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ON THE ALEKSANDROV PROBLEM IN NON-ARCHIMEDEAN 2-NORMED SPACES

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Abstract. This paper show that every generalized area $n-$preserving mapping between non-Archimedean $2-$normed spaces X and Y is a generalized $2-$isometry under some conditions. In addition, we also showed the Alksandrov problem in non-Archimedean $n-$normed spaces under some conditions.

1. Introduction

In 1970, Aleksandrov in [1] posed the question that: whether the exist of the single preserved distance implies that f is an isometry from the metric space X into itself.

Until now, the Alesandrov problem in linear normed spaces has been studied in reference [2-6]. Recently Chu et al in [3] begin to consider the Aleksandrov problem in linear $2-$normed spaces. They introduce the concept of $2-$isometry and prove that Rassias and Semrl's theorem holds under some conditions.

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By utilizing the idea of preserving colinear, the authors give the following conclusion.

Theorem 1.1. ([3]) *Let X and Y be $2-$normed space and $f : X \to Y$, if f is a $2-$Lipschitz mapping with the $2-$Lipschitz constant $K \leq 1$, if x, y and z are colinear implies $f(x), f(y)$ and $f(z)$ are colinear and if f satisfies (AOPP), then f is a $2-$isometry.*

After that, Ren Weiyun in [7] proved that the theorem still hold without the condition of preserving colinear. The author give the following conclusion.

Theorem 1.2. ([7]) *Let X and Y be $2-$normed space and $f : X \to Y$ satisfies (GAnPP) for all $n \in N$, if $\|f(x) - f(z), f(p) - f(q)\| \leq \|x - z, p - q\|$ for all $x, z, p, q \in X$ with $\|x - z, p - q\| \leq 1$, then f is a generalized $2-$isometry.*

A natural question is that: Whether the abover theorem still holds in the non-Archimedean $2-$normed space? In this paper, we prove that the answer is positive if $\|f(x) - f(z), f(p) - f(q)\| \leq \|x - z, p - q\|$ for all $x, z, p, q \in X$.

A non-Archimedean filed [8] is a filed $\mathcal{K}$ equipped with a function (valuation) $|\cdot|$ from $\mathcal{K}$ into $[0, \infty)$ such that $|r| = 0$ if and only if $r = 0, |rs| = |r||s|$, and $|r + s| \leq max\{|r|, |s|\}$ for all $r, s \in \mathcal{K}$. Clearly $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in N$. An example of a non-Archimedean valuation is the mapping $|\cdot|$ taking everything but 0 into 1 and $|0| = 0$. This valuation is called trivial.

Another example of a non-Archimedean valuation is the mapping

$$|r|_1 = \begin{cases} 0, & \text{if} \quad r = 0, \\ \frac{1}{r}, & \text{if} \quad r > 0, \\ -\frac{1}{r}, & \text{if} \quad r < 0, \end{cases}$$

for any $r \in \mathcal{K}$ with the condition that $r = r_1 + r_2$ with $r_1 \cdot r_2 > 0$.

2. The aleksandrov problem in non-archimedean $2-$normed spaces

Definition 2.1. ([9]) Let X be a vector space of dimension greater than 1 over a filed $\mathcal{K}$ with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot, \cdot\| : X \times X \to [0, \infty)$ is said to be a non-Archimedean $2-$norm if it satisfies the following conditions:

(i) $\|x, y\| = 0$ if and only if x, y are linearly dependent;
(ii) $\|x, y\| = \|y, x\|$;
(iii) $\|rx, y\| = |r|\|x, y\| \quad (r \in \mathcal{K}, x, y \in X)$;
(iv) the strong triangle inequality
$\|x, y + z\| \leq max\{\|x, y\|, \|x, z\|\} \quad (x, y, z \in X)$.

Then $(x, \|\cdot, \cdot\|)$ is called a non-Archimedean $2-$normed space.

From now on, we assume that X and Y be non-Archimedean 2-normed linear spaces over a field $\mathcal{K}$ with a non-Archimedean valuation $|\cdot|_1$, f be a mapping from X into Y if without special statements.

Definition 2.2. Let X and Y be non-Archimedean $2-$normed linear spaces and $f : X \to Y$ a mapping. We say that f is a generalized 2-isometry if

$$\|x - w, y - z\| = \|f(x) - f(w), f(y) - f(z)\|$$

for all $x, w, y, z \in X$. In particular if $w = z$, then $f$is said to be a 2-isometry.

Definition 2.3. Let X and Y be non-Archimedean $2-$normed linear spaces and $f : X \to Y$ a mapping. We say that f is a generalized area n preserving property (GAnPP) if

$$\|x - w, y - z\| = n$$

implies that

$$\|f(x) - f(w), f(y) - f(z)\| = n$$

for all $x, w, y, z \in X$. In particular if $n = 1$, then f is said to satisfy the generalized area one preserving property (GAOPP).

Definition 2.4. Let X and Y be non-Archimedean $2-$normed linear spaces and $f : X \to Y$ a mapping. We say that f is 2-Lipschitz mapping if there is a $K > 0$ such that

$$\|f(x) - f(w).f(y) - f(z)\| \leq K\|x - w, y - z\|$$

for all $x, w, y, z \in X$. The smallest such K is called the Lipschitz constant.

Lemma 2.5. ([9]) *Let X be non-Archimedean 2-normed linear spaces, then $\|x, y\| = \|x, y + rx\|$ for all $x, y \in X$ and all $r \in \mathcal{K}$.*

Lemma 2.6. *Let X and Y be non-Archimedean $2-$normed linear spaces and $f : X \to Y$ satisfies GAOPP and*

$$\|f(x) - f(z).f(p) - f(q)\| \leq \|x - z, p - q\|$$

for all $x, z, p, q \in X$ with $\|x - z, p - q\| \leq 1$, then f satisfies

$$\|f(x) - f(z).f(p) - f(q)\| = \|x - z, p - q\|$$

for all $x, z, p, q \in X$ with $\|x - z, p - q\| \leq 1$.

Proof. If

$$\|f(x) - f(z).f(p) - f(q)\| < \|x - z, p - q\|,$$

let

$$w = z - \|x - z, p - q\|(x - z),$$

then

$$\|w - z, p - q\| = \| \|x - z, p - q\|(x - z), p - q\| = 1$$

and

$$\begin{aligned} & \|w - x, p - q\| \\ = \ & \|z - x - \|x - z, p - q\|(x - z), p - q\| \\ = \ & \frac{\|x - z, p - q\|}{1 + \|x - z, p - q\|} \\ < \ & 1. \end{aligned}$$

Hence

$$\|f(w) - f(z), f(p) - f(q)\| = 1$$

and

$$\|f(w) - f(x), f(p) - f(q)\| \leq \|w - x, p - q\| < 1.$$

On the other hand,

$$\begin{aligned} & \|f(w) - f(z), f(p) - f(q)\| \\ \leq \ & \max\{\|f(w) - f(x), f(p) - f(q)\|, \|f(x) - f(z).f(p) - f(q)\|\} \\ < \ & 1. \end{aligned}$$

This contradicts the equality $\|f(w) - f(z).f(p) - f(q)\| = 1$. Hence

$$\|f(x) - f(z).f(p) - f(q)\| = \|x - z, p - q\|$$

for all $x, z, p, q \in X$ with $\|x - z, p - q\| \leq 1$. □

Theorem 2.7. *Let X and Y be non-Archimedean $2-$normed linear spaces and $f : X \to Y$ satisfies GAnPP for all $n \in N$, if f is a 2-Lipschitz mapping with $K = 1$:*

$$\|f(x) - f(z).f(p) - f(q)\| < \|x - z, p - q\|$$

for all $x, z, p, q \in X$, then f is a generalized 2-isometry.

Proof. By Lemma 2.6

$$\|f(x) - f(z), f(p) - f(q)\| = \|x - z, p - q\|$$

for all $x, z, p, q \in X$ with $\|x - z, p - q\| \leq 1$.

In the following, We will show that

$$\|f(x) - f(z).f(p) - f(q)\| = \|x - z, p - q\|$$

if $\|x - z, p - q\| > 1$. Suppose, on the contrary, that

$$\|f(x) - f(z).f(p) - f(q)\| < \|x - z, p - q\|$$

for all $x, z, p, q \in X$ with $\|x - z, p - q\| > 1$. There exists a positive integer n_0 such that $n_0 \leq \|x - z.p - q\| < n_0 + 1$.

Let

$$y = x + \frac{\|x - z.p - q\|}{n_0 + 1}(x - z),$$

then

$$\|y - x, p - q\| = \|\frac{\|x - z.p - q\|}{n_0 + 1}(x - z), p - q\| = n_0 + 1$$

and

$$\begin{aligned} & \|y - z, p - q\| \\ = \; & \|(1 + \frac{\|x - z.p - q\|}{n_0 + 1})(x - z), p - q\| \\ = \; & \frac{n_0 + 1}{n_0 + 1 + \|x - z.p - q\|}\|x - z.p - q\| \\ < \; & \|x - z.p - q\| \\ \leq \; & n_0 + 1. \end{aligned}$$

Hence

$$\|f(y) - f(x), f(p) - f(q)\| = n_0 + 1$$

and

$$\|f(y) - f(z), f(p) - f(q)\| \leq \|y - z, p - q\| < n_0 + 1.$$

On the other hand,

$$\begin{aligned} & \|f(y) - f(x), f(p) - f(q)\| \\ \leq \; & \max\{\|f(y) - f(z), f(p) - f(q)\|, \|f(x) - f(z), f(p) - f(q)\|\} \\ \leq \; & \max\{\|y - z, p - q\|, \|x - z, p - q\|\} \\ < \; & n_0 + 1. \end{aligned}$$

This contradicts the equality

$$\|f(y) - f(x), f(p) - f(q)\| = n_0 + 1.$$

Hence

$$\|f(x) - f(z), f(p) - f(q)\| = \|x - z, p - q\|$$

when $\|x - z, p - q\| > 1$. So f is a generalized 2-isometry. □

3. The aleksandrov problem in non-Archimedean $n-$normed spaces

Definition 3.1. ([10]) Let X be a vector space of dimension greater than $n-1$ over a filed $\mathcal{K}$ with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot, \cdots, \cdot\| : X \times \cdots \times X \to [0, \infty)$ is said to be a non-Archimedean $n-$norm if it satisfies the following conditions:
(i) $\|x_1, \cdots, x_n\| = 0$ if and only if $|x_1, \cdots, x_n$ are linearly dependent;

(ii) $\|\|x_1, \cdots, x_n\| = \|x_{j_1}, \cdots, x_{j_n}\|$ for every permutation $(j_1, \cdots, j_n)$ of $(1, \cdots, n)$;
(iii)$\|rx_1, \cdots, x_n\| = |r|\|x_1, \cdots, x_n\|(r \in \mathcal{K}, x_1, \cdots, x_n \in X)$;
(iv) the strong triangle inequality
$\|x+y, x_2, \cdots, x_n\| \leq max\{\|xx_2, \cdots, x_n\|, \|y, x_2, \cdots, x_n\|\}(x, y, x_2, \cdots, x_n \in X)$.
Then $(x, \|\cdot, \cdots, \cdot\|)$ is called a non-Archimedean $n-$normed space.

From now on, we assume that X and Y be non-Archimedean n-normed linear spaces over a field $\mathcal{K}$ with a non-Archimedean valuation $|\cdot|_1$, f be a mapping from X into Y if without special statements.

Definition 3.2. Let X and Y be non-Archimedean $n-$normed linear spaces and $f : X \to Y$ a mapping. We say that f is a generalized n-isometry if

$$\|x_1 - y_1, ..., x_n - y_n\| = \|f(x_1) - f(y_1), ..., f(x_n) - f(y_n)\|$$

for all $x_1, ..., x_n, y_1, ..., y_n \in X$. In particular if $y_1 = y_2 = ... = y_n$, then f is said to be a n-isometry.

Definition 3.3. Let X and Y be non-Archimedean $n-$normed linear spaces and $f : X \to Y$ a mapping. We say that f is a generalized distance n preserving property (GDnPP) if

$$\|x_1 - y_1, ..., x_n - y_n\| = n$$

implies that

$$\|f(x_1) - f(y_1), ..., f(x_n) - f(y_n)\| = n$$

for all $x_1, ..., x_n, y_1, ..., y_n \in X$. In particular if $n = 1$, then f is said to satisfy the generalized distance one preserving property (GDOPP).

Definition 3.4. Let X and Y be non-Archimedean $n-$normed linear spaces and $f : X \to Y$ a mapping. We say that f is n-Lipschitz mapping if there is a $K > 0$ such that

$$\|f(x_1) - f(y_1), ..., f(x_n) - f(y_n)\| \leq K\|x_1 - y_1, ..., x_n - y_n\|$$

for all $x_1, ..., x_n, y_1, ..., y_n \in X$. The smallest such K is called the n-Lipschitz constant.

Lemma 3.5. ([10]) *Let X be non-Archimedean n-normed linear spaces, x_i be an element of a non-Archimedean n-normed spaces X, for every $i \in \{1, ..., n\}$ and $r \in \mathcal{K}$, then $\|x_1, ..., x_i, ..., x_j, ..., x_n\| = \|x_1, ..., x_i, ..., x_j + rx_i, ..., x_n\|$ for all $x_1, ..., x_n \in X$ and all $1 \leq i \neq j \leq n$.*

Lemma 3.6. *Let X and Y be non-Archimedean $n-$normed linear spaces and $f : X \to Y$ satisfies GDOPP and*

$$\|f(x_1) - f(y_1), ..., f(x_n) - f(y_n)\| \leq \|x_1 - y_1, ..., x_n - y_n\|$$

for all $x_1, ..., x_n, y_1, ..., y_n \in X$ with $\|x_1 - y_1, ..., x_n - y_n\| \leq 1$, then f satisfies

$$\|f(x_1) - f(y_1), ..., f(x_n) - f(y_n)\| = \|x_1 - y_1, ..., x_n - y_n\|$$

for all $x_1, ..., x_n, y_1, ..., y_n \in X$ with $\|x_1 - y_1, ..., x_n - y_n\| \leq 1$.

Proof. If

$$\|f(x_1) - f(y_1), ..., f(x_n) - f(y_n)\| < \|x_1 - y_1, ..., x_n - y_n\|,$$

let

$$y_0 = y_1 - \|x_1 - y_1, ..., x_n - y_n\|(x_1 - y_1),$$

then

$$\|y_0 - y_1, ..., x_n - y_n\| = \|\|x_1 - y_1, ..., x_n - y_n\|(x_1 - y_1), ..., x_n - y_n\| = 1$$

and

$$\begin{aligned} & \|y_0 - x_1, ..., x_n - y_n\| \\ = \; & \|y_1 - x_1 - \|x_1 - y_1, ..., x_n - y_n\|(x_1 - y_1), ..., x_n - y_n\| \\ = \; & \frac{\|(x_1 - y_1), ..., x_n - y_n\|}{1 + \|(x_1 - y_1), ..., x_n - y_n\|} \\ < \; & 1. \end{aligned}$$

Hence

$$\|f(y_0) - f(y_1), ..., f(x_n) - f(y_n)\| = 1$$

and

$$\begin{aligned} & \|f(y_0) - f(x_1), ..., f(x_n) - f(y_n)\| \\ \leq \; & \|y_0 - x_1, ..., x_n - y_n\| \\ < \; & 1. \end{aligned}$$

On the other hand

$$\begin{aligned} & \|f(y_0) - f(y_1), ..., f(x_n) - f(y_n)\| \\ \leq \; & \max\{\|f(y_0) - f(x_1), ..., f(x_n) - f(y_n)\|, \\ & \|f(x_1) - f(y_1), ..., f(x_n) - f(y_n)\|\} \\ < \; & 1. \end{aligned}$$

This contradicts the equality

$$\|f(y_0) - f(y_1), ..., f(x_n) - f(y_n)\| = 1.$$

Hence

$$\|f(x_1) - f(y_1), ..., f(x_n) - f(y_n)\| = \|x_1 - y_1, ..., x_n - y_n\|$$

for all $x_1, ..., x_n, y_1, ..., y_n \in X$ with $\|x_1 - y_1, ..., x_n - y_n\| \leq 1$. □

Theorem 3.7. *Let X and Y be non-Arichimedean n-normed spaces and $f : X \to Y$ satisfies GDnPP for all $n \in N$, if f is a n-Lipschitz mapping with $K = 1$:*

$$\|f(x_1) - f(y_1), ..., f(x_n) - f(y_n)\| \leq \|x_1 - y_1, ..., x_n - y_n\|$$

for all $x_1, ..., x_n, y_1, ..., y_n \in X$, then f is a generalized n-isometry.

Proof. By Lemma 3.6

$$\|f(x_1) - f(y_1), ..., f(x_n) - f(y_n)\| = \|x_1 - y_1, ..., x_n - y_n\|$$

for all $x_1, ..., x_n, y_1, ..., y_n \in X$ with $\|x_1 - y_1, ..., x_n - y_n\| \leq 1$.

In the following, We will show that

$$\|f(x_1) - f(y_1), ..., f(x_n) - f(y_n)\| = \|x_1 - y_1, ..., x_n - y_n\|$$

if $\|x_1 - y_1, ..., x_n - y_n\| > 1$. Suppose, on the contrary, that

$$\|f(x_1) - f(y_1), ..., f(x_n) - f(y_n)\| < \|x_1 - y_1, ..., x_n - y_n\|$$

for all $x_1, ..., x_n, y_1, ..., y_n \in X$ with $\|x_1 - y_1, ..., x_n - y_n\| > 1$. There exists a positive integer n_0 such that $n_0 < \|x_1 - y_1, ..., x_n - y_n\| \leq n_0 + 1$. Let

$$y_0 = y_1 - \frac{\|x_1 - y_1, ..., x_n - y_n\|}{n_0 + 1}(x_1 - y_1),$$

then

$$\begin{aligned} & \|y_0 - y_1, ..., x_n - y_n\| \\ = \; & \| - \frac{\|x_1 - y_1, ..., x_n - y_n\|}{n_0 + 1}(x_1 - y_1), ..., x_n - y_n\| \\ = \; & n_0 + 1, \end{aligned}$$

and

$$\begin{aligned} & \|y_0 - x_1, ..., x_n - y_n\| \\ = \; & \| - (1 + \frac{\|x_1 - y_1, ..., x_n - y_n\|}{n_0 + 1})(x_1 - y_1), ..., x_n - y_n\| \\ = \; & \frac{n_0 + 1}{n_0 + 1 + \|x_1 - y_1, ..., x_n - y_n\|}\|x_1 - y_1, ..., x_n - y_n\| \\ < \; & \|x_1 - y_1, ..., x_n - y_n\| \\ \leq \; & n_0 + 1. \end{aligned}$$

Hence

$$\|f(y_0) - f(y_1), ..., f(x_n) - f(y_n)\| = n_0 + 1$$

and

$$\begin{aligned} & \|f(y_0) - f(x_1), ..., f(x_n) - f(y_n)\| \\ \leq\ & \|y_0 - x_1, ..., x_n - y_n\| \\ <\ & n_0 + 1. \end{aligned}$$

On the other hand,

$$\begin{aligned} & \|f(y_0) - f(y_1), ..., f(x_n) - f(y_n)\| \\ \leq\ & \max\{\|f(y_0) - f(x_1), ..., f(x_n) - f(y_n)\|, \\ & \|f(x_1) - f(y_1), ..., f(x_n) - f(y_n)\|\} \\ <\ & n_0 + 1. \end{aligned}$$

This contradicts the equality

$$\|f(y_0) - f(y_1), ..., f(x_n) - f(y_n)\| = n_0 + 1.$$

Hence

$$\|f(x_1) - f(y_1), ..., f(x_n) - f(y_n)\| = \|x_1 - y_1, ..., x_n - y_n\|$$

when $\|x_1 - y_1, ..., x_n - y_n\| > 1$. So f is a generalized n-isometry. □

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