}$$

We can assume that without loss of generality, $n_k > p$ for all k. Thus by the division algorithm theorem, for every k, there are $\ell_k \in \mathbb{N}$ and $0 \leq v_k \leq p-1$ such that $n_k = p\ell_k + v_k$. Hence, we obtain

$$\lambda_k \mathcal{T}^{n_k}\mathfrak{h} = \lambda_k (\mathcal{T}^p)^{\ell_k + v_k}\mathfrak{h}. \tag{3.2}$$

Since $\{v_k\}_{k\in\mathbb{N}}$ is bounded in $\mathbb{R}$, by the Bolzano-Weierstrass theorem, it has a convergent subsequence that converges to a specific value, say v, where $0 \leq v \leq p-1$. Then, as a consequence of (3.1), (3.2), there exist $\{\ell_{k_i}\} \subset \{\ell_k\}$ and $\{\lambda_{k_i}\}_{i\in\mathbb{N}} \subset \{\lambda_k\}$ such that

$$\lambda_{k_i}\mathcal{T}^{p\ell_{k_i}+v}\mathfrak{h} \longrightarrow \mathfrak{h} \quad \text{as} \quad i \longrightarrow \infty.$$

Now, we want to prove that the vector $\mathfrak{h}$ is $\mathcal{T}^p$-codisk-recurrent. Let U be an open neighborhood of $\mathfrak{h}$. Given that $\lambda_{k_i}\mathcal{T}^{p\ell_{k_i}+v}\mathfrak{h} \longrightarrow \mathfrak{h}$ as $i \longrightarrow \infty$, we can determine $s_1 := \ell_{k_1}$ satisfying $\lambda_{k_1}\mathcal{T}^{ps_1+v}\mathfrak{h} \in U$.

By continuity of $\mathcal{T}^{p\ell_{k_i}+v}$, we have

$$\lambda_{k_i}\lambda_{k_1}\mathcal{T}^{p(\ell_{k_i}+s_1)+2v}\mathfrak{h} = \lambda_{k_i}\lambda_{k_1}\mathcal{T}^{ps_1+v}\mathcal{T}^{p\ell_{k_i}+v}\mathfrak{h} \longrightarrow \lambda_{k_1}\mathcal{T}^{ps_1+v}\mathfrak{h} \text{ in } U.$$

Consequently, we can determine $s_2 = s_1 + \ell_{k_2} > s_1$ satisfying $\lambda_{k_1}\lambda_{k_2}\mathcal{T}^{ps_2+2v}\mathfrak{h}$ in U. Proceeding inductively, we can determine $s_p = s_{p-1} + \ell_{k_p}$ satisfying

$$\lambda_{k_1}\dots\lambda_{k_p}\mathcal{T}^{ps_p+pv}h \text{ in } U.$$

After putting $\lambda = \lambda_{k_1}\dots\lambda_{k_p}$, we get $\lambda(\mathcal{T}^p)^{s_p+v}\mathfrak{h}$ in U, which means that $\mathfrak{h}$ is a codisk-recurrent for $\mathcal{T}^p$. Therefore, $\mathbb{D}^C Rec(\mathcal{T}) \subseteq \mathbb{D}^C Rec(\mathcal{T}^p)$. □

As we investigate the dynamical behavior of the codisk-recurrent vectors set, a natural inquiry arises: Is the set necessarily preserved by the scalar multiplication of the operator?

In each instance, the preservation of this property is conditional on the specific relationship between the scalars. One such instance is described in the following result, which indicates that the preservation of $\mathbb{D}^C Rec(\mathcal{T})$ is guaranteed when the scalar is located on the unit circle.

Theorem 3.2. *Let $\mathcal{T} \in \mathbf{B}(\mathcal{H})$. Then for any unit modulus scalar γ,*

$$\mathbb{D}^C Rec(\mathcal{T}) = \mathbb{D}^C Rec(\gamma\mathcal{T}).$$

Proof. Suppose that $\mathfrak{h} \in \mathbb{D}^C Rec(\mathcal{T})$. To prove that $\mathfrak{h} \in \mathbb{D}^C Rec(\gamma\mathcal{T})$, we establish

$$\mathcal{A} := \{\alpha \in \mathbb{S}(0,1) : \lambda_k(\gamma\mathcal{T})^{n_k}\mathfrak{h} \longrightarrow \alpha\mathfrak{h} \text{ as } k \longrightarrow \infty \text{ for } \{n_k\} \subset \mathbb{N} \text{ and } \{\lambda_k\} \subset \mathbb{B}^c\},$$

this set contains all possible rotation factor α, asymptotically $\lambda_k(\gamma\mathcal{T})^{n_k}$ gets closer to $\alpha\mathfrak{h}$. Since $\mathfrak{h} \in \mathbb{D}^C Rec(\mathcal{T})$, there is a strictly increasing $\{n_k\} \subset \mathbb{N}$, and sequence $\{\lambda_k\} \subset \mathbb{B}^c$, with $\lambda_k\mathcal{T}^{n_k}\mathfrak{h} \longrightarrow \mathfrak{h}$ as $k \longrightarrow \infty$. Since $|\gamma| = 1$, we have $\{\gamma^{n_k}\} \subset \mathbb{S}(0,1)$. Because, $\mathbb{S}(0,1)$ has the compactness property, thus, there is a subsequence $\{\gamma^{n_{k_s}}\}_{s\in\mathbb{N}}$ such that $\gamma^{n_{k_s}} \longrightarrow \beta$ for some $\beta \in \mathbb{S}(0,1)$. Therefore, we obtain

$$\|\gamma^{n_{k_s}}\lambda_k \mathcal{T}^{n_k}\mathfrak{h} - \beta\mathfrak{h}\| \leq |\gamma^{n_{k_s}}| \, \|\lambda_k \mathcal{T}^{n_k}\mathfrak{h} - \mathfrak{h}\| + |\gamma^{n_{k_s}} - \beta| \, \|\mathfrak{h}\| \, .$$

As $k, s \longrightarrow \infty$, we get $\gamma^{n_{k_s}}\lambda_k \mathcal{T}^{n_k}\mathfrak{h} \longrightarrow \beta\mathfrak{h}$. Consequently, $\beta \in \mathcal{A}$, meaning that $\mathcal{A} \neq \emptyset$.

Now, we will prove that $\mathcal{A}$ forms a multiplicative semigroup. In other words, if $\beta_1, \beta_2 \in \mathcal{A}$, then $\beta_1\beta_2 \in \mathcal{A}$. By the definition of $\mathcal{A}$, there are $\{n_{k_1}\}, \{n_{k_2}\}$ in $\mathbb{N}$ and $\{\lambda_{k_1}\}, \{\lambda_{k_2}\}$ in $\mathbb{B}^c$ such that

$$\lambda_{k_1}(\gamma\mathcal{T})^{n_{k_1}}\mathfrak{h} \longrightarrow \beta_1\mathfrak{h} \text{ and } \lambda_{k_2}(\gamma\mathcal{T})^{n_{k_2}}\mathfrak{h} \longrightarrow \beta_2\mathfrak{h} \quad \text{as} \quad k_1, k_2 \longrightarrow \infty.$$

Thus,

$$\lambda_{k_1}\lambda_{k_2}(\gamma\mathcal{T})^{n_{k_1}+n_{k_2}}\mathfrak{h} = \lambda_{k_1}(\gamma\mathcal{T})^{n_{k_1}} \left(\lambda_{k_2}(\gamma\mathcal{T})^{n_{k_2}}\mathfrak{h}\right).$$

And by the continuity of $(\gamma\mathcal{T})^{n_{k_2}}$ as $n_{k_2} \longrightarrow \infty$, we get

$$\lambda_{k_1}(\gamma\mathcal{T})^{n_{k_1}} \left(\lambda_{k_2}(\gamma\mathcal{T})^{n_{k_2}}\mathfrak{h}\right) \longrightarrow \lambda_{k_1}(\gamma\mathcal{T})^{n_{k_1}} \left(\beta_2\mathfrak{h}\right).$$

And since $\lambda_{k_1}(\gamma\mathcal{T})^{n_{k_1}}\mathfrak{h} \longrightarrow \beta_1\mathfrak{h}$ as $n_{k_1} \longrightarrow \infty$, we conclude

$$\lambda_{k_1}(\gamma\mathcal{T})^{n_{k_1}} \left(\beta_2\mathfrak{h}\right) \longrightarrow \beta_1\beta_2\mathfrak{h}.$$

Hence $\beta_1\beta_2 \in \mathcal{A}$ ($\mathcal{A}$ is closed under multiplication).

Applying this condition repeatedly, the semigroup property ensures that for each $\beta \in \mathcal{A}$ and any n, m in $\mathbb{N}$, $\beta^{n+m} \in \mathcal{A}$, that is, all positive integer powers of any element are contained in $\mathcal{A}$.

Now our goal to determine the properties of $\mathcal{A}$ for rational and irrational rotations and that $1 \in \mathcal{A}$ in both cases:

Case1: If $\beta = e^{2\pi i\theta}$, where $\theta = \frac{m}{n} \in \mathbb{Q}; m, n \in \mathbb{Z}$ and $n \neq 0$, then $\beta^n = 1$, this implies that $1 \in \mathcal{A}$.

Case2: If $\beta = e^{2\pi i\theta}$, where $\theta \in \mathbb{Q}^c$, then the set $\{\beta^\omega : \omega \in \mathbb{N}\}$ is dense in $\mathbb{S}(0,1)$. Hence, there is a strictly increasing subsequence $\{\omega_n\}_{n\in\mathbb{N}} \subset \mathbb{N}$ with $\beta^{\omega_n} \longrightarrow 1$. Note that $\mathcal{A}$ is also topologically closed, we deduce that $1 \in \mathcal{A}$. Therefore, there exists a strictly increasing $\{n_k\} \subset \mathbb{N}$ and $\{\lambda_k\} \subset \mathbb{B}^c$ with $\lambda_k(\gamma\mathcal{T})^{n_k}\mathfrak{h} \longrightarrow \mathfrak{h}$ as $k \longrightarrow \infty$, then $\mathfrak{h} \in \mathbb{D}^C Rec(\gamma\mathcal{T})$.

□

The following result discusses the instance where $|\alpha_2| \leq |\alpha_1|$ for any $\alpha_1, \alpha_2 \in \mathbb{C}$ and demonstrates that $\mathbb{D}^C Rec(\alpha_1\mathcal{T})$ is contained within $\mathbb{D}^C Rec(\alpha_2\mathcal{T})$.

Theorem 3.3. *Let $\mathcal{T} \in \mathbf{B}(\mathcal{H})$. Then $\mathbb{D}^C Rec(\alpha_1\mathcal{T}) \subseteq \mathbb{D}^C Rec(\alpha_2\mathcal{T})$ for all $\alpha_1, \alpha_2 \in \mathbb{C}$; $|\alpha_2| \leq |\alpha_1|$.*

Proof. Let $\mathfrak{h} \in \mathbb{D}^C Rec(\alpha_1 \mathcal{T})$. Then, there exists a strictly increasing $\{n_k\} \subset \mathbb{N}$ and $\{\lambda_k\} \subset \mathbb{B}^c$ with

$$\lambda_k \alpha_1^{n_k} \mathcal{T}^{n_k} \mathfrak{h} \longrightarrow \mathfrak{h} \quad \text{as} \quad k \longrightarrow \infty.$$

Since $|\alpha_2| \leq |\alpha_1|$, we can pick α_k; $|\alpha_k| \geq 1$, ensuring that $\alpha_1^{n_k} = \alpha_k \alpha_2^{n_k}$ for all k. Let $\gamma_k = \lambda_k \alpha_k$. Then it follows that $\gamma_k \in \mathbb{B}^c$. Consequently, we get

$$\gamma_k (\alpha_2 \mathcal{T})^{n_k} \mathfrak{h} \longrightarrow \mathfrak{h} \quad \text{as} \quad k \longrightarrow \infty.$$

Therefore, $\mathfrak{h} \in \mathbb{D}^C Rec(\alpha_2 \mathcal{T})$. That is, $\mathbb{D}^C Rec(\alpha_1 \mathcal{T}) \subseteq \mathbb{D}^C Rec(\alpha_2 \mathcal{T})$. □

Recall that $\mathcal{T} \in \mathbf{B}(\mathcal{H})$ is a codisk-recurrent if and only if $\overline{\mathbb{D}^C Rec(\mathcal{T})} = \mathcal{H}$ as stated in [16], one can get the following.

Corollary 3.4. *Let $\mathcal{T} \in \mathbf{B}(\mathcal{H})$. If $\alpha_1 \mathcal{T}$ is a codisk-recurrent operator, then $\alpha_2 \mathcal{T}$ is a codisk-recurrent operator for all $\alpha_1, \alpha_2 \in \mathbb{C}$; $|\alpha_1| \geq |\alpha_2|$.*

In Theorem 3.3, if $|\beta| \leq 1$ we get that $\mathbb{D}^C Rec(\mathcal{T}) \subseteq \mathbb{D}^C Rec(\beta \mathcal{T})$. But, in general, the converse is not true for all $|\beta| < 1$.

Example 3.5. Consider the Hilbert space $\mathcal{H} = \mathbb{C}$. Define the operator $\mathcal{T} : \mathbb{C} \longrightarrow \mathbb{C}$ as $\mathcal{T}\mathfrak{h} = 2h$, and let $\beta = \frac{1}{2}$. For any non-zero $\mathfrak{h}$ in $\mathbb{C}$, $(\beta \mathcal{T})\mathfrak{h} = \mathfrak{h}$.

Now for any $k \in \mathbb{N}$ and $\epsilon > 1$, choose the sequences $\{\lambda_k\} = \left\{1 + \frac{1}{\epsilon^k}\right\}$ and $\{n_k\} = \{k\}$. Note that $\lambda_k \longrightarrow 1$ as $k \longrightarrow \infty$. Then we have $\lambda_k (\beta \mathcal{T})^{n_k} \mathfrak{h} \longrightarrow \mathfrak{h}$. Hence $\mathfrak{h} \in \mathbb{D}^C Rec(\beta \mathcal{T})$.

While, if $\mathfrak{h} \in \mathbb{D}^C Rec(\mathcal{T})$, then there exists sequences $\{\lambda_k\}$, and $\{n_k\}$ such that as $k \longrightarrow \infty$, $\lambda_k \mathcal{T}^{n_k} h \longrightarrow \mathfrak{h}$. Hence $2^{n_k} \lambda_k \mathfrak{h} \longrightarrow \mathfrak{h}$, thus $|\lambda_k| \longrightarrow 2^{-n_k}$. Therefore, $|\lambda_k| \longrightarrow 0$ as $k \longrightarrow \infty$, which is a contradiction.

4. Conclusion

This paper is a complement of the work initiated in [16], where the foundational idea of a codisk-recurrent vector set of linear continuous operators was introduced. The present work aims to investigate the topological structure and dynamical aspects of such sets under transformations induced by polynomials of that operator, powers, and scalar multiplications. It is shown that $\mathbb{D}^C Rec(\mathcal{T})$ forms a G_δ-set. It is also proved that the image of any codisk-recurrent vector is invariant under a polynomial action of the operator. In addition, we study the behavior of these sets under two fundamental operations: raising the operator to positive integer powers and scalar multiplication.

It is proved that applying positive powers of the operator preserves the set of codisk-recurrent vectors, that is,

$$\mathbb{D}^C Rec(\mathcal{T}) = \mathbb{D}^C Rec(\mathcal{T}^p)$$

for all $p \in \mathbb{N}$. As for scalar multiplication, it is established that preservation is guaranteed when the scalar lies on the unit circle; that is, $\mathbb{D}^C Rec(\mathcal{T}) = \mathbb{D}^C Rec(\gamma\mathcal{T})$ for all γ in $\mathbb{C}$ such that $|\gamma| = 1$. However, scalar multiplication does not always preserve codisk-recurrence.

In particular, we prove that $\mathbb{D}^C Rec(\alpha_1\mathcal{T}) \subseteq \mathbb{D}^C Rec(\alpha_2\mathcal{T})$ for all $\alpha_1, \alpha_2 \in \mathbb{C}$ whenever $|\alpha_2| \leq |\alpha_1|$.

Acknowledgments: The author would like to express their gratitude to the referee for their helpful comments.

References

[1] R. Abdulkareem and Z.Z. Jamil, *On the Diskcyclic Criterions*, Iraqi J. Sci., (2020), 2651-2654.

[2] P. Aiena, F. Burderi and S. Triolo, *On the spectrum of supercyclic/hypercyclic operators,* Adv. Oper. Theory, **10**(52) (2025).

[3] M. Amouch and O. Benchiheb, *On linear dynamics of sets of operators,* Turkish J. Math., **43** (2019), 402-411.

[4] N. Bamerni, *D-recurrence of operators on Banach spaces,* Bol. Soc. Parana. Mate., **43** (2025), 1–9.

[5] V. Béhani, *A study of Bishop operators from the point of view of linear dynamics,* J. Math. Anal. Appl., **526**(1) (2023) 127204.

[6] G. Bulancea and H.N. Salas, *Recurrence Operators on Function Spaces,* Complex Anal. Oper. Theory, **19**(6) (2025): 151.

[7] G. Costakis, A. Manoussos and I. Parissis, *Recurrent linear operators,* Complex Anal. Oper. Theory, **8**(8) (2014), 1601-1643. doi: 10.1007/s11785-013-0348-9.

[8] S.O. Dakheel and B.A. Ahmed, *nC-symmetric operators,* Iraqi J. Sci., **63** (2022), 3098-3102.

[9] H. Dymova, *Dynamic Operator Extraction Method, Computer-Integrated Technologies,* Education, Science, Production, (2023), 43-47.

[10] S. Grivaux and A. López-Martnez, *nRecurrence properties for linear dynamical systems: an approach via invariant measures,* J. Math. Pures Appl., **169** (2023), 155-188.

[11] K.G. Grosse-Erdmann, *Hypercyclic and chaotic weighted shifts,* Studia Math., **139**(1) (2000), 47-68.

[12] S.N. Kadhim, M.J.M. Ali and Z.A. Atiya, *A note on s-acts and bounded linear operators,* Ital. J. Pure Appl. Math., **39** (2018), 16-22.

[13] C. Kitai, *Invariant Closed Sets for Linear Operators,* 1982.

[14] Y.X. Liang and Z.H. Zhou, *Disk-cyclic and Codisk-cyclic tuples of the adjoint weighted composition operators on Hilbert spaces,* Bull. Belg. Math. Soc.- Simon Stevin, **23**(2) (2016), 203-215.

[15] Z.H. Maibed, *On multi-coupled points of continuous mappings in generalized metric spaces,* J. Interdiscip. Math., **26**(2) (2023), 573-578.

[16] M.N. Mohammedali and Z.Z. Jamil, *Codisk-recurrent Operators on Hilbert Spaces,* J. Interdiscip. Mat., Accepted on 27-6-2025.

[17] Z.J. Zeana, *Cyclic Phenomena of operators on Hilbert space,* 2002.