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GENERALIZATION OF AN INEQUALITY INVOLVING MAXIMUM MODULI OF A POLYNOMIAL AND ITS POLAR DERIVATIVE

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Abstract. Let $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-} z^{n-\nu}$, $1 \leq \mu \leq n$, be a polynomial of degree n and $D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$ denote the polar derivative of $P(z)$ with respect to a point $\alpha \in \mathbb{C}$. It is known that [3] for every real or complex number α with $|\alpha| \geq k^\mu$,

$$\max_{|z|=1} |D_\alpha P(z)| \geq n \left(\frac{|\alpha| - k^\mu}{1 + k^\mu} \right) \max_{|z|=1} |P(z)| .$$

In this paper, we obtain some generalizations of above inequality by extending it to the sth polar derivative.

1. Introduction

Let $P(z)$ be a polynomial of degree n, then

$$\max_{|z|=1} \left| P'(z) \right| \leq n \max_{|z|=1} |P(z)| . \tag{1.1}$$

Inequality (1.1) is an immediate consequence of S. Bernstein's Theorem on the derivative of a trigonometric polynomial (for reference, see [8, p.531], [9, p.508] or [10]) and the result is best possible with equality holding for the polynomial $P(z) = az^n$, $a \neq 0$.

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For polynomials of degree n having all zeros in $|z| \leq 1$, it was proved by Turán [11] that

$$\max_{|z|=1} \left|P'(z)\right| \geq \frac{n}{2} \max_{|z|=1} |P(z)| . \tag{1.2}$$

The inequality (1.2) is best possible and become equality for polynomial $P(z) = (z+1)^n$.

As an extension of (1.2) Malik [6] proved that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$ where $k \leq 1$, then

$$\max_{|z|=1} \left|P'(z)\right| \geq \frac{n}{1+k} \max_{|z|=1} |P(z)| . \tag{1.3}$$

For the class of polynomials $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$, of degree n having all their zeros in $|z| \leq k$ where $k \leq 1$, Chan and Malik [5] proved that

$$\max_{|z|=1} \left|P'(z)\right| \geq \frac{n}{1+k^{\mu}} \max_{|z|=1} |P(z)| . \tag{1.4}$$

Let $D_\alpha P(z)$ denote the polar derivative of a polynomial $P(z)$ of degree n with respect to a point $\alpha \in \mathbb{C}$, then (see [7])

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z).$$

The polynomial $D_\alpha P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha \to \infty} \frac{D_\alpha P(z)}{\alpha} = P'(z)$$

uniformly with respect to z for $|z| \leq R$, $R > 0$.

Now corresponding to a given nth degree polynomial $P(z)$, we construct a sequence of polar derivatives

$$D_{\alpha_1} P(z) = nP(z) + (\alpha_1 - z)P'(z),$$

$$\begin{aligned} D_{\alpha_s} D_{\alpha_{s-1}} \cdots D_{\alpha_2} D_{\alpha_1} P(z) = {} & (n-s+1) \left\{ D_{\alpha_{s-1}} \cdots D_{\alpha_2} D_{\alpha_1} P(z) \right\} \\ & + (\alpha_s - z) \left\{ D_{\alpha_{s-1}} \cdots D_{\alpha_2} D_{\alpha_1} P(z) \right\}' . \end{aligned}$$

The points $\alpha_1, \alpha_2, \cdots, \alpha_s$, $s = 1, 2, \cdots, n$, may be equal or unequal complex numbers. The sth polar derivative $D_{\alpha_s} D_{\alpha_{s-1}} \cdots D_{\alpha_2} D_{\alpha_1} P(z)$ of $P(z)$ is a polynomial of degree at most $n-s$.

Aziz and Rather [3] extended inequality (1.4) to the polar derivative and proved that if $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$, is a polynomial of degree n having all its zeros in $|z| \leq k$ where $k \leq 1$, then for each complex number α with $|\alpha| \geq k^\mu$

$$\max_{|z|=1} |D_\alpha P(z)| \geq n \left(\frac{|\alpha| - k^\mu}{1+k^\mu} \right) \max_{|z|=1} |P(z)| . \tag{1.5}$$

2. Preliminaries

For the proof of our Theorems, we need the following Lemmas.
The first Lemma follows by repeated application of Laguerre's theorem [1] or [7, p.52].

Lemma 2.1. *If all the zeros of nth degree polynomial lie in circular region C and if none of the points $\alpha_1, \alpha_2, \cdots, \alpha_s$ lie in circular region C, then each of the polar derivatives*

$$D_{\alpha_s} D_{\alpha_{s-1}} \cdots D_{\alpha_1} P(z), \quad s = 1, 2, \cdots, n-1 \tag{2.1}$$

has all its zeros in C.

The next Lemma is due to Aziz and Rather [2].

Lemma 2.2. *If $P(z)$ is a polynomial of degree n having all its zeros in the closed disk $|z| \leq k$, $k \leq 1$, then for each complex number α with $|\alpha| \geq k$ and $|z| = 1$, we have*

$$|D_\alpha P(z)| \geq n \left(\frac{|\alpha| - k}{1 + k} \right) |P(z)|. \tag{2.2}$$

Lemma 2.3. *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n having all its zeros in $|z| \leq k$, $k \leq 1$ then*

$$\frac{1}{n} \left| \frac{a_{n-1}}{a_n} \right| \leq k. \tag{2.3}$$

The above lemma is easy to prove.

Lemma 2.4. *If $P(z)$ be a polynomial of degree n having all zeros in the disk $|z| \leq k$ where $k \leq 1$, then for $\alpha_1, \alpha_2, \cdots, \alpha_s \in \mathbb{C}$ with $|\alpha_1| \geq k, |\alpha_2| \geq k, \cdots, |\alpha_s| \geq k$, $(1 \leq s < n)$, and $|z| = 1$*

$$\begin{aligned} &|D_{\alpha_s} D_{\alpha_{s-1}} \cdots D_{\alpha_2} D_{\alpha_1} P(z)| \\ &\geq n_s \frac{(|\alpha_1| - k)(|\alpha_2| - k) \cdots (|\alpha_s| - k)}{(1+k)^s} |P(z)|, \end{aligned} \tag{2.4}$$

where $n_s := n(n-1)(n-2) \cdots (n-s+1)$.

Proof. The result is trivial if $|\alpha_j| = k$ for at least one j where $j = 1, 2, \cdots, s$. Therefore, we assume $|\alpha_j| > k$ for all $j = 1, 2, \cdots, s$. We shall prove Lemma by principle of mathematical induction. For $s = 1$ the result follows by Lemma 2.2.
We assume that the result is true for $s = q$, which means that for $|z| = 1$, we

have

$$|D_{\alpha_q}D_{\alpha_{q-1}}\cdots D_{\alpha_2}D_{\alpha_1}P(z)| \geq n_q\frac{(|\alpha_1|-k)(|\alpha_2|-k)\cdots(|\alpha_q|-k)}{(1+k)^q}|P(z)|, \tag{2.5}$$

where $q \geq 1$ and $n_q = n(n-1)\cdots(n-q+1)$. Now, we shall prove that the Lemma is also true for $t = q+1$.
Since $D_{\alpha_1}P(z) = (na_n\alpha_1 + a_{n-1})z^{n-1} + \cdots + (na_0 + \alpha_1 a_1)$ and $|\alpha_1| > k$, $D_{\alpha_1}P(z)$ is a polynomial of degree $n-1$. If this is not true, then

$$na_n\alpha_1 + a_{n-1} = 0,$$

which implies

$$|\alpha_1| = \frac{1}{n}\left|\frac{a_{n-1}}{a_n}\right|.$$

By Lemma 2.3, we have

$$|\alpha_1| = \frac{1}{n}\left|\frac{a_{n-1}}{a_n}\right| \leq k.$$

But this is the contradiction to the fact $|\alpha| > k$. Hence, $D_{\alpha_1}P(z)$ is a polynomial of degree $n-1$ and by Lemma 2.1, $D_{\alpha_1}P(z)$ has all its zeros in $|z| \leq k$. Therefore, it follows by similar argument as before, $D_{\alpha_2}D_{\alpha_1}P(z)$ must be a polynomial of degree $n-2$ for $|\alpha_1| > k$, $|\alpha_2| > k$ and all its zeros in $|z| \leq k$. Continuing in this way, we conclude $D_{\alpha_q}D_{\alpha_{q-1}}\cdots D_{\alpha_1}P(z)$ is a polynomial of degree $n-q$ for all $|\alpha_j| > k$, $j = 1, 2, \ldots, q$ and has all zeros in $|z| \leq k$. Applying Lemma 2.2 to $D_{\alpha_q}D_{\alpha_{q-1}}\cdots D_{\alpha_2}D_{\alpha_1}P(z)$, we get for $|\alpha_{q+1}| > k$,

$$|D_{\alpha_{q+1}}D_{\alpha_q}D_{\alpha_{q-1}}\cdots D_{\alpha_2}D_{\alpha_1}P(z)| \geq \frac{(n-q)(|\alpha_{q+1}|-k)}{1+k}|D_{\alpha_q}D_{\alpha_{q-1}}\cdots D_{\alpha_2}D_{\alpha_1}P(z)| \text{ for } |z| = 1. \tag{2.6}$$

Inequality (2.6) in conjunction with (2.5) gives for $|z| = 1$,

$$|D_{\alpha_{q+1}}D_{\alpha_q}D_{\alpha_{q-1}}\cdots D_{\alpha_2}D_{\alpha_1}P(z)| \geq n_{q+1}\frac{(|\alpha_1|-k)(|\alpha_2|-k)\cdots(|\alpha_{q+1}|-k)}{(1+k)^{q+1}}|P(z)| \text{ for } |z| = 1,$$

where $n_{q+1} = n(n-1)\cdots(n-q)$. This shows that the result is true for $s = q+1$ also. This completes the proof of Lemma 2.4. □

We also need the following Lemma due to Aziz and Rather [3].

Lemma 2.5. *Let $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-} z^{n-\nu}$, $1 \le \mu \le n$, be a polynomial of degree n having all its zeros in the disk $|z| \le k$ where $k \le 1$ then for every real or complex number α with $|\alpha| \ge k$*

$$|D_\alpha P(z)| \ge n \left(\frac{|\alpha| - k^\mu}{1 + k^\mu} \right) |P(z)| \qquad for \quad |z| = 1. \tag{2.7}$$

3. Main Results

In this paper, we first obtain the following generalization of inequality (1.5).

Theorem 3.1. *Let $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-} z^{n-\nu}$, $1 \le \mu \le n$, be a polynomial of degree n having all its zeros in the disk $|z| \le k$ where $k \le 1$ then for complex numbers α_j, $j = 1, 2, \cdots, s$ with $|\alpha_j| > k$ if $j = 1, 2, \cdots, \mu - 1$ and $|\alpha_j| \ge k$ if $j = \mu, \mu + 1, \cdots, s$ and for $|z| = 1$*

$$|D_{\alpha_s} D_{\alpha_{s-1}} \cdots D_{\alpha_2} D_{\alpha_1} P(z)| \ge n_s \Lambda(\mu, s) |P(z)|, \tag{3.1}$$

where

$$\Lambda(\mu, s) := \begin{cases} \prod_{j=1}^{s} \left(\frac{|\alpha_j| - k^{\mu-j+1}}{1 + k^{\mu-j+1}} \right) & if \ s \le \mu, \\ \\ \prod_{j=1}^{\mu} \left(\frac{|\alpha_j| - k^{\mu-j+1}}{1 + k^{\mu-j+1}} \right) \prod_{j=\mu+1}^{s} \left(\frac{|\alpha_j| - k}{1 + k} \right) & if \ s > \mu, \end{cases} \tag{3.2}$$

where $n_s = n(n-1)(n-2)\cdots(n-s+1)$.

Proof. If $\mu = 1$, then result follows from Lemma 2.4. So, we assume $\mu > 1$.

Case I: If $s \le \mu$. We prove the result by Principle of Mathematical Induction. Since $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-} z^{n-\nu}$, $1 \le \mu \le n$, has all its zeros in $|z| \le k, k \le 1$, applying Lemma 2.5 to $P(z)$, we get for every $\alpha_1 \in \mathbb{C}$, with $|\alpha_1| > k$,

$$|D_{\alpha_1} P(z)| \ge n \left(\frac{|\alpha_1| - k^\mu}{1 + k^\mu} \right) |P(z)| \qquad \text{for} \quad |z| = 1. \tag{3.3}$$

Again, since $D_{\alpha_1} P(z) = n a_n \alpha_1 z^{n-1} + \mu a_{n-\mu} z^{n-\mu} \cdots + (n a_0 + \alpha_1 a_1)$ is a polynomial of degree $n-1$ and by Lemma 2.1 $D_{\alpha_1} P(z)$ has all its zeros in $|z| \le k$. Therefore, applying Lemma 2.5 to $D_{\alpha_1} P(z)$, then for each $\alpha_2 \in \mathbb{C}$ with $|\alpha_2| > k$ we obtain for $|z| = 1$,

$$\begin{aligned} |D_{\alpha_2} D_{\alpha_1} P(z)| &\ge (n-1) \left(\frac{|\alpha_2| - k^{\mu-1}}{1 + k^{\mu-1}} \right) |D_{\alpha_1} P(z)| \\ &\ge n(n-1) \left(\frac{(|\alpha_1| - k^\mu)(|\alpha_2| - k^{\mu-1})}{(1 + k^\mu)(1 + k^{\mu-1})} \right) |P(z)|. \end{aligned} \tag{3.4}$$

By the similar argument as before, $D_{\alpha_2}D_{\alpha_1}P(z)$ is a polynomial of degree $n-2$ and has all its zeros in $|z|\le k$. We assume that the result is true for $s=q$, that is for every $\alpha_j\in\mathbb{C}$ with $|\alpha_j|>k$, $j=1,2,\cdots,q$

$$\begin{aligned}&|D_{\alpha_q}\cdots D_{\alpha_2}D_{\alpha_1}P(z)|\\&\ge n(n-1)\cdots(n-q+1)\\&\quad\times\left(\frac{(|\alpha_1|-k^{\mu})(|\alpha_2|-k^{\mu-1})\cdots(|\alpha_q|-k^{\mu-q+1})}{(1+k^{\mu})(1+k^{\mu-1})\cdots(1+k^{\mu-q+1})}\right)|P(z)|\quad\text{for}\quad|z|=1.\end{aligned}$$

Proceeding as before, for every $\alpha_j\in\mathbb{C}$ with $|\alpha_j|>k$, $j=1,2,\cdots,q$, we conclude $P_q(z):=D_{\alpha_q}\cdots D_{\alpha_2}D_{\alpha_1}P(z)$ is a polynomial of degree $n-q$ and has all its zeros in $|z|\le k$. Now, $P_q(z)=n_q\prod_{j=1}^{q}\alpha_j a_n z^{n-q}+\mu^q a_{n-\mu}z^{n-\mu}+\cdots+(\alpha_q(\cdots)+(n-q+1)(\cdots))$. Applying Lemma 2.5 to $P_q(z)$, we obtain for every $\alpha_{q+1}\in\mathbb{C}$, with $|\alpha_{q+1}|>k$ and $|z|=1$,

$$\begin{aligned}|D_{\alpha_{q+1}}P_q(z)|&\ge(n-q)\left(\frac{|\alpha_{q+1}|-k^{\mu-q}}{1+k^{\mu-q}}\right)|P_q(z)|\\&\ge n(n-1)\cdots(n-q+1)(n-q)\\&\quad\times\left(\frac{(|\alpha_1|-k^{\mu})(|\alpha_2|-k^{\mu-1})\cdots(|\alpha_{q+1}|-k^{\mu-q})}{(1+k^{\mu})(1+k^{\mu-1})\cdots(1+k^{\mu-q})}\right)|P(z)|.\end{aligned}$$

This show the result is true for $s=q+1$ also. This completes proof of Case I.

Case II: If $s>\mu$, by Case I, for each $\alpha_j\in\mathbb{C}$ with $|\alpha_j|>k$ $j=1,2,\cdots,\mu$, we have for $|z|=1$,

$$\begin{aligned}&|D_{\alpha_\mu}\cdots D_{\alpha_1}P(z)|\\&\ge n(n-1)\cdots(n-\mu+1)\\&\quad\times\left(\frac{(|\alpha_1|-k^{\mu})(|\alpha_2|-k^{\mu-1})\cdots(|\alpha_\mu|-k)}{(1+k^{\mu})(1+k^{\mu-1})\cdots(1+k)}\right)|P(z)|.\end{aligned}\tag{3.5}$$

Since $P_\mu(z):=D_{\alpha_\mu}\cdots D_{\alpha_1}P(z)$ is a polynomial of degree $n-\mu$. Applying Lemma 2.4 to $D_{\alpha_\mu}\cdots D_{\alpha_1}P(z)$, we obtain for $\alpha_{\mu+1},\alpha_{\mu+2},\cdots,\alpha_{s-1},\alpha_s$ with $|\alpha_{\mu+1}|>k,|\alpha_{\mu+2}|>k\cdots,|\alpha_{s-1}|>k,|\alpha_s|>k$

$$\begin{aligned}&|D_{\alpha_s}D_{\alpha_{s-1}}\cdots D_{\alpha_{\mu+1}}P_\mu(z)|\\&\ge(n-\mu)\cdots(n-s)\frac{(|\alpha_s|-k)(|\alpha_{s-1}|-k)\cdots(|\alpha_{\mu+1}|-k)}{(1+k)^{s-\mu}}|P_\mu(z)|\\&\ge n_s\prod_{j=1}^{\mu}\left(\frac{|\alpha_j|-k^{\mu-j+1}}{1+k^{\mu-j+1}}\right)\prod_{j=\mu+1}^{s}\left(\frac{|\alpha_j|-k}{1+k}\right)|P(z)|\quad\text{for}\quad|z|=1.\end{aligned}$$

If $|\alpha_j|=k$ for some $j=\mu,\mu+1,\cdots,s$, then the result follows trivially. This completes the proof of Theorem 3.1. □

If we take $\alpha_1 = \alpha_2 = \cdots = \alpha_s = \alpha$ and divide two sides of inequalities (3.1) and (3.2) by $|\alpha|^s$ and letting $|\alpha| \to \infty$, $j = 1, 2, \cdots, s$, we obtain the following generalization of inequality (1.4).

Corollary 3.2. *Let $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-} z^{n-\nu}$, $1 \leq \mu \leq n$, be a polynomial of degree n having all its zeros in the disk $|z| \leq k$ where $k \leq 1$ then for $|z| = 1$,*

$$|P^{(s)}(z)| \geq n_s \prod_{j=1}^{s} \frac{1}{1 + k^{\mu-j+1}} |P(z)| \quad if \quad s \leq \mu$$

and

$$|P^{(s)}(z)| \geq \frac{n_s}{(1+k)^{s-\mu}} \prod_{j=1}^{\mu} \left(\frac{1}{1 + k^{\mu-j+1}} \right) |P(z)| \quad if \quad s > \mu,$$

where $n_s = n(n-1)(n-2)\cdots(n-s+1)$.

Next as a refinement of Theorem 3.1, we prove:

Theorem 3.3. *Let $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-} z^{n-\nu}$, $1 \leq \mu \leq n$, be a polynomial of degree n having all its zeros in the disk $|z| \leq k$ where $k \leq 1$ then for complex numbers α_j, $j = 1, 2, \cdots, s$ with $|\alpha_j| > k$ if $j = 1, 2, \cdots, \mu - 1$ and $|\alpha_j| \geq k$ if $j = \mu, \mu + 1, \cdots, s$ and for $|z| = 1$,*

$$\begin{aligned} &|D_{\alpha_s} D_{\alpha_{s-1}} \cdots D_{\alpha_2} D_{\alpha_1} P(z)| \\ &\geq n_s \left[\Lambda(\mu, s)|P(z)| + \left\{ |\alpha_1||\alpha_2| \cdots |\alpha_s| - \Lambda(\mu, s) \right\} m \right], \end{aligned} \tag{3.6}$$

where $\Lambda(\mu, s)$ is defined by (3.2), $n_s = n(n-1)(n-2)\cdots(n-s+1)$ and $m = \min_{|z|=k} |P(z)|$.

Proof. We prove Theorem for the case $s > \mu$ and the case $s \leq \mu$ follows on same lines. Let

$$m = \min_{|z|=k} |P(z)|.$$

If $P(z)$ has a zero on $|z| = k$ then result follows from Theorem 3.1. Therefore from now onwards we shall assume that $P(z)$ has all its zeros in $|z| < k$. By Rouché's theorem, the polynomial $F(z) = P(z) - m\beta z^n$ have all its zeros in $|z| < k$, for every β with $|\beta| < 1$. Thereby applying Theorem 3.1 to $F(z)$ then we get for complex numbers α_j, $j = 1, 2, \cdots, s$ with $|\alpha_j| > k$ if $j = 1, 2, \cdots, \mu - 1$ and $|\alpha_j| \geq k$ if $j = \mu, \mu + 1, \cdots, s$ and for $|z| = 1$

$$\begin{aligned} &|D_{\alpha_s} D_{\alpha_{s-1}} \cdots D_{\alpha_2} D_{\alpha_1} F(z)| \\ &\geq n_s \prod_{j=1}^{\mu} \left(\frac{|\alpha_j| - k^{\mu-j+1}}{1 + k^{\mu-j+1}} \right) \prod_{j=\mu+1}^{s} \left(\frac{|\alpha_j| - k}{1 + k} \right) |F(z)|, \qquad s > \mu, \end{aligned}$$

that is

$$\begin{aligned}&\left|D_{\alpha_s}D_{\alpha_{s-1}}\cdots D_{\alpha_2}D_{\alpha_1}P(z)-\beta mn_s\alpha_1\alpha_2\cdots\alpha_s z^{n-s}\right|\\&\geq n_s\prod_{j=1}^{\mu}\left(\frac{|\alpha_j|-k^{\mu-j+1}}{1+k^{\mu-j+1}}\right)\prod_{j=\mu+1}^{s}\left(\frac{|\alpha_j|-k}{1+k}\right)\{|P(z)|-m|\beta|\}\,. \qquad (3.7)\end{aligned}$$

By Lemma 2.1, the polynomial

$$\begin{aligned}&D_{\alpha_s}D_{\alpha_{s-1}}\cdots D_{\alpha_2}D_{\alpha_1}F(z)\\&=D_{\alpha_s}D_{\alpha_{s-1}}\cdots D_{\alpha_2}D_{\alpha_1}P(z)-\beta mn_s\alpha_1\alpha_2\cdots\alpha_s z^{n-s}\end{aligned}$$

has all its zeros in $|z|<k$ and therefore for $|z|\geq 1$,

$$|D_{\alpha_s}D_{\alpha_{s-1}}\cdots D_{\alpha_2}D_{\alpha_1}P(z)|\geq|\beta|mn_s|\alpha_1||\alpha_2|\cdots|\alpha_s||z|^{n-s}. \qquad (3.8)$$

Choosing argument of β in the right hand side of inequality (3.7) such that

$$\begin{aligned}&\left|D_{\alpha_s}D_{\alpha_{s-1}}\cdots D_{\alpha_2}D_{\alpha_1}P(z)-\beta mn_s\alpha_1\alpha_2\cdots\alpha_s z^{n-s}\right|\\&=|D_{\alpha_s}D_{\alpha_{s-1}}\cdots D_{\alpha_2}D_{\alpha_1}P(z)|-|\beta|mn_s|\alpha_1||\alpha_2|\cdots|\alpha_s||z|^{n-s}, \qquad (3.9)\end{aligned}$$

which is possible due to inequality (3.8). Using (3.9) in (3.7), we obtain for $|z|=1$,

$$\begin{aligned}&|D_{\alpha_s}D_{\alpha_{s-1}}\cdots D_{\alpha_2}D_{\alpha_1}P(z)|\\&\geq n_s\Bigg[\prod_{j=1}^{\mu}\left(\frac{|\alpha_j|-k^{\mu-j+1}}{1+k^{\mu-j+1}}\right)\prod_{j=\mu+1}^{s}\left(\frac{|\alpha_j|-k}{1+k}\right)|P(z)|\\&\quad+|\beta|\left\{|\alpha_1||\alpha_2|\cdots|\alpha_s|-\prod_{j=1}^{\mu}\left(\frac{|\alpha_j|-k^{\mu-j+1}}{1+k^{\mu-j+1}}\right)\prod_{j=\mu+1}^{s}\left(\frac{|\alpha_j|-k}{1+k}\right)\right\}m\Bigg].\end{aligned}$$

Letting $|\beta|\to 1$, we get Theorem 3.3. □

If we take $\alpha_1=\alpha_2=\cdots=\alpha_s=\alpha$ and divide two sides of inequalities (3.6) by $|\alpha|^s$ and letting $|\alpha|\to\infty$, $j=1,2,\cdots,s$, we shall obtain a refinement of Corollary 3.2, we omit the details.

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