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CONE METRIC SPACES AND RELATED FIXED POINT THEOREMS

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Abstract. In this paper, we discuss some related properties and fixed point theorems on the complete cone metric spaces, and we obtain some new fixed point theorems for different expanding mappings on the generalized complete cone metric spaces. Especially this paper omittes the normal cone which is required in the similar studies before, and generalizes some new results about the related references.

1. Introduction

It is known to us that the fixed point theory is an important branch of nonlinear analisis, and it is applied in many areas such as physics and economics. Huang and Zhang replace the real numbers by ordering Banach space and define cone metric spaces, and they also prove some fixed point theorems for contractive mappings(see reference [1]). [5] generalizes the results of [1] later.

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Scholars study and obtain many fixed point theorems for different contractive mappings in cone metric spaces, but the normal cone must be used during the process of proving the results.

In this paper, we give some properties of cone metric space and related fixed point theorems about the cone metric space. In the complete $G-$cone metric space, we obtain several new fixed point theorems for different expanding mappings. Especially, this paper omits the normal cone which is required in the similar study before(see references [3] and [8]), and generalizes some related results(see references [6] and [10]).

2. Preliminaries

Let E always be a real Banach space and P a subset of E. P is called a cone if

(i) P is closed, nonempty, and $P \neq \{0\}$;
(ii) $a, b \in R, a, b \geq 0, x, y \in P \Rightarrow ax + by \in P$;
(iii) $x \in P$ and $-x \in P \Rightarrow x = 0$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$but $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$, $intP$ denotes the interior of P.

Definition 2.1. ([1]) Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies

(i) $0 < d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Definition 2.2. ([1]) Let (X, d) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $0 \leq c$, there is N such that for all $n > N$, $d(x_n, x) \leq c$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to x, and x is the limit of $\{x_n\}$. We denote this by

$$\lim_{n\to\infty} x_n = x \quad \text{or} \quad x_n \to x \ \ (n \to \infty).$$

Definition 2.3. ([1]) Let (X, d) be a cone metric space, $\{x_n\}$ be a sequence in X. If for any $c \in E$ with $0 \ll c$, there is N such that for all $n, m > N$, $d(x_m, x_n) \ll c$, then $\{x_n\}$ is called a Cauchy sequence in X.

Definition 2.4. ([1]) Let (X, d) be a cone metric space, if every Cauchy sequence is convergent in X, then X is called a complete cone metric space.

Definition 2.5. ([2]) Let X be a nonempty set. Suppose the mapping $G : X \times X \times X \to E$ satisfies:

(i) $G(x, y, z) = 0$ if $x = y = z$;
(ii) $0 < G(x, x, y)$, whenever $x \neq y$, for all $x, y \in X$;
(iii) $G(x, x, y) \leq G(x, y, z)$, whenever $y \neq z$;
(iv) $G(x, y, z) = G(x, z, y) = G(y, x, z) = ...$ (Symmetric in all three variables);
(v) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$.

Then G is called a generalized cone metric on X, and X is called a generalized cone metric space or more specially a $G-$cone metric space.

From definition, we always have

$$\frac{1}{2}G(x, y, y) \leq G(x, x, y) \leq 2G(x, y, y), \quad \forall\, x, y \in X.$$

Definition 2.6. ([2]) A $G-$cone metric space X is symmetric if

$$G(x, x, y) = G(y, y, x) \quad \text{for all} \quad x, y \in X.$$

Definition 2.7. ([2]) Let X be a $G-$cone metric space and $\{x_n\}$be a sequence in X. We say that $\{x_n\}$ is

(i) *Cauchy sequence* if for every $c \in E$ with $0 \ll c$, there is N such that for all $m, n, l > N$, $G(x_m, x_n, x_l) \ll c$.
(ii) *Convergent sequence* if for every c in E with$0 \ll c$, there is N such that for all $m, n > N$, $G(x_m, x_n, x) \leq c$ for some fixed x in X. Here x is called the limit of a sequence $\{x_n\}$ and is denoted by $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

A $G-$cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X.

Lemma 2.8. *Let (X, d) be a cone metric space, $\{x_m\}$ and $\{y_n\}$ be sequences in X such that $x_m \to x$, $y_n \to y$ as $m, n \to \infty$, then $d(x_m, y_n) \to d(x, y)$ as $m, n \to \infty$.*

Proof. Let $\{x_m\}$, $\{y_n\}$ be sequences in X such that $x_m \to x$, $y_n \to y$ as $m, n \to \infty$. For any $c \in E$ with $0 \ll c$, and any $k \geq 1$, there is N_k, for all $m, n > N_k$, $d(x_m, x) \leq \frac{c}{2k}$, $d(y_n, y) \leq \frac{c}{2k}$. We have

$$d(x_m, y_n) \leq d(x_m, x) + d(x, y_n) \leq d(x_m, x) + d(x, y) + d(y, y_n).$$

Hence, for all $m, n > N_k$,

$$d(x_m, y_n) - d(x, y) \leq d(x_m, x) + d(y, y_n) \leq \frac{c}{2k} + \frac{c}{2k} = \frac{c}{k}.$$

Similarly

$$d(x, y) - d(x_m, y_n) \leq \frac{c}{k}.$$

It implies that $\frac{c}{k} - (d(x_m, y_n) - d(x, y))$ and $\frac{c}{k} - (d(x, y) - d(x_m, y_n))$ are in P.

For any m and n, x_m and y_n is always in X, by (i) in Definition 2.1, we have $0 \leq d(x_m, y_n)$, this means $d(x_m, y_n) \in P$, therefore $\lim_{m,n\to\infty} d(x_m, y_n)$ always exists as $m, n \to \infty$. Since $\frac{c}{k} \to 0$, as $k \to \infty$ and P is closed, therefore we have

$$\{d(x, y) - \lim_{m,n\to\infty} d(x_m, y_n)\} \in P$$

and

$$-\{d(x, y) - \lim_{m,n\to\infty} d(x_m, y_n)\} \in P$$

as $k, m, n \to \infty$. Therefore

$$d(x, y) - \lim_{m,n\to\infty} d(x_m, y_n) = 0,$$

this means

$$\lim_{m,n\to\infty} d(x_m, y_n) = d(x, y).$$ □

Lemma 2.9. ([9]) *Let (X, d) be a cone metric space, P is a cone in E, for all $x, y, z \in E$, we have*

(i) *If $x \leq y, y \ll z$, then $x \ll z$;*
(ii) *If $x \leq y, y \leq z$, then $x \leq z$;*
(iii) *If $x \ll y, y \ll z$, then $x \ll z$.*

Lemma 2.10. ([2]) *Let X be a $G-$cone metric space, $\{x_m\}$, $\{y_n\}$ and $\{z_l\}$ be sequences in X such that $x_m \to x$, $y_n \to y$, $z_l \to z$, then $G(x_m, y_n, z_l) \to G(x, y, z)$, as $m, n, l \to \infty$.*

3. Main Results

Theorem 3.1. *Let X be a cone metric space, P is a cone in E, for all $x, y, g, h \in E$, if $x \leq g$, $y \ll h$, then $x + y \ll g + h$.*

Proof. Given that $x \leq g$, $y \ll h$, we have $g - x \in P$ and $h - y \in intP$. Therefore $g + y - y - x \in P$ and $h + g - g - y \in intP$. This means $(g + y) - (y + x) \in P$ and $(h + g) - (g + y) \in intP$. Thus $x + y \leq g + y$ and $g + y \ll g + h$. By Lemma 2.9, we have $x + y \ll g + h$. □

Theorem 3.2. *Let X be a cone metric space, $P$is a cone in E, $\{x_m\}$ and $\{y_m\}$ be sequences in X, $x_m \to x$, $y_m \to y$ as $m \to \infty$, if for any m, the sequence of $\{y_m - x_m\}$ is convergent in P and $\{y_m - x_m\} \in P$, then $x \leq y$.*

Proof. Since P is a cone in E, therefore P is nonempty, closed and a convex set in E. Thus, for any convergent sequence in P, the limit of the sequence is still in P. We have

$$\begin{aligned}
&\lim_{m\to\infty}(y_m - x_m) \in P,\\
&\lim_{m\to\infty}(y_m - x_m) = \lim_{m\to\infty}(y_m - y + y - x + x - x_m)\\
&\qquad = \lim_{m\to\infty}(y_m - y) + \lim_{m\to\infty}(y - x) + \lim_{m\to\infty}(x - x_m)\\
&\qquad = y - x \in P.
\end{aligned}$$

Therefore $x \leq y$. □

Remark 3.3. The results of Theorem 3.1 and Theorem 3.2 can be generalized to $G-$cone metric space. The process of proof is similar and omitted here.

Theorem 3.4. *Let (X, d) be a complete cone metric space and $T : X \to X$ be a surjective mapping satisfying the following conditions*

$$d(Tx, Ty) \geq ad(x, y) + bd(x, Tx) + cd(y, Ty)$$

for all $x, y \in X$, $x \neq y$, where $a, b, c \geq 0$ and $a + b + c > 1$. Then T must has a fixed point in X, especially when $a > 1$, T has a unique fixed point in X.

Proof. Choose any $x_0 \in X$. Since T is a surjective mapping, therefore there exits $x_1 \in X$ satisfying $Tx_1 = x_0$. Similarly we can obtain a sequence of $\{x_n\}$ in X such that $x_n = Tx_{n+1}$ and $x_n \neq x_{n+1}$ (If there exists a positive integer i satisfying $x_i = x_{i+1}$, then $x_{i+1} = x_i = Tx_{i+1}$, hence x_{i+1} is a fixed point of T). We have

$$\begin{aligned}
d(x_{n-1}, x_n) &= d(Tx_n, Tx_{n+1})\\
&\geq ad(x_n, x_{n+1}) + bd(x_n, x_{n-1}) + cd(x_{n+1}, x_n),
\end{aligned}$$

so,

$$(1 - b)d(x_{n-1}, x_n) \geq (a + c)d(x_n, x_{n+1}). \tag{3.1}$$

If $a+c = 0$, since $a+b+c > 1$, therefore $1-b < 0$, we have $(1-b)d(x_{n-1}, x_n) \leq 0$. But $(a + c)d(x_n, x_{n+1}) \geq 0$. Thus $a + c \neq 0, b < 1$. Using (3.1), we have

$$d(x_n, x_{n+1}) \leq \frac{1 - b}{a + c} d(x_{n-1}, x_n).$$

Let $h = \frac{1-b}{a+c}$, then

$$h = \frac{1 - b}{a + c} < \frac{(a + b + c) - b}{a + c} = 1.$$

Therefore

$$d(x_n, x_{n+1}) \leq hd(x_{n-1}, x_n) \Rightarrow d(x_n, x_{n+1}) \leq h^n d(x_0, x_1).$$

For any positive integer $m > n$, we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m-1}, x_m) \\ &\leq (h^n + h^{n+1} + \ldots + h^{m-1}) d(x_0, x_1) \\ &= \frac{h^n(1 - h^{m-n})}{1 - h} d(x_0, x_1) \\ &\leq \frac{h^n}{1 - h} d(x_0, x_1). \end{aligned}$$

For any $c \in E$ with $0 \ll c$. Choose $t > 0$ such that $\{c\} + N_t(0) \subseteq P$, where $N_t(0) = \{y : ||y|| < t\}$. Choose a positive integer N_1 such that

$$\frac{h^n}{1 - h} d(x_0, x_1) \in N_t(0),$$

for all $n > N_1$. Then, $\frac{h^n}{1-h} d(x_0, x_1) \ll c$, for all $n > N_1$. Thus

$$d(x_n, x_m) \leq \frac{h^n}{1 - h} d(x_0, x_1) \ll c$$

for all $m > n > N_1$. Therefore $\{x_n\}$is a Cauchy sequence in X. Since (X, d) is a complete cone metric space, so there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. Since T is a surjective mapping, there exists $g \in X$ such that $z = Tg$. We have

$$d(x_n, Tg) = d(Tx_{n+1}, Tg) \geq ad(x_{n+1}, g) + bd(x_{n+1}, x_n) + cd(g, Tg). \quad (3.2)$$

It follows from (3.2) and the Lemma 2.8, we have $(a + c)d(g, Tg) \leq 0$ as $n \to \infty$. Since $a + c \neq 0$, so $d(g, Tg) = 0$, Therefore $Tg = g$. So g is one of the fixed points of the mapping T.

While $a > 1$, suppose T also has a fixed point with u, then $Tu = u$. We have

$$d(g, u) = d(Tg, Tu) \geq ad(g, u) + bd(g, Tg) + cd(u, Tu).$$

Therefore

$$(a - 1)d(g, u) \leq 0.$$

Since $a - 1 > 0$, so $d(g, u) = 0$. Therefore $g = u$. It implies that Thas a unique fixed point while $a > 1$. □

Theorem 3.5. *Let (X, d) be a complete cone metric space and let $f : [0, 1) \to (\frac{1}{6}, 1]$ be a nonincreasing and onto function defined by*

$$f(r) = \begin{cases} 1, & 0 \le r \le \frac{\sqrt{3}-1}{2}, \\ \frac{1-2r}{2r^2}, & \frac{\sqrt{3}-1}{2} \le r < \frac{1}{\sqrt{5}}, \\ \frac{1}{4+2r}, & \frac{1}{\sqrt{5}} \le r < 1. \end{cases}$$

Suppose that $T : X \to X$ is a mapping on X and there exists $r \in [0, 1)$ such that

$$f(r)d(x, Tx) \le d(x, y) \Rightarrow d(Tx, Ty) \le rd(x, y)$$

for all $x, y \in X$. Then T has a unique fixed point.

Proof. Since $f(r) \le 1$, hence $f(r)d(x, Tx) \le d(x, Tx)$. Using the hypothesis we have

$$d(Tx, T^2x) \le rd(x, Tx),$$

so

$$d(T^n x, T^{n+1}x) \le r^n d(x, Tx), \tag{3.3}$$

for all $n \in N, x \in X$. Choose $x_0 \in X$, and define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ and $x_n = T^n x_0$. Let $x = x_0$, by (3.3), we have

$$d(x_n, x_{n+1}) \le r^n d(x_0, x_1).$$

Choose a natural number m, while $m > n$, we have

$$\begin{aligned} d(x_n, x_m) &\le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{m-1}, x_m) \\ &\le (r^n + r^{n+1} + ... + r^{m-1})d(x_0, x_1) \\ &= \frac{r^n(1 - r^{m-n})}{1 - r} d(x_0, x_1) \\ &\le \frac{r^n}{1 - r} d(x_0, x_1). \end{aligned}$$

For any $c \in E$ with $0 \ll c$. Choose $t > 0$ such that $\{c\} + N_t(0) \subseteq P$, where $N_t(0) = \{y : ||y|| < t\}$. Choose a natural number N_1 such that $\frac{r^n}{1-r} d(x_0, x_1) \in N_t(0)$, for all $n > N_1$. Therefore $\frac{r^n}{1-r} d(x_0, x_1) \ll c$, for all $n > N_1$. Thus

$$d(x_n, x_m) \le \frac{r^n}{1 - r} d(x_0, x_1) \ll c,$$

for all $m > n > N_1$. Therefore $\{x_n\}$ is a Cauchy sequence in$X$. Since (X, d) is a complete cone metric space, so there exists $z \in X$ such that $x_n \to z$ as

$n \to \infty$. Hence for all $x \in X - \{z\}$, there exists a positive integerN_2, while $n > N_2$, we have $d(x_n, z) \le \frac{d(x,z)}{4}$. Hence

$$\begin{aligned} f(r)d(x_n, Tx_n) &\le d(x_n, x_{n+1}) \le d(x_n, z) + d(x_{n+1}, z) \\ &\le \frac{d(x,z)}{4} + \frac{d(x,z)}{4} = \frac{d(x,z)}{2} \\ &\le \frac{3}{4} d(x,z) = d(x,z) - \frac{d(x,z)}{4} \\ &\le d(x,z) - d(x_n, z) \\ &\le d(x_n, x). \end{aligned}$$

Therefor $f(r)d(x_n, x_{n+1}) \le d(x_n, x)$. By the hypothesis, we have $d(Tx, Tx_n) \le rd(x_n, x)$. Hence

$$d(Tx, z) \le rd(z, x) \tag{3.4}$$

as $n \to \infty$, for all $x \ne z$.

Suppose $T^k z \ne z$, for all positive integer k. By (3.4), we have

$$d(T^{k+1}z, z) \le rd(z, T^k z) \le r^2 d(z, T^{k-1}z) \le ... \le r^k d(z, Tz).$$

(1) If $0 \le r \le \frac{\sqrt{3}-1}{2}$, then $2r^2 + 2r - 1 \le 0$ and $3r^2 < 1$. Suppose $d(T^2z, z) < d(T^2z, T^3z)$, we have

$$\begin{aligned} d(Tz, z) &\le d(Tz, T^2z) + d(T^2z, z) < rd(z, Tz) + d(T^2z, T^3z) \\ &\le rd(z, Tz) + r^2 d(z, Tz) = (r + r^2)d(z, Tz) \\ &\le (2r + 2r^2)d(z, Tz) \le d(z, Tz), \end{aligned}$$

which is a contradiction, since $d(Tz, z) = d(z, Tz)$. Hence the hypothesis that $d(T^2z, z) < d(T^2z, T^3z)$ is wrong, so we have

$$f(r)d(T^2z, T^3z) \le d(T^2z, T^3z) \le d(T^2z, z).$$

We have

$$\begin{aligned} d(Tz, z) &\le d(Tz, T^3z) + d(T^3z, z) \le 2d(Tz, T^3z) + d(T^3z, z) \\ &\le 2rd(T^2z, Tz) + r^2 d(Tz, z) \le 2r^2 d(Tz, z) + r^2 d(Tz, z) \\ &= 3r^2 d(Tz, z) < d(Tz, z), \end{aligned}$$

which is a contradiction, since $d(Tz, z) = d(z, Tz)$.

(2) If $\frac{\sqrt{3}-1}{2} \le r < \frac{1}{\sqrt{5}}$, then $3r^2 < 1$. Suppose $d(T^2z, z) < f(r)d(T^2z, T^3z)$, we have

$$\begin{aligned} d(Tz, z) &\le d(Tz, T^2z) + d(T^2z, z) < d(Tz, T^2z) + f(r)d(T^2z, T^3z) \\ &\le rd(z, Tz) + r^2 f(r)d(z, Tz) \le (2r + 2r^2 f(r))d(z, Tz) = d(z, Tz), \end{aligned}$$

which is a contradiction, since $d(Tz, z) = d(z, Tz)$. Hence the hypothesis that $d(T^2z, z) < f(r)d(T^2z, T^3z)$ is wrong, so we have

$$d(T^2z, z) \geq f(r)d(T^2z, T^3z).$$

We have

$$\begin{aligned} d(Tz, z) &\leq d(Tz, T^3z) + d(T^3z, z) \leq 2d(Tz, T^3z) + d(T^3z, z) \\ &\leq 2rd(T^2z, Tz) + r^2d(Tz, z) \leq 2r^2d(Tz, z) + r^2d(Tz, z) \\ &= 3r^2d(Tz, z) < d(Tz, z), \end{aligned}$$

which is a contradiction, since $d(Tz, z) = d(z, Tz)$.

(3) If $\frac{1}{\sqrt{5}} \leq r < 1$, then $f(r)d(x, Tx) \leq d(x, y)$ or $f(r)d(Tx, T^2x) \leq d(Tx, y)$ for all $x, y \in X$. Since if $f(r)d(x, Tx) > d(x, y)$ and $f(r)d(Tx, T^2x) > d(Tx, y)$, then we have

$$\begin{aligned} d(Tx, x) &\leq d(Tx, y) + d(y, x) \leq d(Tx, y) + 2d(y, x) \\ &< f(r)d(Tx, T^2x) + 2f(r)d(x, Tx) \leq rf(r)d(x, Tx) + 2f(r)d(x, Tx) \\ &\leq f(r)(2r + 4)d(x, Tx) = d(Tx, x), \end{aligned}$$

which is a contradiction, since $d(Tx, x) = d(x, Tx)$. Hence for all positive integer n, we have

$$f(r)d(x_{2n}, x_{2n+1}) \leq d(x_{2n}, z)$$

or

$$f(r)d(x_{2n+1}, x_{2n+2}) \leq d(x_{2n+1}, z).$$

Hence we have

$$d(x_{2n+1}, Tz) \leq rd(x_{2n}, z)$$

or

$$d(x_{2n+2}, Tz) \leq rd(x_{2n+1}, z).$$

Since $x_n \to z$ as $n \to \infty$, therefore there exists a subsequence of $\{x_{n_i}\}$ such that $x_{n_i} \to Tz$ as $n_i \to \infty$, where i is a positive integer. Hence $Tz = z$, which is a contradiction with the hypothesis that $T^kz \neq z$ for any positive integer k. By (3.1), (3.2) and (3.3), we obtain that the previous hypothesis that $T^kz \neq z$ for any positive integer k is wrong. Hence $T^jz = z$, for some positive integer j. Hence z is the fixed point of T^j.

Next we show that the fixed point of T^j is unique. Suppose that $w \neq z$ is also a fixed point of T^j, then $T^jw = w$. By (3.4), we have

$$d(T^jw, T^jz) \leq rd(T^{j-1}w, T^jz) \leq \cdots \leq r^jd(w, T^jz) = r^jd(T^jw, T^jz).$$

Hence $(1 - r^j)d(T^jw, T^jz) \leq 0$. Since $r \in [0, 1)$, so $1 - r^j > 0$. Hence $d(T^jw, T^jz) = 0$. Thus we have $T^jw = T^jz$. Hence $w = z$, T^j has a unique fixed point. Since $T^jz = z$, so $T^j(Tz) = (Tz)$. Hence Tz is also a fixed point of T^j. Since the fixed point of T^j is unique, therefore $Tz = z$, z is a fixed point of T.

Suppose that Talso has a fixed point v, then $Tv = v$. By (3.4), we have

$$d(z, v) = d(z, Tv) \leq rd(z, v).$$

Hence $(1 - r)d(z, v) \leq 0$. Since $0 < 1 - r < 1$, therefore $d(z, v) = 0$. Hence $z = v$, T has a unique fixed point. □

Theorem 3.6. *Let (X, G) be a complete symmetric $G-$cone metric space and $T : X \to X$ be a surjective mapping satisfying the following conditions*

$$G(T^p x, T^p x, T^q y) \geq hG(x, x, y)$$

for all $x, y \in X$ and $h > 1$, where p and q are positive integers. Then T has a unique fixed point.

Proof. Choose any $x_0 \in X$. Since T is a surjective mapping, there exists $x_1 \in X$ such that $Tx_1 = x_0$. Similarly, we obtain a sequence of $\{x_n\}$ such that $x_n = Tx_{n+1}$ in X. Let $y_0 = x_0$, $y_1 = x_q$, $y_2 = x_{p+q}$, $y_{2n-1} = x_{(n-1)(p+q)+q}$, $y_{2n} = x_{n(p+q)}$, then we have

$$\begin{aligned} G(y_{2n-1}, y_{2n-1}, y_{2n}) &= G(T^p y_{2n}, T^p y_{2n}, T^q y_{2n+1}) \\ &\geq hG(y_{2n}, y_{2n}, y_{2n+1}). \end{aligned}$$

Hence

$$G(y_{2n}, y_{2n}, y_{2n+1}) \leq \frac{1}{h} G(y_{2n-1}, y_{2n-1}, y_{2n}). \tag{3.5}$$

Similarly, we have

$$\begin{aligned} &G(y_{2n-2}, y_{2n-2}, y_{2n-1}) \\ &= G(T^q y_{2n-1}, T^q y_{2n-1}, T^p y_{2n}) = G(T^p y_{2n}, T^p y_{2n}, T^q y_{2n-1}) \\ &\geq hG(y_{2n}, y_{2n}, y_{2n-1}) = hG(y_{2n-1}, y_{2n-1}, y_{2n}). \end{aligned}$$

Hence

$$G(y_{2n-1}, y_{2n-1}, y_{2n}) \leq \frac{1}{h} G(y_{2n-2}, y_{2n-2}, y_{2n-1}). \tag{3.6}$$

It follows from (3.5) and (3.6) that

$$G(y_n, y_n, y_{n+1}) \leq \frac{1}{h} G(y_{n-1}, y_{n-1}, y_n) \leq \cdots \leq \frac{1}{h^n} G(y_0, y_0, y_1).$$

Let $k = \frac{1}{h}$. Since $h > 1$, therefore $0 < k < 1$. Choose a natural number $m > n$, we have

$$\begin{aligned} &G(y_n, y_n, y_m) \\ &\leq G(y_n, y_n, y_{n+1}) + G(y_{n+1}, y_{n+1}, y_m) \\ &\leq G(y_n, y_n, y_{n+1}) + G(y_{n+1}, y_{n+1}, y_{n+2}) + \ldots + G(y_{m-1}, y_{m-1}, y_m) \\ &\leq (k^n + k^{n+1} + \ldots + k^{m-1})G(y_0, y_0, y_1) \\ &= \frac{k^n(1-k^{m-n})}{1-k} G(y_0, y_0, y_1) \\ &\leq \frac{k^n}{1-k} G(y_0, y_0, y_1). \end{aligned}$$

For any $c \in E$ with $0 \ll c$. Choose $t > 0$ such that $\{c\} + N_t(0) \subseteq P$, where $N_t(0) = \{y : ||y|| < t\}$. Choose a positive integer N_1 such that $\frac{k^n}{1-k} G(y_0, y_0, y_1) \in N_t(0)$, for all $n > N_1$. Then, $\frac{k^n}{1-k} G(y_0, y_0, y_1) \ll c$, for all $n > N_1$. Thus

$$G(y_n, y_n, y_m) \leq \frac{k^n}{1-k} G(y_0, y_0, y_1) \ll c$$

for all $m > n > N_1$. Therefore $\{y_n\}$ is a Cauchy sequence in X. Since (X, G) is a complete cone metric space, so there exists $z \in X$ such that $y_n \to z$ as $n \to \infty$. Since T is a surjective mapping, so T^p is also a surjective mapping. Hence there exists $u \in X$ such that $T^p u = z$. We have

$$G(z, z, y_{2n}) = G(T^p u, T^p u, T^q y_{2n+1}) \geq hG(u, u, y_{2n+1}).$$

Hence $hG(u, u, z) \leq 0$ as $n \to \infty$. Since $h > 1$, so $G(u, u, z) = 0$. Hence $u = z$. Since $T^p u = z$, so $T^p u = z = u$. Therefore z is a fixed point of T^p. Similarly, z is also a fixed point of T^q. Hence z is the common fixed point of T^p and T^q.

Now we show that the common fixed point of T^p and T^q is unique. We assume that $e \in X$ is also a common fixed point of T^p and T^q, then

$$G(z, z, e) = G(T^p z, T^p z, T^q e) \geq hG(z, z, e).$$

Hence $(h-1)G(z, z, e) \leq 0$. Since $h - 1 > 0$, so $G(z, z, e) = 0$. Hence $z = e$. Since $T^p z = z$, so $T^p(Tz) = Tz$. Hence Tz is also a fixed point of T^p. Similarly, Tz is also a fixed point of T^q. Hence Tz is a common fixed point of T^p and T^q. Since the common fixed point of T^p and T^q is unique, so $Tz = z$. Hence z is a fixed point of T and z is also the unique fixed point of T. □

Theorem 3.7. *Let (X, G) be a complete $G-$cone metric space and $T : X \to X$ be a surjective mapping satisfying the following conditions*

$$G(Tx, Ty, Ty) \geq aG(x, y, y) + bG(x, x, Tx) + cG(y, y, Ty),$$

for all $x, y \in X$ and $x \neq y$, where $a, b, c \geq 0$ and $a+b+c > 1$. Then T has at least one fixed point. Especially, while $a > 1$, T has a unique fixed point.

Proof. Choose any $x_0 \in X$. Since T is a surjective mapping, so there exists $x_1 \in X$ such that $Tx_1 = x_0$. Similarly we can obtain a sequence of $\{x_n\}$ in X such that $x_n = Tx_{n+1}$ and$x_n \neq x_{n+1}$ (if there exists a positive integer i satisfying $x_i = x_{i+1}$, then $x_{i+1} = x_i = Tx_{i+1}$, hence x_{i+1} is a fixed point of T). We have

$$\begin{aligned} & G(x_{n-1}, x_n, x_n) \\ & = G(Tx_n, Tx_{n+1}, Tx_{n+1}) \\ & \geq aG(x_n, x_{n+1}, x_{n+1}) + bG(x_n, x_n, x_{n-1}) + cG(x_{n+1}, x_{n+1}, x_n). \end{aligned} \tag{3.7}$$

Hence

$$(1-b)G(x_{n-1}, x_n, x_n) \geq (a+c)G(x_n, x_{n+1}, x_{n+1}).$$

Suppose that $a + c = 0$, since $a+b+c > 1$, so $1-b < 0$, we have $(1-b)G(x_{n-1}, x_n, x_n) \leq 0$. But $(a+c)G(x_n, x_{n+1}, x_{n+1}) \geq 0$. Hence $a+c \neq 0, b < 1$. It follows from (3.7) that we have

$$G(x_n, x_{n+1}, x_{n+1}) \leq \frac{1-b}{a+c} G(x_{n-1}, x_n, x_n).$$

Let $h = \frac{1-b}{a+c}$, then $h = \frac{1-b}{a+c} < \frac{(a+b+c)-b}{a+c} = 1$. Hence

$$G(x_n, x_{n+1}, x_{n+1}) \leq hG(x_{n-1}, x_n, x_n).$$

Thus we have

$$G(x_n, x_{n+1}, x_{n+1}) \leq h^n G(x_0, x_1, x_1).$$

For any positive integer $m > n$, we have

$$\begin{aligned} & G(x_n, x_m, x_m) \\ & \leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_m, x_m) \\ & \leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + G(x_{m-1}, x_m, x_m) \\ & \leq (h^n + h^{n+1} + \cdots + h^{m-1})G(x_0, x_1, x_1) \\ & = \frac{h^n(1-h^{m-n})}{1-h} G(x_0, x_1, x_1) \\ & \leq \frac{h^n}{1-h} G(x_0, x_1, x_1). \end{aligned}$$

For any $c \in E$ with $0 \ll c$. Choose $t > 0$ such that $\{c\} + N_t(0) \subseteq P$, where $N_t(0) = \{y : ||y|| < t\}$. Choose a positive integer N_1 such that $\frac{h^n}{1-h} G(x_0, x_1, x_1) \in N_t(0)$, for all $n > N_1$. Then, $\frac{h^n}{1-h} G(x_0, x_1, x_1) \ll c$, for all $n > N_1$. Thus

$$G(x_n, x_m, x_m) \leq \frac{h^n}{1-h} G(x_0, x_1, x_1) \ll c$$

for all $m > n > N_1$. Therefore $\{x_n\}$ is a Cauchy sequence in X. Since (X, G) is a complete cone metric space, so there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. Since T is a surjective mapping, so there exists $g \in X$ such that $z = Tg$. We have

$$\begin{aligned} G(Tg, x_n, x_n) &= G(Tg, Tx_{n+1}, Tx_{n+1}) \\ &\geq aG(g, x_{n+1}, x_{n+1}) + bG(g, g, Tg) + cG(x_{n+1}, x_{n+1}, x_n), \end{aligned}$$

by Lemma 2.10, we have

$$\begin{aligned} 0 &\geq aG(g, Tg, Tg) + bG(g, g, Tg) \\ &\geq aG(g, Tg, Tg) + \frac{b}{2}G(g, Tg, Tg) \\ &= (a + \frac{b}{2})G(g, Tg, Tg) \end{aligned}$$

as $n \to \infty$. Since $a + \frac{b}{2} > 0$, so $G(g, Tg, Tg) = 0$. Hence $Tg = g$, g is a fixed point of T.

While $a > 1$, suppose that u is another fixed point of T, then $Tu = u$. We have

$$\begin{aligned} G(g, u, u) &= G(Tg, Tu, Tu) \\ &\geq aG(g, u, u) + bG(g, Tg, Tg) + cG(u, Tu, Tu). \end{aligned}$$

Hence $(a - 1)G(g, u, u) \leq 0$. Since $a - 1 > 0$, hence $G(g, u, u) = 0$. So $g = u$. Hence we have that T has a unique fixed point while $a > 1$. □

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