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INEQUALITIES CONCERNING TO THE POLAR DERIVATIVE OF A POLYNOMIAL

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Abstract. If $P(z) := \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n, having all its zeros in $|z| \leq k$, $k \geq 1$, then it was proved by Aziz and Rather [*J. Math. Ineq. Appl.*, 1 (1998), 231-238] that for every real or complex number α with $|\alpha| \geq k$,

$$\max_{|z|=1} |D_\alpha P(z)| \geq n\Big(\frac{|\alpha| - k}{k^n + 1}\Big) \max_{|z|=1} |P(z)|.$$

In this paper, we sharpen above result for the polynomials $P(z)$ of degree $n \geq 3$.

1. Introduction

Let $P(z) := \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n and $P'(z)$ its derivative, then

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|. \tag{1.1}$$

Inequality (1.1) is a famous result due to Bernstein and is best possible with equality holding for the polynomial $P(z) = \lambda z^n$, where λ is a complex number.

If we restricted ourselves to a class of polynomial having no zeros in $|z| < 1$, then the above inequality can be sharpened. In fact, Erdös conjectured and later Lax [7] proved that if $P(z) \neq 0$ in $|z| < 1$, then

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$$2\max_{|z|=1}|P'(z)| \leq n\max_{|z|=1}|P(z)|. \tag{1.2}$$

On the other hand, it was proved by Turán [10] that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then

$$2\max_{|z|=1}|P'(z)| \geq n\max_{|z|=1}|P(z)|. \tag{1.3}$$

The inequalities (1.2) and (1.3) are also best possible and become equality for polynomials which have all zeros on $|z| = 1$.

For the class of polynomials having all the zeros in $|z| \leq k$, Malik [8](See also Govil [6]) proved that if $P(z)$ is a polynomial of degree n having all zeros lie in $|z| \leq k$, then

$$(1+k)\max_{|z|=1}|P'(z)| \geq n\max_{|z|=1}|P(z)|, \quad if\ k \leq 1, \tag{1.4}$$

where as Govil [6] showed that

$$(1+k^n)\max_{|z|=1}|P'(z)| \geq n\max_{|z|=1}|P(z)|, \quad if\ k \geq 1. \tag{1.5}$$

Both the inequalities are best possible, with equality in (1.4) holding for $P(z) = (z+k)^n$ and in (1.5) the equality holds for the polynomial $P(z) = (z^n + k^n)$.

Let $D_\alpha P(z)$ denote the polar derivative of the polynomial $P(z)$ of degree n with respect to α, then

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z).$$

The polynomial $D_\alpha P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty}\frac{D_\alpha P(z)}{\alpha} = P'(z).$$

Aziz and Rather [2] extended (1.5) to the polar derivative of a polynomial and proved the following:

Theorem 1.1. *If the polynomial $P(z) := \sum_{j=0}^{n} a_j z^j$ has all its zeros in $|z| \leq k$, $k \geq 1$, then for every real or complex number α with $|\alpha| \geq k$,*

$$\max_{|z|=1}|D_\alpha P(z)| \geq n\left(\frac{|\alpha| - k}{k^n + 1}\right)\max_{|z|=1}|P(z)|. \tag{1.6}$$

2. Lemmas

We need the following lemmas:

The first lemma is due to Bhat [3].

Lemma 2.1. *Let* $P(z) := \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n*. Then for* $R > 1$,

$$\max_{|z|=R} |P(z)| \le R^n \max_{|z|=1} |P(z)| - |a_1| \Big[\Big\{ \frac{(c_n+1)(R^n-1)}{n} \Big\} - \Big\{ \frac{R^{n-2}-1}{n-2} \Big\} \Big], \ n > 2 \tag{2.1}$$

and

$$\max_{|z|=R} |P(z)| \le R^2 \max_{|z|=1} |P(z)| - |a_1| \Big\{ \frac{(R-1)(R+1-\sqrt{2})}{\sqrt{2}} \Big\}, \ n = 2, \tag{2.2}$$

where $c_2 = \sqrt{2} - 1$, $c_3 = \frac{1}{\sqrt{2}}$ *and for* $n \ge 4$, c_n *is the unique positive root of the equation*

$$16n - 8(3n+2)x^2 - 16x^3 + (n+4)x^4 = 0$$

lying in $(0, 1)$.

Frapper [5] showed that the coefficient c_n defined in the above Lemma is given by

$$c_n := \tfrac{2n}{n-4} \sqrt{\tfrac{2(n+2)}{n}} - 1 \quad \text{for } n \ge 4.$$

Lemma 2.2. *If* $P(z) := \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *having all its zeros in* $|z| \le 1$*, then for every* α *with* $|\alpha| \ge 1$*,*

$$\max_{|z|=1} |D_\alpha P(z)| \ge \frac{n}{2} \Big\{ (|\alpha| - 1) \max_{|z|=1} |P(z)| + (|\alpha| + 1) \min_{|z|=1} |P(z)| \Big\}.$$

This lemma is due to Aziz and Dawood [1].

Lemma 2.3. *If* $P(z) := \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *having no zeros in* $|z| \le 1$*, then for* $R \ge 1$*,*

$$
\begin{aligned}
&\max_{|z|=R}|P(z)| \\
&\leq \Big(\frac{R^n+1}{2}\Big)\max_{|z|=1}|P(z)| - \Big(\frac{R^n-1}{2}\Big)\min_{|z|=1}|P(z)| \\
&\quad - \frac{2|P'(0)|}{(n+1)}\Big[\frac{(R^n-1)}{n} - (R-1)\Big] \\
&\quad - 2|P''(0)|\Big[\Big(\frac{(R^n-1)-n(R-1)}{n(n-1)}\Big) - \Big(\frac{(R^{n-2}-1)-(n-2)(R-1)}{(n-2)(n-3)}\Big)\Big]
\end{aligned} \tag{2.3}
$$

for $n > 3$ *and*

$$
\begin{aligned}
\max_{|z|=R}|P(z)| \leq \Big(\frac{R^n+1}{2}\Big)\max_{|z|=1}|P(z)| - \Big(\frac{R^n-1}{2}\Big)\min_{|z|=1}|P(z)| \\
- \frac{2|P'(0)|}{(n+1)}\Big[\frac{(R^n-1)}{n} - (R-1)\Big] - \frac{2|P''(0)|}{n(n-1)}(R-1)^n
\end{aligned} \tag{2.4}
$$

for $n = 3$.

The above result is a special case of a result due to Dewan, Singh and Mir [4] with k=1.

Remark 2.4. Here we note that for the proof of this result an additional hypothesis that $P(0) \neq 0$ is required. A simple counter example in this case is $P(z) = z^n$.

3. Main Result

In this paper, we prove the following result which is a refinement as well as generalization of Theorem 1.1.

Theorem 3.1. *Let* $P(z) := \sum_{j=0}^{n} a_j z^j$, $a_n a_0 \neq 0$ *be a polynomial of degree* $n \geq 3$, *having all its zeros in* $|z| \leq k$, $k \geq 1$. *Then for every real or complex number* α *with* $|\alpha| \geq k$,

$$
\begin{aligned}
&\max_{|z|=1}|D_\alpha P(z)| \\
&\geq n\Big(\frac{|\alpha|-k}{1+k^n}\Big)\Big\{\max_{|z|=1}|P(z)| + \frac{(k^n-1)}{2k^n}\min_{|z|=k}|P(z)| \\
&\quad + 2\frac{|a_{n-1}|}{(n+1)}\Big[\frac{(k^n-1)}{n} - (k-1)\Big] \\
&\quad + 2|a_{n-2}|\Big[\Big(\frac{(k^n-1)-n(k-1)}{n(n-1)}\Big) - \Big(\frac{(k^{n-2}-1)-(n-2)(k-1)}{(n-2)(n-3)}\Big)\Big]\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&+ n\frac{(|\alpha|+k)}{2k^n}\min_{|z|=k}|P(z)| \\
&+ \frac{1}{k^{n-1}}\Big[\Big(\frac{(c_{n-1}+1)(k^{n-1}-1)}{n-1}\Big)-\Big(\frac{k^{n-3}-1}{n-3}\Big)\Big]|(n-1)a_1+2\alpha a_2|
\end{aligned} \tag{3.1}
$$

for $n>3$ *and*

$$
\begin{aligned}
\max_{|z|=1}|D_\alpha P(z)| \geq n\Big(\frac{|\alpha|-k}{1+k^n}\Big)\Big\{&k^{n-3}\max_{|z|=1}|P(z)|+\frac{(k^n-1)}{2k^3}\min_{|z|=k}|P(z)| \\
&+2k^{n-3}\frac{|a_{n-1}|}{(n+1)}\Big[\frac{(k^n-1)}{n}-(k-1)\Big] \\
&+2k^{n-3}\frac{|a_{n-2}|}{n(n-1)}(k-1)^n\Big\}+n\frac{(|\alpha|+k)}{2k^3}\min_{|z|=k}|P(z)| \\
&+\frac{1}{k^2}|(n-1)a_1+2\alpha a_2|\Big[\frac{(k-1)(k+1-\sqrt{2})}{\sqrt{2}}\Big]
\end{aligned} \tag{3.2}
$$

for $n=3$*, where* $c_2=\sqrt{2}-1$*,* $c_3=\frac{1}{\sqrt{2}}$ *and* $c_n:=\frac{2n}{n-4}\sqrt{\frac{2(n+2)}{n}}-1$*, for* $n\geq 4$*.*

Proof. Since the polynomial $P(z)$ has all its zeros in $|z|\leq k$, $k\geq 1$. If $Q(z)=z^nP(\frac{1}{z})$ be the reciprocal polynomial of $P(z)$. Then the polynomial $Q(\frac{z}{k})$ has all its zeros in $|z|\geq 1$. Hence by applying (2.3) of Lemma 2.3 to the polynomial $Q(\frac{z}{k})$, $k\geq 1$, we get

$$
\begin{aligned}
&\max_{|z|=k}|Q(\frac{z}{k})| \\
&\leq \frac{(k^n+1)}{2}\max_{|z|=1}|Q(\frac{z}{k})|-\frac{(k^n-1)}{2}\min_{|z|=1}|Q(\frac{z}{k})|-\frac{2|a_{n-1}|}{(n+1)}\Big[\frac{(k^n-1)}{n}-(k-1)\Big] \\
&\quad -2|a_{n-2}|\Big[\Big\{\frac{(k^n-1)-n(k-1)}{n(n-1)}\Big\}-\Big\{\frac{(k^{n-2}-1)-(n-2)(k-1)}{(n-2)(n-3)}\Big\}\Big].
\end{aligned}
$$

This in particular gives

$$
\begin{aligned}
&\max_{|z|=1}|P(z)| \\
&\leq \frac{(k^n+1)}{2k^n}\max_{|z|=k}|P(z)|-\frac{(k^n-1)}{2k^n}\min_{|z|=k}|P(z)|-\frac{2|a_{n-1}|}{(n+1)}\Big[\frac{(k^n-1)}{n}-(k-1)\Big] \\
&\quad -2|a_{n-2}|\Big[\Big\{\frac{(k^n-1)-n(k-1)}{n(n-1)}\Big\}-\Big\{\frac{(k^{n-2}-1)-(n-2)(k-1)}{(n-2)(n-3)}\Big\}\Big].
\end{aligned}
$$

Which is equivalent to

$$
\max_{|z|=k}|P(z)| \geq \frac{2k^n}{(k^n+1)}\max_{|z|=1}|P(z)|+\frac{(k^n-1)}{(k^n+1)}\min_{|z|=k}|P(z)|
$$

$$
\begin{aligned}
&+\frac{4k^n}{(k^n+1)}\frac{|a_{n-1}|}{(n+1)}\Big[\frac{(k^n-1)}{n}-(k-1)\Big]\\
&+\frac{4k^n}{(k^n+1)}|a_{n-2}|\Big[\Big\{\frac{(k^n-1)-n(k-1)}{n(n-1)}\Big\}\\
&-\Big\{\frac{(k^{n-2}-1)-(n-2)(k-1)}{(n-2)(n-3)}\Big\}\Big].
\end{aligned}
\tag{3.3}
$$

By hypothesis that the polynomial $P(z)$ has all its zeros in $|z| \leq k$, $k \geq 1$, therefore all the zeros of the polynomial $T(z) = P(kz)$ lie in $|z| \leq 1$. As $\frac{|\alpha|}{k} \geq 1$ and by applying Lemma 2.2 to the polynomial $T(z)$, we get

$$\max_{|z|=1}|D_{\frac{\alpha}{k}}T(z)| \geq \frac{n}{2}\Big\{\Big(|\frac{\alpha}{k}|-1\Big)\max_{|z|=1}|T(z)|+\Big(|\frac{\alpha}{k}|+1\Big)\min_{|z|=1}|T(z)|\Big\},$$

that is,

$$\max_{|z|=k}|D_\alpha P(z)| \geq \frac{n}{2}\Big\{\Big(\frac{|\alpha|-k}{k}\Big)\max_{|z|=k}|P(z)|+\Big(\frac{|\alpha|+k}{k}\Big)\min_{|z|=k}|P(z)|\Big\}. \tag{3.4}$$

The polynomial $P(z)$ is of degree $n > 3$ and so $D_\alpha P(z)$ is the polynomial of degree $n-1$, where $n-1 > 2$, hence by applying (2.1) of Lemma 2.1 to the polynomial $D_\alpha P(z)$, we obtain for $k \geq 1$,

$$
\begin{aligned}
&\max_{|z|=k}|D_\alpha P(z)|\\
&\leq k^{n-1}\max_{|z|=1}|D_\alpha P(z)|\\
&\quad -|(n-1)a_1+2\alpha a_2|\Big[\Big\{\frac{(c_{n-1}+1)(k^{n-1}-1)}{n-1}\Big\}-\Big\{\frac{k^{n-3}-1}{n-3}\Big\}\Big],
\end{aligned}
\tag{3.5}
$$

for $n > 3$, where c_n is defined as in the Theorem.

Combining (3.4) and (3.5), we get

$$
\begin{aligned}
&k^{n-1}\max_{|z|=1}|D_\alpha P(z)|-|(n-1)a_1+2\alpha a_2|\Big[\Big\{\frac{(c_{n-1}+1)(k^{n-1}-1)}{n-1}\Big\}-\Big\{\frac{k^{n-3}-1}{n-3}\Big\}\Big]\\
&\geq \frac{n}{2}\Big\{\Big(\frac{|\alpha|-k}{k}\Big)\max_{|z|=k}|P(z)|+\Big(\frac{|\alpha|+k}{k}\Big)\min_{|z|=k}|P(z)|\Big\}
\end{aligned}
$$

or,

$$
\begin{aligned}
&\max_{|z|=1}|D_\alpha P(z)|\\
&\geq \frac{n}{2}\Big\{\Big(\frac{|\alpha|-k}{k^n}\Big)\max_{|z|=k}|P(z)|+\Big(\frac{|\alpha|+k}{k^n}\Big)\min_{|z|=k}|P(z)|\Big\}\\
&\quad +\frac{1}{k^{n-1}}|(n-1)a_1+2\alpha a_2|\Big[\Big\{\frac{(c_{n-1}+1)(k^{n-1}-1)}{n-1}\Big\}-\Big\{\frac{k^{n-3}-1}{n-3}\Big\}\Big].
\end{aligned}
\tag{3.6}
$$

Again, combining (3.6) and (3.3), we get the desired result. This completes the proof of inequality (3.1). The proof of the Theorem 3.1 in the case $n = 3$ follows along the same lines as the proof of (3.1) but instead of inequalities (2.1) and (2.3), we use inequalities (2.2) and (2.4) respectively. □

Remark 3.2. For $k = 1$, Theorem 3.1 provides a refinement of a theorem proved by Shah [9].

Remark 3.3. For $k > 1$, and for $y > 1$, $\frac{[(k^y-1)-y(k-1)]}{y(y-1)}$ and $\frac{(k^y-1)}{y}$ are both increasing functions of y and so the expressions

$$\Big[\Big\{\frac{(k^n-1)-n(k-1)}{n(n-1)}\Big\} - \Big\{\frac{(k^{n-2}-1)-(n-2)(k-1)}{(n-2)(n-3)}\Big\}\Big]$$

and

$$\Big[\frac{(k^n-1)}{n} - (k-1)\Big]$$

are always non-negative so that for polynomials of degree $n \geq 3$, Theorem 3.1 is an improvement of Theorem 1.1.

Dividing both sides of (3.1) and (3.2) by $|\alpha|$ and letting $|\alpha| \to \infty$, we get the following:

Corollary 3.4. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$, $a_n a_0 \neq 0$ be a polynomial of degree $n \geq 3$, having all its zeros in $|z| \leq k$, $k \geq 1$, then for every real or complex number α with $|\alpha| \geq k$,*

$$\begin{aligned}
&\max_{|z|=1}|P'(z)| \\
&\geq \frac{n}{1+k^n}\Big\{\max_{|z|=1}|P(z)| + \frac{(k^n-1)}{2k^n}\min_{|z|=k}|P(z)| + 2\frac{|a_{n-1}|}{(n+1)}\Big[\frac{(k^n-1)}{n} - (k-1)\Big] \\
&\quad + 2|a_{n-2}|\Big[\Big(\frac{(k^n-1)-n(k-1)}{n(n-1)}\Big) - \Big(\frac{(k^{n-2}-1)-(n-2)(k-1)}{(n-2)(n-3)}\Big)\Big]\Big\} \\
&\quad + \frac{n}{2k^n}\min_{|z|=k}|P(z)| \\
&\quad + \frac{2|a_2|}{k^{n-1}}\Big[\Big(\frac{(c_{n-1}+1)(k^{n-1}-1)}{n-1}\Big) - \Big(\frac{k^{n-3}-1}{n-3}\Big)\Big]
\end{aligned}$$

for $n > 3$ and

$$\begin{aligned}
\max_{|z|=1}|P'(z)| \geq \frac{n}{1+k^n}\Big\{&k^{n-3}\max_{|z|=1}|P(z)| + \frac{(k^n-1)}{2k^3}\min_{|z|=k}|P(z)| \\
&+ 2k^{n-3}\frac{|a_{n-1}|}{(n+1)}\Big[\frac{(k^n-1)}{n} - (k-1)\Big]
\end{aligned}$$

$$+ 2k^{n-3}\frac{|a_{n-2}|}{n(n-1)}(k-1)^n\Big\} + \frac{n}{2k^3}\min_{|z|=k}|P(z)|$$
$$+ \frac{2|a_2|}{k^2}\Big[\frac{(k-1)(k+1-\sqrt{2})}{\sqrt{2}}\Big] \ for \ n = 3,$$

where $c_2 = \sqrt{2} - 1$, $c_3 = \frac{1}{\sqrt{2}}$ *and* $c_n := \frac{2n}{n-4}\sqrt{\frac{2(n+2)}{n}} - 1$, *for* $n \geq 4$.

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