Nonlinear Functional Analysis and Applications
Vol. 19, No. 2 (2014), pp. 193-211

http://nfaa.kyungnam.ac.kr/jour-nfaa.htm

A NONLINEAR VOLTERRA-HAMMERSTEIN INTEGRAL EQUATION IN THREE VARIABLES

Le Thi Phuong Ngoc[1], **Tran Minh Thuyet**[2] **and Nguyen Thanh Long**[3]

[1]Nhatrang Educational College
01 Nguyen Chanh Str., Nhatrang City, Vietnam
e-mail: ngoc1966@gmail.com

[2]Department of Mathematics, University of Economics of Ho Chi Minh City
Ho Chi Minh City, Vietnam
e-mail: tmthuyet@ueh.edu.vn

[3]Department of Mathematics and Computer Science, University of Natural Science,
Vietnam National University Ho Chi Minh City
227 Nguyen Van Cu Str., Dist. 5, Ho Chi Minh City, Vietnam
e-mail: longnt2@gmail.com

Abstract. Motivated by recent known results about the solvability of nonlinear functional integral equations in one, two or n variables, this paper establishes the existence of asymptotically stable solutions for a Volterra-Hammerstein integral equation in three variables. The proofs are completed via a fixed point theorem of Krasnosel'skii type, a condition for the relative compactness of a subset in certain space and integral inequalities with explicit estimates.

[0]Received October 22, 2013. Revised April 7, 2014.
[0]2010 Mathematics Subject Classification: 47H10, 45G10, 47N20, 65J15.
[0]Keywords: The fixed point theorem of Krasnosel'skii type, Volterra-Hammerstein integral equation in three variables, contraction mapping, completely continuous, asymptotically stable solution.

1. Introduction

In this paper, we consider the nonlinear Volterra-Hammerstein integral equation in three variables of the form

$$\begin{aligned} u(x,y,z) &= q(x,y,z) + f(x,y,z;u(x,y,z)) \\ &\quad + \int_0^x \int_0^y \int_0^z V\left(x,y,z;s,r,t;u(s,r,t)\right) dsdrdt \qquad (1.1) \\ &\quad + \int_0^\infty \int_0^\infty \int_0^\infty F\left(x,y,z;s,r,t;u(s,r,t)\right) dsdrdt, \end{aligned}$$

for $(x,y,z) \in \mathbb{R}_+^3$, where $\mathbb{R}_+ = [0,\infty)$, $q : \mathbb{R}_+^3 \to E$; $f : \mathbb{R}_+^3 \times E \to E$; $F : \mathbb{R}_+^6 \times E \to E$; $V : \Delta \times E \to E$ are supposed to be continuous, in which E is a Banach space with norm $|\cdot|$ and $\Delta = \{(x,y,z;s,r,t) \in \mathbb{R}_+^6 : s \leq x,\ r \leq y,\ t \leq z\}$.

Nonlinear integral equations of various types and kinds appear in the mathematical description of the applications in other fields of science, such as economics, mechanics and physics. Solving such equations and proving the existence of their solutions have been extensively interested by many authors, see [1]-[11] and the references given therein. In general, the main results have been obtained via the fundamental methods in which the fixed point theorems are often applied.

In [2], using a fixed point theorem of Krasnosel'skii, Avramescu and Vladimirescu have proved the existence of asymptotically stable solutions to the following integral equation

$$u(t) = q(t) + \int_0^t K(t,s,u(s))ds + \int_0^\infty G(t,s,u(s))ds,\ t \in \mathbb{R}_+, \qquad (1.2)$$

where the functions given with real values are supposed to be continuous satisfying suitable conditions.

In case the Banach space E is arbitrary, recently in [8], [9], the existence of asymptotically stable solutions to the following integral equations

$$x(t) = q(t) + f(t,x(t)) + \int_0^t V(t,s,x(s))ds + \int_0^\infty G(t,s,x(s))ds,\ t \in \mathbb{R}_+, \quad (1.3)$$

or

$$\begin{aligned} u(x,y) &= q(x,y) + f(x,y,u(x,y)) + \int_0^x \int_0^y V\left(x,y,s,t,u(s,t)\right) dsdt \\ &\quad + \int_0^\infty \int_0^\infty F\left(x,y,s,t,u(s,t)\right) dsdt, \qquad (1.4) \end{aligned}$$

$(x,y) \in \mathbb{R}_+^2$, also have been proved by using the fixed point theorem of Krasnosel'skii type [7] as follows.

Theorem 1.1. *Let* $(X, |\cdot|_n)$ *be a Fréchet space and let* U, $C : X \to X$ *be two operators. Assume that*

(i) U *is a* $k-$*contraction operator,* $k \in [0, 1)$ *(depending on* n*), with respect to a family of seminorms* $\|\cdot\|_n$ *equivalent with the family* $|\cdot|_n$;

(ii) C *is completely continuous;*

(iii) $\lim\limits_{|x|_n \to \infty} \frac{|Cx|_n}{|x|_n} = 0$, $\forall n \in \mathbb{N}$.

Then $U + C$ *has a fixed point.*

In [6], Lungu and Rus established some results relative to existence, uniqueness, integral inequalities and data dependence for the solutions of the following functional Volterra-Fredholm integral equation in two variables with deviating argument in a Banach space by Picard operators technique

$$u(x, y) = g(x, y, h(u)(x, y)) + \int_0^x \int_0^y K(x, y, s, t, u(s, t))\, dsdt, \tag{1.5}$$

for $(x, y) \in \mathbb{R}_+^2$.

In [10], based on the applications of the Banach fixed point theorem coupled with Bielecki type norm and the integral inequality with explicit estimates, B. G. Pachpatte studied some basic properties of solutions of the Fredholm type integral equation in two variables as follows

$$u(x, y) = f(x, y) + \int_0^a \int_0^b g(x, y, s, t, u(s, t), D_1u(s, t), D_2u(s, t))\, dtds. \tag{1.6}$$

With the same methods, in [11], the existence, uniqueness and other properties of solutions of certain Volterra integral and integrodifferential equations in two variables were considered.

In [1], M.A. Abdou et al. investigated a mixed nonlinear integral equation of the second kind in n-dimensional. Using the Banach fixed point theorem, the existence of a unique solution of this equation was proved.

Also applying the Banach fixed point theorem, in [4], El-Borai et al. have proved the existence of a unique solution of a nonlinear integral equation of type Volterra-Hammerstein in n-dimensional.

Motivated by the mentioned works as above, we extend the results of [9] to several dimensions concentrating on three dimensions since this is the first case where new techniques or ideas are need. Applying Theorem 1.1, under some suitable conditions, we also get the same results for (1.1) as those for (1.4) in [9]. The proofs are completed by combination of the arguments in [9], a condition for the relative compactness of a subset in certain space and the integral inequalities with explicit estimates (see Lemmas 2.2, 2.3, 3.1 as below).

The paper consists of three sections. In section 2, the existence of solutions are proved. In section 3, we present the existence of asymptotically stable solutions for (1.1). The results obtained here may be considered as the generalizations of those in [9] and can be useful for seeking the corresponding results in n variables.

2. The Existence Result

Let $X = C(\mathbb{R}_+^3; E)$ be the space of all continuous functions on $\mathbb{R}_+^3$ to E which equipped with the numerable family of seminorms

$$|u|_n = \sup_{0\le x,\ y,\ z\le n} |u(x,y,z)|\,,\ n\ge 1.$$

Then $(X, |\cdot|_n)$ is complete in the metric

$$d(u,v) = \sum\nolimits_{n=1}^{\infty} 2^{-n}\frac{|u-v|_n}{1+|u-v|_n}$$

and X is the Fréchet space. Consider in X the other family of seminorms $\|\cdot\|_n$ is defined as follows

$$\|u\|_n = |u|_{\gamma_n} + |u|_{h_n}\,,\ n\ge 1,$$

where

$$\begin{aligned}|u|_{\gamma_n} &= \sup_{0\le x,\ y,\ z\le n,\ x+y+z\le\gamma_n} |u(x,y,z)|\,,\\ |u|_{h_n} &= \sup_{0\le x,\ y,\ z\le n,\ x+y+z\ge\gamma_n} e^{-h_n(x+y+z-\gamma_n)}\,|u(x,y,z)|\,,\end{aligned}$$

$\gamma_n\in(0,n)$ and $h_n>0$ are arbitrary numbers, which is equivalent to $|u|_n$, since

$$e^{-h_n(3n-\gamma_n)}\,|u|_n \le \|u\|_n \le 2\,|u|_n\,,\ \forall u\in X,\ \forall n\ge 1.$$

We make the following assumptions.

(A_1) $q\in X$.

(A_2) There exists a constant $L\in[0,1)$ such that

$$|f(x,y,z,u)-f(x,y,z,v)| \le L\,|u-v|\,,\ \forall u,v\in E,\ \forall (x,y,z)\in\mathbb{R}_+^3.$$

(A_3) There exists a continuous function $\omega_1:\Delta\to\mathbb{R}_+$ such that

$$|V(x,y,z;s,r,t;u)-V(x,y,z;s,r,t;v)| \le \omega_1(x,y,z;s,r,t)\,|u-v|\,,\quad \forall (x,y,z;s,r,t)\in\Delta,\ \forall u,v\in E.$$

(A_4) F is completely continuous such that for all bounded subsets I_1, I_2 of $\mathbb{R}_+^3$ and for any bounded subset J of E, for all $\varepsilon>0$, there exists

$\delta > 0$, such that $\forall (x_1, y_1, z_1), (x_2, y_2, z_2) \in I_1$,

$$\begin{aligned}&|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2| < \delta \\ &\Longrightarrow |F(x_1, y_1, z_1; s, r, t, u) - F(x_2, y_2, z_2; s, r, t, u)| < \varepsilon,\end{aligned}$$

for all $(s, r, t, u) \in I_2 \times J$.

(A_5) There exists a continuous function $\omega_2 : \mathbb{R}_+^6 \to \mathbb{R}_+$ such that for each bounded subset I of $\mathbb{R}_+^3$,

$$\iiint_{\mathbb{R}_+^3} \sup_{(x,y,z)\in I} \omega_2(x, y, z; s, r, t) dtdrds < \infty,$$

and

$$|F(x, y, z; s, r, t; u)| \le \omega_2(x, y, z; s, r, t),$$

for all $(x, y, z; s, r, t; u) \in I \times \mathbb{R}_+^3 \times E$.

Theorem 2.1. *Let* $(A_1) - (A_5)$ *hold. Then the equation* (1.1) *has a solution on* $\mathbb{R}_+^3$.

Proof. The proof consists of four steps.

Step 1. In X, we consider the equation

$$\begin{aligned} u(x, y, z) &= q(x, y, z) + f(x, y, z; u(x, y, z)) \\ &\quad + \int_0^x \int_0^y \int_0^z V(x, y, z; s, r, t; u(s, r, t))\, dsdrdt, \end{aligned} \tag{2.1}$$

for $(x, y, z) \in \mathbb{R}_+^3$.

Lemma 2.2. *Let* $(A_1) - (A_3)$ *hold. Then equation* (2.1) *has a unique solution* $u = \xi$.

Proof. We rewrite (2.1) as follows

$$u(x, y, z) = \Phi u(x, y, z),\ (x, y, z) \in \mathbb{R}_+^3, \tag{2.2}$$

where

$$\begin{aligned} \Phi u(x, y, z) &= q(x, y, z) + f(x, y, z; u(x, y, z)) \\ &\quad + \int_0^x \int_0^y \int_0^z V(x, y, z; s, r, t; u(s, r, t))\, dsdrdt, \end{aligned} \tag{2.3}$$

for $(x, y, z, u) \in \mathbb{R}_+^3 \times X$. By the assumptions (A_2), (A_3), we have, for all u, $v \in X$,

$$\begin{aligned}
&|\Phi u(x,y,z) - \Phi v(x,y,z)| \\
\leq\ & |f(x,y,z,u(x,y,z)) - f(x,y,z,v(x,y,z))| \\
&+ \int_0^x \int_0^y \int_0^z |V(x,y,z;s,r,t;u(s,r,t)) \\
&- V(x,y,z;s,r,t;v(s,r,t))|\, dsdrdt \\
\leq\ & L\,|u(x,y,z) - v(x,y,z)| \qquad (2.4)\\
&+ \int_0^x \int_0^y \int_0^z \omega_1(x,y,z;s,r,t)\,|u(s,r,t) - v(s,r,t)|\, dsdrdt.
\end{aligned}$$

Let $n \in \mathbb{N}$ be fixed. For all $x, y, z \in [0,n]$, $0 \leq x+y+z \leq \gamma_n$, with $\gamma_n \in (0,n)$ chosen later, we have

$$\begin{aligned}
&|\Phi u(x,y,z) - \Phi v(x,y,z)| \\
\leq\ & L\,|u(x,y,z) - v(x,y,z)| \\
&+ \int_0^x \int_0^y \int_0^z \omega_1(x,y,z;s,r,t)\,|u(s,r,t) - v(s,r,t)|\, dsdrdt \\
\leq\ & L\,|u-v|_{\gamma_n} + \gamma_n^3 \tilde{\omega}_{1n}\,|u-v|_{\gamma_n} \\
=\ & \left(L + \gamma_n^3 \tilde{\omega}_{1n}\right)|u-v|_{\gamma_n}, \qquad (2.5)
\end{aligned}$$

where

$$\begin{aligned}
\tilde{\omega}_{1n} &= \sup\{\omega_1(x,y,z;s,r,t) : (x,y,z;s,r,t) \in \Delta_n\}, \qquad (2.6)\\
\Delta_n &= \{(x,y,z;s,r,t) : 0 \leq s \leq x \leq n,\ 0 \leq r \leq y \leq n,\ 0 \leq t \leq z \leq n\}.
\end{aligned}$$

So

$$|\Phi u - \Phi v|_{\gamma_n} \leq \left(L + \gamma_n^3 \tilde{\omega}_{1n}\right)|u-v|_{\gamma_n}. \qquad (2.7)$$

On the other hand, for all $x, y, z \in [0,n]$, $x+y+z \geq \gamma_n$, we have

$$\begin{aligned}
&|\Phi u(x,y,z) - \Phi v(x,y,z)| \\
\leq\ & L\,|u(x,y,z) - v(x,y,z)| \\
&+ \int_0^x \int_0^y \int_0^z \omega_1(x,y,z;s,r,t)\,|u(s,r,t) - v(s,r,t)|\, dsdrdt \\
\leq\ & L\,|u(x,y,z) - v(x,y,z)| + \tilde{\omega}_{1n} \int_0^x \int_0^y \int_0^z |u(s,r,t) - v(s,r,t)|\, dsdrdt \\
=\ & L\,|u(x,y,z) - v(x,y,z)| \qquad (2.8)\\
&+ \tilde{\omega}_{1n} \Bigg[\underset{s+r+t \leq \gamma_n}{\int_0^x \int_0^y \int_0^z} |u(s,r,t) - v(s,r,t)|\, dsdrdt
\end{aligned}$$

$$+\int_0^x\int_0^y\int_0^z\limits_{s+r+t\geq\gamma_n} |u(s,r,t)-v(s,r,t)|\,dsdrdt\Bigg].$$

By the inequality

$$0<e^{-h_n(x+y+z-\gamma_n)}\leq 1,\quad \forall x,y,z\in[0,n],\quad x+y+z\geq\gamma_n, \tag{2.9}$$

with $h_n>0$ is also chosen later, we get

$$\begin{aligned}
& |\Phi u(x,y,z)-\Phi v(x,y,z)|\,e^{-h_n(x+y+z-\gamma_n)} \\
\leq\ & L\,|u-v|_{h_n} \\
& +\tilde{\omega}_{1n}e^{-h_n(x+y+z-\gamma_n)}\Bigg[\int_0^x\int_0^y\int_0^z\limits_{s+r+t\leq\gamma_n}|u(s,r,t)-v(s,r,t)|\,dsdrdt \\
& +\int_0^x\int_0^y\int_0^z\limits_{s+r+t\geq\gamma_n}e^{h_n(s+r+t-\gamma_n)}e^{-h_n(s+r+t-\gamma_n)}\,|u(s,r,t)-v(s,r,t)|\,dsdrdt\Bigg] \\
\leq\ & L\,|u-v|_{h_n}+\tilde{\omega}_{1n}e^{-h_n(x+y+z-\gamma_n)}\Bigg[|u-v|_{\gamma_n}\int_0^x\int_0^y\int_0^z\limits_{s+r+t\leq\gamma_n}dsdrdt \\
& +|u-v|_{h_n}\int_0^x\int_0^y\int_0^z\limits_{s+r+t\geq\gamma_n}e^{h_n(s+r+t-\gamma_n)}dsdrdt\Bigg] \\
=\ & L\,|u-v|_{h_n}+\tilde{\omega}_{1n}e^{-h_n(x+y+z-\gamma_n)}\Big[|u-v|_{\gamma_n}I_1+|u-v|_{h_n}I_2\Big].
\end{aligned} \tag{2.10}$$

On the other hand

$$I_1=\int_0^x\int_0^y\int_0^z\limits_{s+r+t\leq\gamma_n}dsdrdt\leq\iiint\limits_{s+r+t\leq\gamma_n}dsdrdt=\frac{1}{6}\gamma_n^3, \tag{2.11}$$

$$\begin{aligned}
I_2 &= \int_0^x\int_0^y\int_0^z\limits_{s+r+t\geq\gamma_n}e^{h_n(s+r+t-\gamma_n)}dsdrdt \\
&\leq \int_0^x\int_0^y\int_0^z e^{h_n(s+r+t-\gamma_n)}dsdrdt \\
&= \frac{1}{h_n^3}e^{h_n(x+y+z-\gamma_n)}\left(1-e^{-h_nx}\right)\left(1-e^{-h_ny}\right)\left(1-e^{-h_nz}\right).
\end{aligned} \tag{2.12}$$

Hence

$$\begin{aligned}
&\left|\Phi u(x,y,z)-\Phi v(x,y,z)\right| e^{-h_n(x+y+z-\gamma_n)} \\
\leq\; & L\left|u-v\right|_{h_n}+\tilde{\omega}_{1n}e^{-h_n(x+y+z-\gamma_n)}\left[\frac{1}{6}\gamma_n^3\left|u-v\right|_{\gamma_n}\right. \\
&\left.+\left|u-v\right|_{h_n}\frac{1}{h_n^3}e^{h_n(x+y+z-\gamma_n)}\left(1-e^{-h_n x}\right)\left(1-e^{-h_n y}\right)\left(1-e^{-h_n z}\right)\right] \\
\leq\; & L\left|u-v\right|_{h_n}+\frac{1}{6}\gamma_n^3\tilde{\omega}_{1n}\left|u-v\right|_{\gamma_n}+\frac{1}{h_n^3}\tilde{\omega}_{1n}\left|u-v\right|_{h_n} \\
=\; & \frac{1}{6}\gamma_n^3\tilde{\omega}_{1n}\left|u-v\right|_{\gamma_n}+\left(L+\frac{1}{h_n^3}\tilde{\omega}_{1n}\right)\left|u-v\right|_{h_n}. \qquad (2.13)
\end{aligned}$$

So

$$\left|\Phi u-\Phi v\right|_{h_n}\leq\frac{1}{6}\gamma_n^3\tilde{\omega}_{1n}\left|u-v\right|_{\gamma_n}+\left(L+\frac{1}{h_n^3}\tilde{\omega}_{1n}\right)\left|u-v\right|_{h_n}. \qquad (2.14)$$

Consequently,

$$\begin{aligned}
&\left\|\Phi u-\Phi v\right\|_n \\
=\; & \left|\Phi u-\Phi v\right|_{\gamma_n}+\left|\Phi u-\Phi v\right|_{h_n} \\
\leq\; & \left(L+\gamma_n^3\tilde{\omega}_{1n}\right)\left|u-v\right|_{\gamma_n}+\frac{1}{6}\gamma_n^3\tilde{\omega}_{1n}\left|u-v\right|_{\gamma_n}+\left(L+\frac{1}{h_n^3}\tilde{\omega}_{1n}\right)\left|u-v\right|_{h_n} \\
\leq\; & \left(L+\frac{7}{6}\gamma_n^3\tilde{\omega}_{1n}\right)\left|u-v\right|_{\gamma_n}+\left(L+\frac{1}{h_n^3}\tilde{\omega}_{1n}\right)\left|u-v\right|_{h_n} \\
\leq\; & L_n\left\|x-y\right\|_n, \qquad (2.15)
\end{aligned}$$

where $L_n=\max\left\{L+\frac{7}{6}\gamma_n^3\tilde{\omega}_{1n},\ L+\frac{1}{h_n^3}\tilde{\omega}_{1n}\right\}$. Choosing h_n, γ_n such that

$$h_n>\sqrt[3]{\frac{1}{1-L}\tilde{\omega}_{1n}}\quad\text{and}\quad 0<\gamma_n<\min\left\{\sqrt[3]{\frac{6(1-L)}{7\tilde{\omega}_{1n}}},\ n\right\},$$

then we have $L_n<1$, so Φ is a L_n- contraction operator on the Fréchet space $(X,\left\|\cdot\right\|_n)$, the Lemma 2.2 follows via the known Banach's contraction principle. □

By the transformation $u=v+\xi$, we can write the equation (1.1) in the form

$$v(x,y,z)=Uv(x,y,z)+Cv(x,y,z),\ (x,y,z)\in\mathbb{R}_+^3, \qquad (2.16)$$

where

$$\begin{cases} Uv(x,y,z) = q(x,y,z) + f(x,y,z; v(x,y,z) + \xi(x,y,z)) - \xi(x,y,z) \\ \qquad\qquad + \int_0^x \int_0^y \int_0^z V(x,y,z;s,r,t; v(s,r,t) + \xi(s,r,t))\, dsdrdt, \\ Cv(x,y,z) = \iiint_{\mathbb{R}_+^3} F(x,y,z;s,r,t; v(s,r,t) + \xi(s,r,t))\, dsdrdt, \end{cases} \tag{2.17}$$

for $(x,y,z) \in \mathbb{R}_+^3$.

Step 2. The operator U is a L_n-contraction, with respect to a family of seminorms $\|\cdot\|_n$. Indeed, fixed an arbitrary positive integer $n \in \mathbb{N}$. We have

$$\begin{aligned} & Uv(x,y,z) - U\tilde{v}(x,y,z) \\ = \; & f(x,y,z; v(x,y,z) + \xi(x,y,z)) - f(x,y,z; \tilde{v}(x,y,z) + \xi(x,y,z)) \\ & + \int_0^x \int_0^y \int_0^z [V(x,y,z;s,r,t; v(s,r,t) + \xi(s,r,t)) \\ & - V(x,y,z;s,r,t; \tilde{v}(s,r,t) + \xi(s,r,t))]\, dsdrdt, \end{aligned} \tag{2.18}$$

so using the similar estimates as in the proof of Lemma 2.2, the results are as follows. For all $x,y,z \in [0,n]$, $0 \le x+y+z \le \gamma_n$,

$$\begin{aligned} & |Uv(x,y,z) - U\tilde{v}(x,y,z)| \\ \le \; & L\,|v(x,y,z) - \tilde{v}(x,y,z)| \\ & + \int_0^x \int_0^y \int_0^z \omega_1(x,y,z;s,r,t)\, |v(s,r,t) - \tilde{v}(s,r,t)|\, dsdrdt \\ \le \; & \left(L + \gamma_n^3 \tilde{\omega}_{1n}\right) |v - \tilde{v}|_{\gamma_n}. \end{aligned} \tag{2.19}$$

This implies that

$$|Uv - U\tilde{v}|_{\gamma_n} \le \left(L + \gamma_n^3 \tilde{\omega}_{1n}\right) |v - \tilde{v}|_{\gamma_n}. \tag{2.20}$$

For all $x,y,z \in [0,n]$, $x+y+z \ge \gamma_n$,

$$\begin{aligned} & |Uv(x,y,z) - U\tilde{v}(x,y,z)|\, e^{-h_n(x+y+z-\gamma_n)} \\ \le \; & \frac{1}{6}\gamma_n^3 \tilde{\omega}_{1n}\, |v - \tilde{v}|_{\gamma_n} + \left(L + \frac{1}{h_n^3}\tilde{\omega}_{1n}\right) |v - \tilde{v}|_{h_n}. \end{aligned} \tag{2.21}$$

It follows that

$$|Uv - U\tilde{v}|_{h_n} \le \frac{1}{6}\gamma_n^3 \tilde{\omega}_{1n}\, |v - \tilde{v}|_{\gamma_n} + \left(L + \frac{1}{h_n^3}\tilde{\omega}_{1n}\right) |v - \tilde{v}|_{h_n}. \tag{2.22}$$

Consequently

$$\|Uv - U\tilde{v}\|_n \le L_n \|v - \tilde{v}\|_n. \tag{2.23}$$

and then U is a L_n-contraction operator with respect to $\|\cdot\|_n$.

Step 3. The operator $C : X \to X$ is completely continuous. First, let us give the following condition for the relative compactness of a subset in X.

Lemma 2.3. *Let $X = C(\mathbb{R}_+^3; E)$ be the Fréchet space defined as above and A be a subset of X. For each $n \in \mathbb{N}$, let $X_n = C([0,n]^3; E)$ be the Banach space of all continuous functions $u : [0,n]^3 \to E$ with the norm*

$$|u|_n = \sup_{(x,y,z)\in[0,n]^3} |u(x,y,z)|$$

and $A_n = \{u|_{[0,n]^3} : u \in A\}$. The set A in X is relatively compact if and only if for each $n \in \mathbb{N}$, A_n is equicontinuous in X_n and for every $(x,y,z) \in [0,n]^3$, the set $A_n(x,y,z) = \{u(x,y,z) : u \in A_n\}$ is relatively compact in E.

The proof of this condition is similar to that in Appendix of [7], it follows from the Ascoli-Arzela's Theorem(see [5], p.211).

By the assumptions (A_4), (A_5), using the method as in [9] in which Lemma 2.3 and the arguments of density are applied, the Step 3 is proved. Let us sum up the main points of this proof as follows.

(i) For any $v_0 \in X$, let $\{v_m\}$ be a sequence in X such that $\lim_{m\to\infty} v_m = v_0$.

Let $n \in \mathbb{N}$ be fixed. For any given $\varepsilon > 0$, by

$$\iiint_{\mathbb{R}_+^3} \sup_{(x,y,z)\in[0,n]^3} \omega_2(x,y,z;s,r,t)dtdrds < \infty,$$

there exists $T_n \in \mathbb{N}(T_n$ is big enough) such that

$$\begin{aligned} &\iiint_{\mathbb{R}_+^3\setminus B_n} \omega_2(x,y,z;s,r,t)dtdrds \\ \leq\ &\iiint_{\mathbb{R}_+^3\setminus B_n} \sup_{(x,y,z)\in[0,n]^3} \omega_2(x,y,z;s,r,t)dtdrds \\ <\ &\frac{\varepsilon}{4}, \quad \forall (x,y,z) \in [0,n]^3, \end{aligned} \tag{2.24}$$

where $B_n = \{(s,r,t) \in \mathbb{R}_+^3 : s^2 + r^2 + t^2 \leq T_n^2\}$.

Put $K = \{(v_m + \xi)(s,r,t) : (s,r,t) \in B_n,\ m \in \mathbb{Z}_+\}$, then K is compact in E. For $\varepsilon > 0$ be given as above, by F is continuous on the compact set $[0,n]^3 \times B_n \times K$, there exists $\delta > 0$ such that for every $u,\ v \in K$, $|u - v| < \delta$,

$$\begin{aligned} &|F(x,y,z;s,r,t;u) - F(x,y,z;s,r,t;v)| \\ <\ &\frac{3\varepsilon}{\pi T_n^3}, \quad \forall (x,y,z;s,r,t) \in [0,n]^3 \times B_n. \end{aligned} \tag{2.25}$$

By $\lim_{m\to\infty} \sup_{(s,r,t)\in B_n} |(v_m + \xi)(s,r,t) - (v_0 + \xi)(s,r,t)| = 0$, there exists m_0 such that for $m > m_0$,

$$|(v_m + \xi)(s,r,t) - (v_0 + \xi)(s,r,t)| < \delta, \quad \forall (s,r,t) \in B_n. \tag{2.26}$$

This implies that for all $(x,y,z)\in[0,n]^3$, for all $m>m_0$,

$$\begin{aligned}|Cv_m(x,y,z)-Cv_0(x,y,z)| &\leq \iiint_{B_n}|F(x,y,z;s,r,t;(v_m+\xi)(s,r,t))\\ &\quad -F(x,y,z;s,r,t;(v_0+\xi)(s,r,t))|dsdrdt\\ &\quad +2\iiint_{\mathbb{R}^3_+\setminus B_n}\omega_2(x,y,z;s,r,t)dsdrdt\\ &< \frac{\pi}{6}T_n^3\times\frac{3\varepsilon}{\pi T_n^3}+2\frac{\varepsilon}{4}=\varepsilon, \qquad (2.27)\end{aligned}$$

so $|Cv_m-Cv_0|_n<\varepsilon$, for all $m>m_0$, and the continuity of C is proved.
(ii) It remains to show that C maps bounded sets into relatively compact sets.
Let Ω be a bounded subset of X. We have to prove that for $n\in\mathbb{N}$,

(a) the set $(C\Omega)_n$ is equicontinuous in X_n,

(b) for every $(x,y,z)\in[0,n]^3$, the set

$$(C\Omega)_n(x,y,z)=\{Cv|_{[0,n]^3}(x,y,z):v\in\Omega\}$$

is relatively compact in E.

Let $n\in\mathbb{N}$ be fixed. Consider any $\varepsilon>0$ given. Then, there exists $T_n\in\mathbb{N}$(T_n is big enough) such that (2.24) is valid.

Proof of (a) : For any $v\in\Omega$, for all (x_1,y_1,z_1), $(x_2,y_2,z_2)\in[0,n]^3$,

$$\begin{aligned}&|Cv(x_1,y_1,z_1)-Cv(x_2,y_2,z_2)|\\ \leq\ &\iiint_{B_n}|F(x_1,y_1,z_1;s,r,t;v(s,r,t)+\xi(s,r,t)) \qquad (2.28)\\ &-F(x_2,y_2,z_2;s,r,t;v(s,r,t)+\xi(s,r,t))|\,dtdrds\\ &+\iiint_{\mathbb{R}^3_+\setminus B_n}(\omega_2(x_1,y_1,z_1;s,r,t)+\omega_2(x_2,y_2,z_2;s,r,t))\,dtdrds.\end{aligned}$$

According to (2.24), (2.28) and the hypothesis (A_4), $(C\Omega)_n$ is equicontinuous on X_n.

Proof of (b) : Let $\{Cv_k|_{[0,n]^3}(x,y,z)\}_k$, $v_k\in\Omega$, be a sequence in $(C\Omega)_n(x,y,z)$. We have to show that there exists a convergent subsequence of $\{Cv_k|_{[0,n]^3}(x,y,z)\}_k$. Put

$$S=\{(v+\xi)(x,y,z):v\in\Omega,\ (x,y,z)\in B_n\}. \qquad (2.29)$$

Then S is bounded in E and consequently the set $F([0,n]^3\times B_n\times S)$ is relatively compact in E, since F is completely continuous. The sequence $\{F(x,y,z;s,r,t;(v_k+\xi)(s,r,t))\}_k$ belongs to $F([0,n]^3\times B_n\times S)$, so there exists a subsequence $\left\{F\left(x,y,z;s,r,t;(v_{k_j}+\xi)(s,r,t)\right)\right\}_j$ and $\Psi(x,y,z;s,r,t)\in E$, such that

$$\left|F\left(x,y,z;s,r,t;(v_{k_j}+\xi)(s,r,t)\right)-\Psi(x,y,z;s,r,t)\right|\to 0 \text{ as } j\to\infty. \qquad (2.30)$$

On the other hand, by (A_5),

$$\left|F\left(x,y,z;s,r,t;(v_{k_j}+\xi)(s,r,t)\ \right)\right| \leq \omega_2(x,y,z;s,r,t),$$

for all $(x,y,z;s,r,t) \in [0,n]^3 \times B_n$. Hence

$$\begin{aligned} & \left|F\left(x,y,z;s,r,t;(v_{k_j}+\xi)(s,r,t)\ \right) - \Psi(x,y,z;s,r,t)\right| \\ \leq\ & \left|F\left(x,y,z;s,r,t;(v_{k_j}+\xi)(s,r,t)\ \right)\right| + |\Psi(x,y,z;s,r,t)| \\ \leq\ & 2\omega_2(x,y,z;s,r,t), \end{aligned} \tag{2.31}$$

for all $(x,y,z;s,r,t) \in [0,n]^3 \times B_n$, $\omega_2(x,y,z;\cdot) \in L^1(B_n)$. Using the dominated convergence theorem, (2.30) and (2.31) yield

$$\iiint_{B_n} \left|F\left(x,y,z;s,r,t;(v_{k_j}+\xi)(s,r,t)\ \right) - \Psi(x,y,z;s,r,t)\right| dsdrdt \to 0,$$

as $j \to \infty$. It means that, for given $\varepsilon > 0$, there exists j_0 such that for $j > j_0$,

$$\iiint_{B_n} \left|F\left(x,y,z;s,r,t;(v_{k_j}+\xi)(s,r,t)\ \right) - \Psi(x,y,z;s,r,t)\right| dsdrdt < \frac{3\varepsilon}{4}.$$

Consequently, for $j > j_0$,

$$\begin{aligned} & \left|Cv_{k_j}(x,y,z) - \iiint_{B_n} \Psi(x,y,z;s,r,t)\ dsdrdt\right| \\ =\ & \left|\iiint_{\mathbb{R}^3_+} F\left(x,y,z;s,r,t;(v_{k_j}+\xi)(s,r,t)\ \right)\ dsdrdt \right. \\ & \left. - \iiint_{B_n} \Psi(x,y,z;s,r,t)\ dsdrdt\right| \\ \leq\ & \left|\iiint_{B_n} F\left(x,y,z;s,r,t;(v_{k_j}+\xi)(s,r,t)\ \right) dsdrdt \right. \\ & \left. - \iiint_{B_n} \Psi(x,y,z;s,r,t)\ dsdrdt\right| \\ & + \left|\iiint_{\mathbb{R}^3_+\setminus B_n} F\left(x,y,z;s,r,t;(v_{k_j}+\xi)(s,r,t)\right)\ dsdrdt\right| \\ \leq\ & \iiint_{B_n} \left|F\left(x,y,z;s,r,t;(v_{k_j}+\xi)(s,r,t)\ \right) - \Psi(x,y,z;s,r,t)\right| dsdrdt \\ & + \iiint_{\mathbb{R}^3_+\setminus B_n} \left|F\left(x,y,z;s,r,t;(v_{k_j}+\xi)(s,r,t)\ \right)\right| dsdrdt \\ \leq\ & \frac{\varepsilon}{2} + \iiint_{\mathbb{R}^3_+\setminus B_n} \omega_2(x,y,z;s,r,t)\ dsdrdt < \frac{3\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon. \end{aligned} \tag{2.32}$$

Note that $\left\{Cv_{k_j}(x,y,z)\right\}_j$ is a subsequence of $\{Cv_k(x,y,z)\}_k$. Then, $(C\Omega)_n(x,$ $y,z)$ is relatively compact in E. In view of Lemma 2.3, $C(\Omega)$ is relatively compact in X. Therefore, C is completely continuous. Step 3 is proved.

Step 4. Finally, we show that $\forall n \in \mathbb{N}$,

$$\lim_{|v|_n\to\infty} \frac{|Cv|_n}{|v|_n} = 0. \tag{2.33}$$

By the assumption (A_4), for all $(x,y,z) \in [0,n]^3$, we get

$$\begin{aligned} & |Cv(x,y,z)| \\ \leq & \iiint_{\mathbb{R}^3_+} |F(x,y,z;s,r,t;v(s,r,t)+\xi(s,r,t))| \ dsdrdt \\ \leq & \iiint_{\mathbb{R}^3_+} \sup_{(x,y,z)\in[0,n]^3} \omega_2(x,y,z;s,r,t)\ dsdrdt < \infty. \end{aligned} \tag{2.34}$$

It follows that

$$\lim_{|v|_n\to\infty} \frac{|Cv|_n}{|v|_n} = 0. \tag{2.35}$$

By applying Theorem 1.1, the operator $U + C$ has a fixed point v in X. Then the equation (1.1) has a solution $u = v + \xi$ on $\mathbb{R}^3_+$. The Theorem 2.1 is proved. □

3. The Asymptotically Stable Solutions

We now consider the asymptotically stable solutions for (1.1) defined as follows.

Definition 3.1. A function $\tilde{u}$ is said to be an asymptotically stable solution of (1.1) if for any solution u of (1.1),

$$\lim_{x^2+y^2+z^2\to+\infty} |u(x,y,z) - \tilde{u}(x,y,z)| = 0. \tag{3.1}$$

In this section, we assume $(A_1)-(A_5)$ hold. Then, by the Theorem 2.1, the equation (1.1) has a solution on $\mathbb{R}^3_+$.

On the other hand, if u is a solution of (1.1) then, as step 1 of the proof the Theorem 2.1, $v = u - \xi$ satisfies (2.16). This implies that for all $(x,y,z) \in \mathbb{R}^3_+$,

$$|v(x,y,z)| \leq |Uv(x,y,z)| + |Cv(x,y,z)|, \tag{3.2}$$

where $Uv(x,y,z)$, $Cv(x,y,z)$ as in (2.17). Using $(A_1)-(A_5)$ and note that

$$\begin{aligned}\xi(x,y,z) &= q(x,y,z)+f(x,y,z;\xi(x,y,z)) \\ &\quad +\int_0^x\int_0^y\int_0^z V\left(x,y,z;s,r,t,\xi(s,r,t)\right)dsdrdt, \qquad (3.3)\end{aligned}$$

we obtain for all $(x,y,z)\in\mathbb{R}_+^3$,

$$\begin{aligned}&|v(x,y,z)| \\ \leq\ & L\,|v(x,y,z)|+\int_0^x\int_0^y\int_0^z \omega_1(x,y,z;s,r,t)|v(s,r,t)|dsdrdt \\ &+\iiint_{\mathbb{R}_+^3}\omega_2(x,y,z;s,r,t)dsdrdt. \qquad (3.4)\end{aligned}$$

It follows that

$$|v(x,y,z)|\leq\int_0^x\int_0^y\int_0^z r(x,y,z;s,r,t)|v(s,r,t)|dsdrdt+a(x,y,z), \qquad (3.5)$$

where

$$\begin{cases} a(x,y,z)=\frac{1}{1-L}\iiint_{\mathbb{R}_+^3}\omega_2(x,y,z;s,r,t)dsdrdt, \\ r(x,y,z;s,r,t)=\frac{1}{1-L}\omega_1(x,y,z;s,r,t). \end{cases} \qquad (3.6)$$

The following properties of the function $w(x,y,z)=|v(x,y,z)|\in C(\mathbb{R}_+^3;\mathbb{R}_+)$ are needed for the proof of our main result in this section.

Lemma 3.2. *Let* w, $a\in C(\mathbb{R}_+^3;\mathbb{R}_+)$ *and* $r\in C(\Delta;\mathbb{R}_+)$,

$$r(x,y,z;s,r,t)\leq r(x,y,z;0,0,0)\leq r(0,0,0;0,0,0),$$

$$\forall(x,y,z;s,r,t)\in\Delta=\{(x,y,z;s,r,t)\in\mathbb{R}_+^6: s\leq x, r\leq y, t\leq z\}.$$

If

$$w(x,y,z)\leq a(x,y,z)+\int_0^x\int_0^y\int_0^z r(x,y,z;s,r,t)w(s,r,t)\,dtdrds, \qquad (3.7)$$

for all $(x,y,z)\in\mathbb{R}_+^3$, *then*

(i) $w(x,y,z)\leq a(x,y,z)$

$$+\sum_{k=0}^{\infty}\frac{(xyzR(x,y,z))^k}{(k!)^3}R(x,y,z)\int_0^x\int_0^y\int_0^z a(s,r,t)\,dsdrdt,$$

(ii) $w(x,y,z)\leq a(x,y,z)$

$$+R(x,y,z)\exp(xyzR(x,y,z))\int_0^x\int_0^y\int_0^z a(s,r,t)\,dsdrdt, \qquad (3.8)$$

for all $(x,y,z) \in \mathbb{R}_+^3$, *where*

$$R(x,y,z) = r(x,y,z;0,0,0). \tag{3.9}$$

Proof. Put

$$Aw(x,y,z) = \int_0^x \int_0^y \int_0^z r(x,y,z;s,r,t)w(s,r,t)\, dsdrdt, \tag{3.10}$$

then

$$Aw(x,y,z) \le R(x,y,z) \int_0^x \int_0^y \int_0^z w(s,r,t)\, dsdrdt, \tag{3.11}$$

for all $w \in C(\mathbb{R}_+^3;\mathbb{R}_+)$. Combining (3.7), (3.9) and (3.10), we get

$$\begin{aligned} w(x,y,z) &\le a(x,y,z) + Aw(x,y,z) \\ &\le a(x,y,z) + A\left(a + Aw\right)(x,y,z) \\ &= a(x,y,z) + Aa(x,y,z) + A^2 w(x,y,z) \\ &\le \cdots \le a(x,y,z) + \sum_{k=0}^{n-1} A^{k+1} a(x,y,z) + A^{n+1} w(x,y,z). \end{aligned} \tag{3.12}$$

By induction, the result is

$$\begin{aligned} &A^{k+1}w(x,y,z) \\ &\le \frac{(R(0,0,0)xyz)^k}{(k!)^3} R(x,y,z) \int_0^x \int_0^y \int_0^z w(s,r,t) dsdrdt. \end{aligned} \tag{3.13}$$

Thus

$$\begin{aligned} &w(x,y,z) \\ &\le a(x,y,z) + \sum_{k=0}^{n-1} A^k a(x,y,z) + A^{n+1} w(x,y,z) \\ &\le a(x,y,z) \\ &\quad + R(x,y,z) \sum_{k=0}^{n-1} \frac{(R(0,0,0)xyz)^k}{(k!)^3} \int_0^x \int_0^y \int_0^z a(s,r,t) dsdrdt \\ &\quad + \frac{(R(0,0,0)xyz)^n}{(n!)^3} R(x,y,z) \int_0^x \int_0^y \int_0^z w(s,r,t) dsdrdt. \end{aligned} \tag{3.14}$$

For $X_0 > 0$, $Y_0 > 0$, $Z_0 > 0$ are given, we have

$$\left| \frac{(R(0,0,0)xyz)^k}{(k!)^3} \right| \le \frac{(R(0,0,0)X_0Y_0Z_0)^k}{(k!)^3}, \tag{3.15}$$

for all $(x,y,z) \in [0,X_0] \times [0,Y_0] \times [0,Z_0]$, $\forall k \in \mathbb{N}$. The positive series $\sum_{k=0}^{\infty} \frac{(R(0,0,0)X_0Y_0Z_0)^k}{(k!)^3}$ converges (via a standard of D'Alembert) and then

$\sum_{k=0}^{\infty} \frac{(R(0,0,0)xyz)^k}{(k!)^3}$ converges uniformly on $[0, X_0] \times [0, Y_0] \times [0, Z_0]$ (via a standard of Weierstrass). By the continuity of the function $(x, y, z) \longmapsto \frac{(R(0,0,0)xyz)^k}{(k!)^3}$ on $[0, X_0] \times [0, Y_0] \times [0, Z_0]$, the sum of the series $\sum_{k=0}^{\infty} \frac{(R(0,0,0)xyz)^k}{(k!)^3}$ is continuous on $[0, X_0] \times [0, Y_0] \times [0, Z_0]$. On the other hand, because $X_0 > 0$, $Y_0 > 0$, $Z_0 > 0$ are arbitrary, the sum of this series is continuous on $\mathbb{R}_+^3$.

Note that $\frac{(R(0,0,0)xyz)^n}{(n!)^3} \to 0$ as $n \to \infty$, for all $(x, y, z) \in \mathbb{R}_+^3$, consequently, (3.14) leads to

$$\begin{aligned} & w(x,y,z) \\ \leq\ & a(x,y,z) + R(x,y,z) \sum_{k=0}^{\infty} \frac{(R(0,0,0)xyz)^k}{(k!)^3} \int_0^x \int_0^y \int_0^z a(s,r,t)dsdrdt, \end{aligned} \tag{3.16}$$

for all $(x, y, z) \in \mathbb{R}_+^3$. The inequality (3.8)(i) follows.

Next, the inequality (3.8)(ii) is also obtained by

$$0 \leq \frac{(R(0,0,0)xyz)^k}{(k!)^3} \leq \frac{(R(0,0,0)xyz)^k}{k!}, \ \forall (x,y,z) \in \mathbb{R}_+^3, \tag{3.17}$$

consequently

$$\sum_{k=0}^{\infty} \frac{(R(0,0,0)xyz)^k}{(k!)^3} \leq \sum_{k=0}^{\infty} \frac{(R(0,0,0)xyz)^k}{k!} = \exp(R(0,0,0)xyz), \tag{3.18}$$

for all $(x, y, z) \in \mathbb{R}_+^3$. Therefore

$$\begin{aligned} & w(x,y,z) \\ \leq\ & a(x,y,z) + R(x,y,z) \exp\left(R(0,0,0)xyz\right) \int_0^x \int_0^y \int_0^z a(s,r,t)dsdrdt, \end{aligned} \tag{3.19}$$

for all $(x, y, z) \in \mathbb{R}_+^3$. Lemma 3.2 is proved. □

Using the inequality (3.8) (ii), with

$$\begin{aligned} w(x,y,z) &= |v(x,y,z)|, a(x,y,z) \\ &= \frac{1}{1-L} \iiint_{\mathbb{R}_+^3} \omega_2(x,y,z;s,r,t)\ dsdrdt, \\ r(x,y,z;s,r,t) &= \frac{1}{1-L} \omega_1(x,y,z;s,r,t), \end{aligned}$$

we obtain the following properties of $|v(x,y,z)|$, for all $(x, y, z) \in \mathbb{R}_+^3$:

$$\begin{aligned} & |v(x,y,z)| \\ \leq\ & a(x,y,z) + R(x,y,z) \exp\left(R(0,0,0)xyz\right) \int_0^x \int_0^y \int_0^z a(s,r,t)dsdrdt, \end{aligned} \tag{3.20}$$

where

$$\begin{cases} R(x,y,z) = r(x,y,z;0,0,0) = \frac{1}{1-L}\omega_1(x,y,z;0,0,0), \\ a(x,y,z) = \frac{1}{1-L}\iiint_{\mathbb{R}^3_+} \omega_2(x,y,z;s,r,t)dsdrdt. \end{cases} \tag{3.21}$$

Then we have the following theorem about the asymptotically stable solutions.

Theorem 3.3. *Let* $(A_1)-(A_5)$ *hold. If*

$$\lim_{x^2+y^2+z^2\to\infty} \Big[a(x,y,z) \tag{3.22}$$

$$+R(x,y,z)\exp\left(R(0,0,0)xyz\right)\int_0^x\int_0^y\int_0^z a(s,r,t)dsdrdt\Big] = 0,$$

where

$$\begin{cases} a(x,y,z) = \frac{1}{1-L}\iiint_{\mathbb{R}^3_+} \omega_2(x,y,z;s,r,t)\, dsdrdt, \\ R(x,y,z) = r(x,y,z;0,0,0) = \frac{1}{1-L}\omega_1(x,y,z;0,0,0), \end{cases} \tag{3.23}$$

then every solution u *to* (1.1) *is an asymptotically stable solution. Furthermore,*

$$\lim_{x^2+y^2+z^2\to\infty} |u(x,y,z)-\xi(x,y,z)| = 0. \tag{3.24}$$

Proof. Combining (3.20) and (3.22), we obtain

$$\lim_{x^2+y^2+z^2\to+\infty} |v(x,y,z)| = \lim_{x^2+y^2+z^2\to+\infty} |u(x,y,z)-\xi(x,y,z)| = 0.$$

Theorem 3.3 is proved. □

Remark 3.4. Assume that there exist the continuous functions $\bar{\alpha}$, β_1, $\beta_2 \in C(\mathbb{R}^3_+;\mathbb{R}_+)$, such that

$$\begin{cases} \omega_1(x,y,z;s,r,t) \le C\bar{\alpha}(x,y,z)\beta_1(s,r,t), \\ \omega_2(x,y,z;s,r,t) \le C\bar{\alpha}(x,y,z)\beta_2(s,r,t), \\ \lim\limits_{x^2+y^2+z^2\to+\infty} \bar{\alpha}(x,y,z)\exp\left(R(0,0,0)xyz\right) = 0, \\ \iiint_{\mathbb{R}^3_+} \bar{\alpha}(x,y,z)dxdydz < \infty, \quad \iiint_{\mathbb{R}^3_+} \beta_2(s,r,t)dsdrdt < \infty, \\ \sup\limits_{0\le s\le x,\ 0\le r\le y,\ 0\le t\le z} \beta_1(s,r,t) \le C, \text{ for all } (x,y,z)\in\mathbb{R}^3_+, \end{cases} \tag{3.25}$$

with C always indicating a constant independent of x,y,z,s,r,t. Then (3.22) holds.

Indeed, by (3.23), (3.25), we obtain

$$\begin{aligned} a(x,y,z) &= \frac{1}{1-L}\iiint_{\mathbb{R}^3_+}\omega_2(x,y,z;s,r,t)dsdrdt \\ &\le \frac{C}{1-L}\bar{\alpha}(x,y,z)\iiint_{\mathbb{R}^3_+}\beta_2(s,r,t)dsdrdt \\ &= C_2\bar{\alpha}(x,y,z)\to 0,\ \text{as } x^2+y^2+z^2\to+\infty. \end{aligned} \tag{3.26}$$

$$\begin{aligned} R(x,y,z) &= r(x,y,z;0,0,0) = \frac{1}{1-L}\omega_1(x,y,z;0,0,0) \\ &\le \frac{C}{1-L}\beta_1(0,0,0)\bar{\alpha}(x,y,z)\le C\bar{\alpha}(x,y,z), \end{aligned}$$

$$\begin{aligned} & R(x,y,z)\exp\left(R(0,0,0)xyz\right)\int_0^x\int_0^y\int_0^z a(s,r,t)dsdrdt \\ \le\ & CC_2\bar{\alpha}(x,y,z)\exp\left(R(0,0,0)xyz\ \right)\int_0^x\int_0^y\int_0^z\bar{\alpha}(x,y,z)a(s,r,t)dsdrdt \\ \le\ & \left(CC_2\int_0^x\int_0^y\int_0^z\bar{\alpha}(x,y,z)a(s,r,t)dsdrdt\right)\bar{\alpha}(x,y,z)\exp\left(R(0,0,0)xyz\right) \\ \le\ & C\bar{\alpha}(x,y,z)\exp\left(R(0,0,0)xyz\right)\to 0, \end{aligned} \tag{3.27}$$

as $x^2+y^2+z^2\to+\infty$. Hence

$$a(x,y,z)+R(x,y,z)\exp\left(R(0,0,0)xyz\right)\int_0^x\int_0^y\int_0^z a(s,r,t)dsdrdt\to 0,$$

as $x^2+y^2+z^2\to+\infty$. Then (3.22) holds.

Acknowledgements: The authors wish to express their sincere thanks to the referees and the Editor for their valuable comments. This research is funded by Vietnam National University Ho Chi Minh City (VNU-HCM) under Grant no. B2013-18-05.

References

[1] M.A. Abdou, A.A. Badr and M.M. El-Kojok, *On the solution of a mixed nonlinear integral equation*, Applied Mathematics and Computation, **217(12)** (2011), 5466–5475.

[2] C. Avramescu and C. Vladimirescu, *An existence result of asymptotically stable solutions for an integral equation of mixed type*, Electronic J. Qualitative Theory of Diff. Equat., **25** (2005), 1–6.

[3] C. Corduneanu, *Integral equations and applications*, Cambridge University Press, New York, 1991.

[4] M.M. El-Borai, M.A. Abdou and M.M. El-Kojok, *On a discussion of nonlinear integral equation of type Volterra-Hammerstein*, J. Korea Soc. Math. Educ., Ser. B, Pure Appl. Math., **15(1)** (2008), 1–17.

[5] S. Lang, *Analysis II,* Addison-Wesley, Reading, Mass., California London, 1969.

[6] N. Lungu and I.A. Rus, *On a functional Volterra–Fredholm integral equation via Picard operator*, Journal of Mathematical Inequalities, **3(4)** (2009), 519–527.

[7] L.T.P. Ngoc and N.T. Long, *On a fixed point theorem of Krasnosel'skii type and application to integral equations*, Fixed Point Theory and Applications, Vol. **2006** (2006), Article ID 30847, 24 pages.

[8] L.T.P. Ngoc and N.T. Long, *Applying a fixed point theorem of Krasnosel'skii type to the existence of asymptotically stable solutions for a Volterra–Hammerstein integral equation*, Nonlinear Anal. TMA., **74(11)** (2011), 3769–3774.

[9] L.T.P. Ngoc and N.T. Long, *On a nonlinear Volterra–Hammerstein integral equation in two variables*, Acta Mathematica Scientia, **33B(2)** (2013), 484–494.

[10] B.G. Pachpatte, *On Fredholm type integral equation in two variables*, Differential Equations & Applications, **1(1)** (2009), 27–39.

[11] B.G. Pachpatte, *Volterra integral and integrodifferential equations in two variables*, J. Inequal. Pure and Appl. Math., **10(4)** (2009), Art. 108, 10 pp.