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THE NEW FASTEST HYBRID ITERATION PROCESS

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Abstract. For a class of contractive operators defined on an arbitrary Banach space, it has been shown recently that the Picard-Mann iteration process converges faster than the Picard iteration process. In this paper, we introduce a new hybrid iteration process which can be seen as a " double Picard " or a hybrid of Picard iteration process with itself. We first give the convergence analysis of our new iteration process. Thereafter, we prove that the new process converges faster than Picard-Mann iteration process for contractive-like operators in $K-$normed linear space.

1. Introduction

Approximation of fixed point of nonlinear operators is one of the most important problems in numerical analysis. Much attention has been given to developing several iteration processes to approximate fixed point of some classess of contractive operators, see [3, 4, 6, 7] and the references therein. We introduce a new hybrid iteration process which we will call a Picard-Picard hybrid iteration process and prove that the new process converges faster than the recently announced Picard-Mann iteration process.

We need the following definition of $K-$normed space [10].

Definition 1.1. Let X be a real linear space. Then, X is said to be $K-$normed if operator $].[: X \to X$ satisfies:

1. $]x[\geq 0,\ (x \in X)$,
2. $]x[= 0 \implies x = 0$,
3. $]\mu x[= |\mu|]x[,\ (x \in X, \mu \in R)$,

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4. $]x+y[\leq]x[+]y[,\ (x,y\in X)$.

Definition 1.2. A sequence $\{y_n\}(n\geq 0)$ in X is said to be

(a) convergent to a limit $y\in X$ if

$$\lim_{n\to\infty}]y_n-y[=0,\ in\ X,$$

and we write

$$(X)-\lim_{n\to\infty}y_n=y;$$

(b) a Cauchy sequence if

$$\lim_{m,n\to\infty}]y_m-y_n[=0.$$

The space X is complete if every Cauchy sequence is convergent.

Now, let X be a $K-$normed linear space and C be a non-empty subset of X. Let $T:C\to C$ be a mapping. The iteration process called Picard is defined as follows:

Picard iteration process(see, [1-4, 9]):
Step 1. For initial guess x_1, a tolerance $\epsilon>0$, for iterations n, set $k=1$.
Step 2. Calculate $x_2,x_3,...$, such that

$$x_{n+1}=Tx_n,\quad n\geq 1,\tag{1.1}$$

Step 3. For given $\epsilon>0$, if $|x_{k+1}-x_k|<\epsilon$, or $k>n$, then stop.
Step 4. Set $k=k+1$ and go to Step 2.

The following definitions about the rate of convergence are due to Berinde [1], see also [2].

Definition 1.3. Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ be two sequences of real numbers that converge to a and b, respectively, assume that there exists

$$l=\lim_{n\to\infty}\frac{|a_n-a|}{|b_n-b|}.$$

(i) If $l=0$, then we say that $\{a_n\}_{n=1}^{\infty}$ converges faster to a than $\{b_n\}_{n=1}^{\infty}$ to b.

(ii) If $0<l<\infty$, then we say that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ have the same rate of convergence.

Suppose that for two fixed point iteration processes $\{u_n\}$ and $\{v_n\}$ both converging to the same fixed point ρ, the following error estimates

$$]u_n-\rho[\leq a_n,\ n\geq 1,$$

$$]v_n - \rho[\leq b_n,\ n \geq 1$$

are available where $\{a_n\}$ and $\{b_n\}$ are two sequences of positive numbers converging to zero. If $\{a_n\}$ converges faster than $\{b_n\}$, then $\{u_n\}$ converges faster than $\{v_n\}$ to ρ.

Olatinwo and Imoru [8] employed the following more general contractive definition: there exist $\delta \in [0,1)$ and a motone increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that for each $x, y \in E$,

$$\|Tx - Ty\| \leq \delta\|x - y\| + \Phi(\|Tx - x\|). \tag{1.2}$$

Several researchers have studied the existence and convergence theorems for finding the fixed point of the class of operator defined in (1.2), see for instance [6-8].

Consider a $K-$normed linear space X, equation (1.2) becomes

$$]Tx - Ty[\leq \delta]x - y[+\Phi(]Tx - x[), \quad for\ each\ x, y \in X. \tag{1.3}$$

In 2013, Khan [4] introduced the following Picard-Mann hybrid iteration process for single map T.

Picard-Mann hybrid iteration process (due to Khan [4]):
Step 1. For initial guess x_1, a tolerance $\epsilon > 0$, for iterations n, set $k = 1$.
Step 2. Calculate $x_2, x_3, ...,$ such that

$$\begin{aligned} y_n &= (1 - b_n)x_n + b_n T x_n, \\ x_{n+1} &= T y_n, \qquad n \geq 1, \end{aligned} \tag{1.4}$$

where $\{b_n\}_{n=1}^{\infty}$ is sequence in $(0, 1)$.
Step 3. For given $\epsilon > 0$, if $|x_{k+1} - x_k| < \epsilon$, or $k > n$, then stop.
Step 4. Set $k = k + 1$ and go to Step 2.

He showed that Picard-Mann hybrid iteration process converges faster than the Picard iteration process in the sense of Berinde [1] for contraction. A natural question that arises is: Can it be possible to develop an iteration process whose rate of convergence is even faster than the Picard-Mann hybrid iteration process (1.4)?

As an answer, we introduced the following iteration process.

New iteration (Picard-Picard hybrid) process:
Step 1. For initial guess x_1, a tolerance $\epsilon > 0$, for iterations n, set $k = 1$.
Step 2. Calculate $x_2, x_3, ...,$ such that

$$y_n = Tx_n,$$

$$x_{n+1} = Ty_n, \quad n \geq 1. \tag{1.5}$$

Step 3. For given $\epsilon > 0$, if $|x_{k+1} - x_k| < \epsilon$, or $k > n$, then stop.
Step 4. Set $k = k + 1$ and go to Step 2.

Our new iteration can be seen as a " double Picard " or a hybrid of Picard iteration process with itself.

In this paper, we prove strong convergence theorem for contractive-like operator using (1.3). We also prove that our new process (1.5) converges faster than (1.4) and (1.1) in the sense of Berinde [1].

2. Convergence Analysis

We will first give the convergence analysis of our new iteration process.

Theorem 2.1. *Let $(X,].[)$ be a $K-$normed linear space, $T : X \to X$ be a contractive-like operator satisfying (1.3). Suppose T has a fixed point ρ. For arbitrary $x_1 \in X$, define the sequence $\{x_n\}_{n=1}^{\infty}$ iteratively by (1.5). Then, the iteration process (1.5) converges to fixed point ρ.*

Proof. Let ρ be a fixed point of T. Then, using (1.5), we have

$$]x_{n+1} - \rho[\quad = \quad]Ty_n - \rho[. \tag{2.1}$$

Now using (1.3) with $x = \rho$, $y = y_n$, we obtain the following two inequality,

$$\begin{aligned}]Ty_n - \rho[&\leq \delta]y_n - \rho[+ \Phi(]T\rho - \rho[) \\ &= \delta]y_n - \rho[. \end{aligned} \tag{2.2}$$

In a similar way, by using (1.3), we can have

$$\|y_n - \rho\| \leq \delta]x_n - \rho[. \tag{2.3}$$

Substituting (2.2) and (2.3) into (2.1) to have

$$\begin{aligned}]x_{n+1} - \rho[&\leq \delta^2]x_n - \rho[\\ &\vdots \\ &\leq \delta^{2n}]x_1 - \rho[. \end{aligned} \tag{2.4}$$

Using the facts that $0 \leq \delta < 1$. It is clear that $]x_{n+1} - \rho[= 0$. Consequently, $x_n \to \rho$. This completes the proof. □

3. Convergence Analysis

In this section, we prove that our new iteration process converges faster than the two (Picard and Picard-Mann) iterations.

Theorem 3.1. *Let* $(X,].[)$ *be a* $K-$*normed linear space,* T *be a contractive-like operator satisfying* (1.3) *with a fixed point* ρ*. Let* $\{u_n\}_{n=1}^{\infty}$ *be defined by the iteration process*

$$\begin{aligned} u_1 &= x \in X, \\ u_{n+1} &= Tu_n, \quad n \geq 1. \end{aligned}$$

$\{x_n\}_{n=1}^{\infty}$ *be defined by the iteration process*

$$\begin{aligned} x_1 &= x \in X, \\ y_n &= Tx_n, \\ x_{n+1} &= Ty_n, \quad n \geq 1. \end{aligned}$$

$\{v_n\}_{n=1}^{\infty}$ *be defined by the iteration process*

$$\begin{aligned} v_1 &= x \in X, \\ w_n &= (1-\alpha_n)v_n + \alpha_n Tv_n, \\ v_{n+1} &= Tw_n, \quad n \geq 1, \end{aligned}$$

where $\{\alpha_n\}$ *is in* $[\lambda, 1-\lambda]$ *for all* $n \geq 1$ *and for some* λ *in* $(0,1)$*. Then, the sequence* $\{x_n\}$ *converges faster than* $\{v_n\}$*. That is, our new iteration* $\{x_n\}$ *converges faster than Picard-Mann iteration* $\{v_n\}$*.*

Proof. As proved in Theorem 2.1 of Khan [4], $]v_{n+1}-\rho[\leq [\delta(1-(1-\delta)\lambda)]^n]v_1-\rho[$ for all $n \geq 1$. Let

$$a_n = [\delta(1-(1-\delta)\lambda)]^n]v_1 - \rho[.$$

Also, from our Theorem 2.1,

$$]x_{n+1} - \rho[\leq \delta^{2n}]x_1 - \rho[.$$

Let

$$b_n = \delta^{2n}]x_1 - \rho[.$$

Then,

$$\frac{b_n}{a_n} = \frac{\delta^{2n}]x_1 - \rho[}{[\delta(1-(1-\delta)\lambda)]^n]v_1 - \rho[}$$

$$\to 0, \quad n \to \infty.$$

Consequently, $\{x_n\}$ converges faster than $\{v_n\}$.
In a similar way, from Theorem 2.1 of Khan [4], $]u_{n+1} - \rho[\leq \delta^n]u_1 - \rho[$ for all $n \geq 1$. Let

$$c_n = \delta^n]u_1 - \rho[.$$

Then,

$$\frac{b_n}{c_n} = \frac{\delta^{2n}]x_1 - \rho[}{\delta^n]u_1 - \rho[}$$

$$\to 0, \quad n \to \infty.$$

Thus, $\{x_n\}$ converges faster than $\{u_n\}$. □

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