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THE ALEKSANDROV PROBLEM IN 2-FUZZY N-NORMED LINEAR SPACES

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Abstract. In this paper, we obtain some results for the Aleksandrov problem in 2-fuzzy n-normed linear spaces using the concepts of n-isometry, n-collinearity, n-Lipschitz mapping and 2-fuzzy n-normed linear spaces which was introduced by Park and Alaca [7].

1. Introduction

Let X and Y be metric spaces. A mapping $f : X \to Y$ is called an isometry if f satisfies $d_Y(f(x), f(y)) = d_X(x, y)$ for every $x, y \in X$, Where $d_X(\cdot,\cdot)$ and $d_Y(\cdot,\cdot)$ denote the metrics in the spaces X and Y, respectively. For some fixed number $r > 0$, suppose that f preserves distance r; i.e., for all x, y in X with $d_X(x, y) = r$, we have $d_Y(f(x), f(y)) = r$. Then r is called a conservative(or preserved) distance for the mapping f. The basic problem of conservative distances is whether the existence of a single conservative distance for some f implies that f is an isometry of X into Y. It is called the Aleksandrov problem. Some results about this problem can be seen in [10-14].

In 1984, Katsaras [1] and Wu and Fang [2] introduced a notion of a fuzzy norm. Different authors introduced the definitions of fuzzy norms on a linear space. Cheng and Mordeson [3] and Bag and Samanta [4] introduced a concept of fuzzy norm on a linear space. The concept of fuzzy n-normed linear spaces

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has been studied by many authors(see [5], [6]). In 2012, Park and Alaca [7] introduced the concept of 2-fuzzy n-normed linear space or fuzzy n-normed linear space of the set of all fuzzy sets of a non-empty set.

In this paper, we obtain some results for the Aleksandrov problem in 2-fuzzy n-normed linear spaces using the concepts of n-isometry, n-collinearity, n-Lipschitz mapping and 2-fuzzy n-normed linear spaces which was introduced Park and Alaca [7].

2. Preliminaries

Definition 2.1. ([8]) Let $n \in \mathbb{N}$ and let X be a real vector space of dimension $d \geq n$.(Here we allow d to be infinite). A real-valued function$\|\cdot, \dots, \cdot\|$ on $X \times X \cdots \times X$ satisfies the following properties:

(1) $\|x_1, x_2, \dots, x_n\| = 0$ if and only if x_1, x_2,..., x_n are linearly dependent,
(2) $\|x_1, x_2, \dots, x_n\|$ is invariant under any permutation,
(3) $\|x_1, x_2, \dots, \alpha x_n\| = |\alpha| \cdot \|x_1, x_2, \dots, x_n\|$, for any $\alpha \in \mathbb{R}$,
(4) $\|x_1, x_2, \dots, x_{n-1}, y+z\| \leq \|x_1, x_2, \dots, x_{n-1}, y\| + \|x_1, x_2, \dots, x_{n-1}, z\|$,

is called an n-normed on X and the pair $(X, \|\cdot, \dots, \cdot\|)$ is called an *n-normed linear space.*

Definition 2.2. ([6]) Let X be a linear space over a field $\mathcal{K}$. A fuzzy subset N of $X^n \times \mathbb{R}$($\mathbb{R}$, the set of real numbers) is called a fuzzy n-normed on X if and only if :

(N1) For all $t \leq 0, N(x_1, x_2, \cdots, x_n, t) = 0$,
(N2) For all $t > 0$, $N(x_1, x_2, \cdots, x_n, t) = 1$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent,
(N3) $N(x_1, x_2, \cdots, x_n, t)$ is invariant under any permutation of x_1, x_2, $\cdots$, x_n,
(N4) $N(x_1, x_2, \cdots, cx_n, t) = N(x_1, x_2, \cdots, x_n, \frac{t}{|c|})$, for $c \neq 0$ and $c \in \mathcal{K}$,
(N5) For all $s, t \in \mathbb{R}$,

$$N(x_1, x_2, \cdots, x_n + x_n^{'}, s+t) \\ \geq min\{N(x_1, x_2, \cdots, x_n, s), N(x_1, x_2, \cdots, x_n^{'}, t)\},$$

(N6) $N(x_1, x_2, \cdots, x_n, t)$ is a nondecreasing function of $t \in \mathbb{R}$ and

$$\lim_{t\to\infty} N(x_1, x_2, \cdots, x_n, t) = 1.$$

The pair (X, N) is called a *fuzzy n-normed linear space* or in short f-n-NLS.

Theorem 2.3. ([6]) *Let (X, N) be an f-n-NLS. Assume that*

(N7) $N(x_1, x_2, \cdots, x_n, t) > 0$ *for all* $t > 0$ *implies that* $x_1, x_2, \cdots, x_n$ *are linearly dependent.*

Define

$$\|x_1, x_2, \ldots, x_n\|_\alpha = inf\{t : N(x_1, x_2, \cdots, x_n, t) \geq \alpha, \;\; \alpha \in (0, 1)\}.$$

Then$\{\|\cdot, \cdot, \ldots, \cdot\|_\alpha : \alpha \in (0, 1)\}$ *is an ascending family of n-normed on* X.

We call these n-norms as α-n-norms on X corresponding to the fuzzy n-norm on X.

Definition 2.4. ([9]) Let X be any non-empty set and $\Im(X)$ be the set of all fuzzy sets on X. For $U, V \in \Im(X)$ and $\lambda \in \mathcal{K}$ the field of real numbers, define

$$U + V = \{(x + y, v \wedge \mu) : (x, v) \in U, \;\; (y, \mu) \in V\}$$

and $\lambda U = \{(\lambda x, v) : (x, v) \in U\}$.

Definition 2.5. ([9]) A fuzzy linear space $\widehat{X} = X \times (0, 1]$ over the number field $\mathcal{K}$, where the addition and scalar multiplication operation on X are defined by

$$(x, v) + (y, \mu) = (x + y, v \wedge \mu), \quad \lambda(x, v) = (\lambda x, v)$$

is a fuzzy normed space if to every $(x, v) \in \widehat{X}$ there is associated a non-negative real number, $\|(x, v)\|$, called the fuzzy norm of (x, v), in such a way that

(1) $\|(x, v)\| = 0$ if $x = 0$ the zero element of X, $v \in (0, 1]$,
(2) $\|\lambda(x, v)\| = |\lambda|\|(x, v)\|$ for all $(x, v) \in \widehat{X}$ and $\lambda \in \mathcal{K}$,
(3) $\|(x, v) + (y, \mu)\| \leq \|(x, v \wedge \mu)\| + \|(y, v \wedge \mu)\|$ for all $(x, v), (y, \mu) \in \widehat{X}$,
(4) $\|(x, \vee_t v_t)\| = \wedge_t \|x, v_t\|$ for all $v_t \in (0, 1]$.

3. 2-fuzzy n-normed linear space

In this section, we introduce the concepts of 2-fuzzy n-normed linear space and α-n-norms on the set of all fuzzy sets of a non-empty set.

Definition 3.1. ([7]) Let X be any non-empty set and $\Im(X)$ be the set of all fuzzy sets on X. If $f \in \Im(X)$ then $f = \{(x, \mu) : x \in X \text{ and } \mu \in (0, 1]\}$. Clearly f is a bounded function for $|f(x)| \leq 1$. Let $\mathcal{K}$ be the space of real numbers, then $\Im(X)$ is a linear space over the field $\mathcal{K}$ where the addition and scalar multiplication are defined by

$$f + g = \{(x, \mu) + (y, \eta)\} = \{(x + y, \mu \wedge \eta) : (x, \mu) \in f, \;\; (y, \eta) \in g\}$$

and

$$\lambda f = \{(\lambda x, \mu) : (x, \mu) \in f\}, \quad \lambda \in \mathcal{K}.$$

The linear space $\Im(X)$ is said to be normed linear space if , for every $f \in \Im(X)$, there exists an associated non-negative real number $\|f\|$ (called the norm of f) which satisfies

(1) $\|f\| = 0$ if and only if $f = 0$. For

$$\begin{aligned} \|f\| &= 0 \\ &\Leftrightarrow \{\|(x,\mu)\| : (x,\mu) \in f\} = 0 \\ &\Leftrightarrow x = 0, \ \ \mu \in (0,1] \Leftrightarrow f = 0. \end{aligned}$$

(2) $\|\lambda f\| = |\lambda|\|f\|, \ \ \lambda \in \mathcal{K}$. For

$$\begin{aligned} \|\lambda f\| &= \{\|\lambda(x,\mu)\| : (x,\mu) \in f, \ \ \lambda \in \mathcal{K}\} \\ &= \{|\lambda|\|(x,\mu)\| : (x,\mu) \in f\} = |\lambda|\|f\|. \end{aligned}$$

(3) $\|f+g\| \leq \|f\| + \|g\|$ for every $f, g \in \Im(X)$. For

$$\begin{aligned} \|f+g\| &= \{\|(x,\mu)+(y,\eta)\| : x,y \in X, \ \ \mu,\eta \in (0,1]\} \\ &\leq \{\|(x,\mu\wedge\eta)\| + \|(y,\mu\wedge\eta)\| : (x,\mu) \in f, \ \ (y,\eta) \in g\} \\ &= \|f\| + \|g\|. \end{aligned}$$

Then $(\Im(X), \|\cdot\|)$ is a *normed linear space.*

Definition 3.2. ([7]) A 2-fuzzy set on X is a fuzzy set on $\Im(X)$.

Definition 3.3. ([7]) Let X be a real vector space of dimension $d \geq n (n \in \mathbb{N})$ and $\Im(X)$ be the set of all fuzzy sets in X. Here we allow d to be infnite. Assume that a [0,1]-valued function $\|\cdot, \ldots, \cdot\|$ on $\Im(X) \times \cdots \times \Im(X)$ satisfies the following properties

(1) $\|f_1, f_2, \cdots, f_n\| = 0$ if and only if $f_1, f_2, \cdots, f_n$ are linearly dependent,
(2) $\|f_1, f_2, \cdots, f_n\|$ is invariant under any permutation of $f_1, f_2, \cdots, f_n$,
(3) $\|f_1, f_2, \cdots, \lambda f_n\| = |\lambda|\|f_1, f_2, \cdots, f_n\|$, for any $\lambda \in \mathcal{K}$,
(4) $\|f_1, f_2, \cdots, f_{n-1}, y+z\| \leq \|f_1, f_2, \cdots, f_{n-1}, y\| + \|f_1, f_2, \cdots, f_{n-1}, z\|$.

Then $(\Im(X), \|\cdot, \ldots, \cdot\|)$ is an *n-normed linear space* or $(X, \|\cdot, \cdots, \cdot\|)$ is a *2-n-normed linear space.*

Definition 3.4. ([7]) Let $\Im(X)$ be a linear space over the field $\mathcal{K}$. A fuzzy subset N of $\Im(X) \times \cdots \times \Im(X) \times \mathbb{R}$ is called a 2-fuzzy n-norm on X (or fuzzy n-norm on $\Im(X)$) if and only if

(2-N1) for all $t \in \mathbb{R}$ with $t \leq 0$, $N(f_1, f_2, \cdots, f_n, t) = 0$,
(2-N2) for all $t \in \mathbb{R}$ with $t > 0$, $N(f_1, f_2, \cdots, f_n, t) = 1$, if and only if $f_1, f_2, \cdots, f_n$ are linearly dependent,
(2-N3) $N(f_1, f_2, \cdots, f_n, t)$ is invariant under any permutation of $f_1, f_2, \cdots, f_n$,

(2-N4) for all $t \in \mathbb{R}$ with $t > 0$, $N(f_1, f_2, \cdots, \lambda f_n, t) = N(f_1, f_2, \cdots, f_n, \frac{t}{|\lambda|})$, if $\lambda \neq 0$, $\lambda \in \mathcal{K}$,

(2-N5) for all $s, t \in \mathbb{R}$,

$$N(f_1, f_2, \cdots, f_n + f_n^{'}, s + t) \\ \geq min\{N(f_1, f_2, \cdots, f_n, s), N(f_1, f_2, \cdots, f_n^{'}, t)\},$$

(2-N6) $N(f_1, f_2, \cdots, f_n, \cdot) : (0, \infty) \to [0, 1]$ is continuous,

(2-N7) $\lim_{t\to\infty} N(f_1, f_2, \cdots, f_n, t) = 1$.

Then $(\Im(X), N)$ is a *fuzzy n-normed linear space* or (X, N) is a *2-fuzzy n-normed linear space.*

Remark 3.5. ([7]) In a 2-fuzzy n-normed linear space (X, N), $N(f_1, f_2, \cdots, f_n, \cdot)$ is a nondecreasing function of $\mathbb{R}$ for all $f_1, f_2, \cdots, f_n \in \Im(X)$.

The following example agrees with our notion of 2-fuzzy n-normed linear space.

Example 3.6. $(\Im(X), \|\cdot, \ldots, \cdot\|)$ be an n-normed linear space as in Definition 3.3. Define

$$N(f_1, f_2, \cdots, t) = \begin{cases} \frac{t}{t+\|f_1, f_2, \cdots, f_n\|}, & \text{if } t > 0, \ t \in \mathbb{R}; \\ \\ 0, & \text{if } t \leq 0. \end{cases}$$

for all $(f_1, f_2, \cdots, f_n) \in \Im(X) \times \cdots \times \Im(X)$. Then (X, N) is a 2- fuzzy n-normed linear space.

Theorem 3.7. ([7]) *Let $(\Im(X), N)$ be a fuzzy n-normed linear space. Assume that*

(2-N8) *$N(f_1, f_2, \cdots, f_n, t) > 0$ for all $t > 0$ implies $f_1, f_2, \cdots, f_n$ are linearly dependent.*

Define

$$\|f_1, f_2, \cdots, f_n\|_\alpha = inf\{t : N(f_1, f_2, \cdots, f_n, t) \geq \alpha, \ \alpha \in (0, 1)\}.$$

Then $\{\|\cdot, \cdot, \ldots, \cdot\|_\alpha : \alpha \in (0, 1)\}$ is an ascending family of n-norms on $\Im(X)$.

Proof. The proof of Theorem is clear from [7, Theorem 3.1]. □

Remark 3.8. In Theorem 3.7, these n-norms are called α-n-norms on $\Im(X)$ corresponding to the 2-fuzzy n-norm on X.

4. On the Aleksandrov problem

In this section, we give a new generalization of the Aleksandrov problem when X is a 2-fuzzy n-normed linear space or $\Im(X)$ is a fuzzy n-normed linear space. Hereafter we use the notion of fuzzy n-normed linear space on $\Im(X)$ instead of 2-fuzzy n-normed linear space on X .

Definition 4.1. Let $\Im(X)$ and $\Im(Y)$ are fuzzy n-normed linear spaces and Ψ: $\Im(X) \to \Im(Y)$ is a mapping .We call Ψ an *n-isometry* if

$$\|f_1 - f_0, \cdots, f_n - f_0\|_\alpha = \|\Psi(f_1) - \Psi(f_0), \cdots, \Psi(f_n) - \Psi(f_0)\|_\beta$$

for all $f_0, f_1, f_2, \cdots, f_n \in \Im(X)$ and $\alpha, \beta \in (0, 1)$.

For a mapping Ψ, consider the following condition which is called the *n-distance one preserving property* (nDOPP).
Let $f_0, f_1, f_2, \cdots, f_n \in \Im(X)$ with $\|f_1 - f_0, \cdots, f_n - f_0\|_\alpha = 1$, then

$$\|\Psi(f_1) - \Psi(f_0), \cdots, \Psi(f_n) - \Psi(f_0)\|_\beta = 1. \qquad (nDOPP)$$

Lemma 4.2. *Let* $f_0, f_1, f_2, \cdots, f_n \in \Im(X)$, $\alpha \in (0, 1)$ *and* $\lambda \in \mathbb{R}$. *Then,*

$$\|f_1, \cdots, f_i, \cdots, f_j, \cdots, f_n\|_\alpha = \|f_1, \cdots, f_i, \cdots, f_j + \lambda f_i, \cdots, f_n\|_\alpha,$$

for all $1 \le i \ne j \le n$.

Proof. It is obviously true. □

Definition 4.3. The elements $f_0, f_1, f_2, \cdots, f_n \in \Im(X)$ are said to be *n-collinear* if for every i, $\{f_j - f_i : 0 \le j \ne i \le n\}$ is linearly dependent.

Definition 4.4. The elements f_0, f_1 and f_2 are said to be *2-collinear* if and only if $f_2 - f_0 = r(f_1 - f_0)$ for some real number r.

Definition 4.5. We call Ψ an *n-Lipschitz mapping* if there is a $k \ge 0$ such that

$$\|\Psi(f_1) - \Psi(f_0), \cdots, \Psi(f_n) - \Psi(f_0)\|_\beta \le k\|f_1 - f_0, \cdots, f_n - f_0\|_\alpha$$

for all $f_0, f_1, f_2, \cdots, f_n \in \Im(X)$ and $\alpha, \beta \in (0, 1)$. The smallest such k is called the n-Lipschitz constant.

We only consider in this paper the n-Lipschitz constant $k \le 1$.

Definition 4.6. We call Ψ a *locally n-Lipschitz mapping* if there is a $k \ge 0$ such that

$$\|\Psi(f_1) - \Psi(f_0), \cdots, \Psi(f_n) - \Psi(f_0)\|_\beta \le k\|f_1 - f_0, \cdots, f_n - f_0\|_\alpha,$$

whenever $\|f_1 - f_0, \cdots, f_n - f_0\|_\alpha \leq 1$, for all $f_0, f_1, f_2, \cdots, f_n \in \Im(X)$ and $\alpha, \beta \in (0, 1)$.

Theorem 4.7. (see [7]) *Let Ψ be n-Lipschitz mapping with n-Lipschitz constant $k \leq 1$. Assume that if $f_0, f_1, \cdots, f_m$ are m-collinear then $\Psi(f_0)$, $\Psi(f_1)$, $\cdots$, $\Psi(f_m)$ are m-collinear, $m = 2, n$, and that Ψ satisfies $(nDOPP)$, then Ψ is an n-isometry.*

Theorem 4.8. (see [7]) *Assume that f_0, f_1 and f_2 are 2-collinear then $\Psi(f_0)$, $\Psi(f_1)$ and $\Psi(f_2)$ are 2-collinear, and Ψ satisfies $(nDOPP)$. Then Ψ preserves the n-distance k for each $k \in \mathbb{N}$.*

Lemma 4.9. *If a mapping $\Psi \colon \Im(X) \to \Im(Y)$ is locally n-Lipschitz mapping, then Ψ is a n-Lipschitz mapping.*

Proof. We may assume that $\|f_1 - f_0, \cdots, f_n - f_0\|_\alpha > 1$, then there is $n_0 \in \mathbb{N}$ such that $n_0 - 1 < \|f_1 - f_0, \cdots, f_n - f_0\|_\alpha \leq n_0$. Let $g_i = f_0 + \frac{i}{n_0}(f_1 - f_0)$. Then

$$\begin{aligned}\|g_i - g_{i-1}, f_2 - g_{i-1}, \cdots, f_n - g_{i-1}\|_\alpha &= \|g_i - g_{i-1}, f_2 - f_0, \cdots, f_n - f_0\|_\alpha \\ &= \|\frac{f_1 - f_0}{n_0}, f_2 - f_0, \cdots, f_n - f_0\|_\alpha \\ &= \frac{\|f_1 - f_0, f_2 - f_0, \cdots, f_n - f_0\|_\alpha}{n_0} \\ &\leq 1.\end{aligned}$$

And

$$\begin{aligned}&\|\Psi(g_i) - \Psi(g_{i-1}), \Psi(f_2) - \Psi(f_0), \cdots, \Psi(f_n) - \Psi(f_0)\|_\beta \\ &= \|\Psi(g_i) - \Psi(g_{i-1}), \Psi(f_2) - \Psi(g_{i-1}), \cdots, \Psi(f_n) - \Psi(g_{i-1})\|_\beta \\ &\leq \|g_i - g_{i-1}, f_2 - g_{i-1}, \cdots, f_n - g_{i-1}\|_\alpha \\ &= \frac{\|f_1 - f_0, f_2 - f_0, \cdots, f_n - f_0\|_\alpha}{n_0},\end{aligned}$$

$$\begin{aligned}&\|\Psi(f_1) - \Psi(f_0), \Psi(f_2) - \Psi(f_0), \cdots, \Psi(f_n) - \Psi(f_0)\|_\beta \\ &= \|\sum_{i=1}^{n_0}(\Psi(g_i) - \Psi(g_{i-1})), \Psi(f_2) - \Psi(f_0), \cdots, \Psi(f_n) - \Psi(f_0)\|_\beta \\ &\leq \sum_{i=1}^{n_0} \|\Psi(g_i) - \Psi(g_{i-1}), \Psi(f_2) - \Psi(f_0), \cdots, \Psi(f_n) - \Psi(f_0)\|_\beta\end{aligned}$$

$$= \sum_{i=1}^{n_0} \frac{\|f_1 - f_0, f_2 - f_0, \cdots, f_n - f_0\|_\alpha}{n_0}$$
$$= \|f_1 - f_0, f_2 - f_0, \cdots, f_n - f_0\|_\alpha.$$

□

Remark 4.10. Assume that Ψ is a locally n-Lipschitz mapping and $f_0, f_1, \cdots, f_n$ are n-collinear, then $\Psi(f_0)$, $\Psi(f_1), \cdots, \Psi(f_n)$ are n-collinear.

Indeed $f_0, f_1, \cdots, f_n$ are n-collinear if and only if $\|f_1 - f_0, f_2 - f_0, \cdots, f_n - f_0\|_\alpha = 0$. Since

$$\|\Psi(f_1) - \Psi(f_0), \Psi(f_2) - \Psi(f_0), \cdots, \Psi(f_n) - \Psi(f_0)\|_\beta$$
$$\leq \|f_1 - f_0, f_2 - f_0, \cdots, f_n - f_0\|_\alpha$$

thus

$$\|\Psi(f_1) - \Psi(f_0), \Psi(f_2) - \Psi(f_0), \cdots, \Psi(f_n) - \Psi(f_0)\|_\beta = 0,$$

it followings that $\Psi(f_0), \Psi(f_1), \cdots, \Psi(f_n)$ are n-collinear.

So Theorem 4.7 and Theorem 4.8 [7] can be simplified as Theorem 4.11 and Theorem 4.12:

Theorem 4.11. *Assume that Ψ is a locally n-Lipschitz mapping and satisfies $(nDOPP)$, then Ψ is an n-isometry.*

Theorem 4.12. *Assume that Ψ is a locally n-Lipschitz mapping and satisfies $(nDOPP)$, then Ψ preserves the n-distance k for each $k \in \mathbb{N}$.*

Theorem 4.13. *If $f_0, f_1, \cdots, f_n \in \Im(X)$ and $\|f_1 - f_0, f_2 - f_0, \cdots, f_n - f_0\|_\alpha \neq 0$, there exists a $w \in \Im(X)$ such that*

$$\|f_0 - w, f_1 - w, f_2 - w, \cdots, f_{n-1} - w\|_\alpha$$
$$= \|f_1 - w, f_2 - w, f_3 - w, \cdots, f_n - w\|_\alpha = 1.$$

Proof. By hypothesis, $\gamma = \|f_1 - f_0, f_2 - f_0, \cdots, f_n - f_0\|_\alpha > 0$. Set $w = f_1 + \frac{2}{\gamma}(\frac{f_0+f_n}{2} - f_1)$, we have

$$\begin{aligned}
&\|f_0-w, f_1-w, f_2-w,\cdots, f_{n-1}-w\|_\alpha \\
&=\|f_0-f_1-\frac{2}{\gamma}(\frac{f_0+f_n}{2}-f_1), -\frac{2}{\gamma}(\frac{f_0+f_n}{2}-f_1),\cdots, \\
&\qquad f_{n-1}-f_1-\frac{2}{\gamma}(\frac{f_0+f_n}{2}-f_1)\|_\alpha \\
&=\|f_0-f_1, -\frac{2}{\gamma}(\frac{f_0+f_n}{2}-f_1), f_2-f_1,\cdots, f_{n-1}-f_1\|_\alpha \\
&=\frac{1}{\gamma}\|f_0-f_1, f_0-f_1+f_n-f_1, f_2-f_1,\cdots, f_{n-1}-f_1\|_\alpha \\
&=\frac{1}{\gamma}\|f_0-f_1, f_n-f_1, f_2-f_1,\cdots, f_{n-1}-f_1\|_\alpha \\
&=\frac{1}{\gamma}\|f_1-f_0, f_2-f_0,\cdots, f_n-f_0\|_\alpha=1,
\end{aligned}$$

$$\begin{aligned}
&\|f_1-w, f_2-w, f_3-w,\cdots, f_n-w\|_\alpha \\
&=\|-\frac{2}{\gamma}(\frac{f_0+f_n}{2}-f_1), f_2-f_1-\frac{2}{\gamma}(\frac{f_0+f_n}{2}-f_1),\cdots, \\
&\qquad f_n-f_1-\frac{2}{\gamma}(\frac{f_0+f_n}{2}-f_1)\|_\alpha \\
&=\frac{1}{\gamma}\|f_0-f_1+f_n-f_1, f_2-f_1, f_3-f_1,\cdots, f_n-f_1\|_\alpha \\
&=\frac{1}{\gamma}\|f_0-f_1, f_2-f_1, f_3-f_1,\cdots, f_n-f_1\|_\alpha \\
&=\frac{1}{\gamma}\|f_1-f_0, f_2-f_0,\cdots, f_n-f_0\|_\alpha=1.
\end{aligned}$$

This completes the proof. □

Lemma 4.14. *Assume that if f_0, f_1 and f_2 are 2-collinear then $\Psi(f_0), \Psi(f_1)$ and $\Psi(f_2)$ are 2-collinear, and that Ψ satisfies $(nDOPP)$. Then Ψ preserves the n-distance $\frac{1}{k}$ for any positive integer k.*

Proof. Suppose that there exist $f_0, f_1 \in \Im(X)$ with $f_0 \neq f_1$ such that $\Psi(f_0) = \Psi(f_1)$. Since $dim\Im(X) \geq n$, there are $f_2, \cdots, f_n \in \Im(X)$ such that $f_1 - f_0$, $f_2 - f_0$, $\cdots, f_n - f_0$ are linearly dependent. Since $\|f_1 - f_0,\ f_2 - f_0,\ \cdots,\ f_n - f_0\|_\alpha \neq 0$, we can set

$$g_2 = f_0 + \frac{f_2 - f_0}{\|f_1 - f_0, f_2 - f_0, \cdots, f_n - f_0\|_\alpha}.$$

Then we have

$$\begin{aligned}&\|f_1-f_0,g_2-f_0,f_3-f_0\cdots,f_n-f_0\|_\alpha\\&=\|f_1-f_0,\frac{f_2-f_0}{\|f_1-f_0,f_2-f_0,\cdots,f_n-f_0\|_\alpha},f_3-f_0\cdots,f_n-f_0\|_\alpha\\&=1.\end{aligned}$$

Since Ψ preserves the unit n-distance ,

$$\|\Psi(f_1)-\Psi(f_0),\Psi(g_2)-\Psi(f_0),\Psi(f_3)-\Psi(f_0)\cdots,\Psi(f_n)-\Psi(f_0)\|_\beta=1$$

But it follows from $\Psi(f_0)=\Psi(f_1)$ that

$$\|\Psi(f_1)-\Psi(f_0),\Psi(g_2)-\Psi(f_0),\Psi(f_3)-\Psi(f_0)\cdots,\Psi(f_n)-\Psi(f_0)\|_\beta=0$$

which is a contradiction. Hence, Ψ is a injective.

By Theorem 4.8 , Ψ preserves the n-distance k for each positive integer k. We claim that Ψ preserves the n-distance k for each positive integer $\frac{1}{k}$. Let $f_0,f_1,\cdots,f_n\in\Im(X)$ satisfies

$$\|f_1-f_0,f_2-f_0,\cdots,f_n-f_0\|_\alpha=\frac{1}{k}.$$

By Theorem 4.13, there is an element w of $\Im(X)$ such that

$$\begin{aligned}&\|f_0-w,f_1-w,f_2-w,\cdots,f_{n-1}-w\|_\alpha\\&=\|f_0-w,f_1-w,f_2-w,\cdots,f_{n-1}-w\|_\alpha=1.\end{aligned}$$

Let $p_{ij}=w+j(f_i-w)$ for each positive integer j and each $i=0,1,\cdots,n$. First, we show that

$$\Psi(p_{ij})=\Psi(w)+j(\Psi(f_i)-\Psi(w))$$

for each positive integer j and each $i=0,1,\cdots,n$. We prove it by the induction on j. When $j=1$, it is clear. Assume that the above statement holds for all positive integers less than $j+1$. Let $i=0$. Since

$$\begin{aligned}&\|p_{0j+1}-p_{0j},f_1-w,\cdots,f_{n-1}-w\|_\alpha\\&=\|f_0-w,f_1-w,\cdots,f_{n-1}-w\|_\alpha=1,\end{aligned}$$

we have

$$\begin{aligned}&\|\Psi(p_{0j+1})-\Psi(p_{0j}),\Psi(f_1)-\Psi(w),\cdots,\Psi(f_{n-1})-\Psi(w)\|_\beta\\&=\|\Psi(f_0)-\Psi(w),\Psi(f_1)-\Psi(w),\cdots,\Psi(f_{n-1})-\Psi(w)\|_\beta\\&=1.\end{aligned}$$

By the inductive hypothesis, $\Psi(p_{0j})=\Psi(w)+j(\Psi(f_0)-\Psi(w))$. Since w,f_0, p_{0j+1} are 2-collinear, $\Psi(w),\Psi(f_0),\Psi(p_{0j+1})$ are 2-collinear. Let

$$\Psi(p_{0j+1})=\Psi(w)+\alpha(\Psi(f_0)-\Psi(w)).$$

Then we have

$$\Psi(p_{0j+1})-\Psi(p_{0j})=(\alpha-j)(\Psi(f_0)-\Psi(w))$$

and

$$\begin{aligned}1 &= \|\Psi(p_{0j+1}) - \Psi(p_{0j}), \Psi(f_1) - \Psi(w), \cdots, \Psi(f_{n-1}) - \Psi(w)\|_\beta \\ &= \|(\alpha - j)(\Psi(f_0) - \Psi(w)), \Psi(f_1) - \Psi(w), \cdots, \Psi(f_{n-1}) - \Psi(w)\|_\beta \\ &= |\alpha - j| \|\Psi(f_0) - \Psi(w), \Psi(f_1) - \Psi(w), \cdots, \Psi(f_{n-1}) - \Psi(w)\|_\beta \\ &= |\alpha - j|.\end{aligned}$$

Assume that $\alpha - j = -1$, that is $\alpha = j - 1$. Then

$$\Psi(p_{0j+1}) = \Psi(w) + (j-1)(\Psi(f_0) - \Psi(w)) = \Psi(p_{0j-1}).$$

Since Ψ is a injective, which is a contradition. Thus we have $\alpha = j + 1$. Hence

$$\Psi(p_{0j+1}) = \Psi(w) + (j+1)(\Psi(f_0) - \Psi(w)).$$

By induction

$$\Psi(p_{0j}) = \Psi(w) + j(\Psi(f_0) - \Psi(w))$$

for all positive integers j. Similarly,

$$\Psi(p_{ij}) = \Psi(w) + j(\Psi(f_i) - \Psi(w))$$

for all positive integers j and each $i = 0, 1, \cdots, n$. Thus we obtain that

$$\begin{aligned}&\|p_{1k} - p_{0k}, p_{2k} - p_{0k}, \cdots, p_{nk} - p_{0k}\|_\alpha \\ &= \|w + k(f_1 - w) - (w + k(f_0 - w)), w + k(f_2 - w) - (w + k(f_0 - w)), \\ &\qquad \cdots, w + k(f_n - w) - (w + k(f_0 - w))\|_\alpha \\ &= \|k(f_1 - f_0), k(f_2 - f_0), \cdots, k(f_n - f_0)\|_\alpha \\ &= k^n \|f_1 - f_0, f_2 - f_0, \cdots, f_n - f_0\|_\alpha \\ &= k^n \cdot \frac{1}{k} = k^{n-1}.\end{aligned}$$

By Theorem 4.8,

$$\begin{aligned}k^{n-1} &= \|\Psi(p_{1k}) - \Psi(p_{0k}), \Psi(p_{2k}) - \Psi(p_{0k}), \cdots, \Psi(p_{nk}) - \Psi(p_{0k})\|_\beta \\ &= \|\Psi(w) + k(\Psi(f_1) - \Psi(w)) - (\Psi(w) + k(\Psi(f_0) - \Psi(w))), \\ &\qquad \cdots, \Psi(w) + k(\Psi(f_n) - \Psi(w)) - (\Psi(w) + k(\Psi(f_0) - \Psi(w)))\|_\beta \\ &= \|k(\Psi(f_1) - \Psi(f_0)), k(\Psi(f_2) - \Psi(f_0)), \cdots, k(\Psi(f_n) - \Psi(f_0))\|_\beta \\ &= k^n \|\Psi(f_1) - \Psi(f_0), \Psi(f_2) - \Psi(f_0), \cdots, \Psi(f_n) - \Psi(f_0)\|_\beta.\end{aligned}$$

Therefore,

$$\|\Psi(f_1) - \Psi(f_0), \Psi(f_2) - \Psi(f_0), \cdots, \Psi(f_n) - \Psi(f_0)\|_\beta = \frac{1}{k}$$

which completes the proof. □

Theorem 4.15. *Assume that if $f_0, f_1, \cdots, f_m$ are m-collinear then $\Psi(f_0)$, $\Psi(f_1)$, $\cdots$, $\Psi(f_m)$ are m-collinear, $m = 2, n$, and that if $g_1 - g_2 = \lambda(g_3 - g_2)$ for some $\lambda \in (0, 1]$ then $\Psi(g_1) - \Psi(g_2) = \eta(\Psi(g_3) - \Psi(g_2))$ for some $\eta \in (0, 1]$. If Ψ satisfies (nDOPP), then Ψ is an n-isometry.*

Proof. For $f_0, f_1, \cdots, f_n \in \Im(X)$, there are two cases depending upon whether $\|f_1 - f_0, f_2 - f_0, \cdots, f_n - f_0\|_\alpha = 0$ or not.

In the case $\|f_1 - f_0, f_2 - f_0, \cdots, f_n - f_0\|_\alpha = 0$, $f_1 - f_0, f_2 - f_0, \cdots, f_n - f_0$ are linearly dependent, that is, n-collinear. Thus $\Psi(f_0), \Psi(f_1), \cdots, \Psi(f_n)$ are n-collinear. Thus $\Psi(f_1) - \Psi(f_0), \Psi(f_2) - \Psi(f_0), \cdots, \Psi(f_n) - \Psi(f_0)$ are linearly dependent, Hence

$$\|\Psi(f_1) - \Psi(f_0), \Psi(f_2) - \Psi(f_0), \cdots, \Psi(f_n) - \Psi(f_0)\|_\beta = 0.$$

In the case $\|f_1 - f_0, f_2 - f_0, \cdots, f_n - f_0\|_\alpha > 0$, let

$$\frac{s-1}{r} < \|f_1 - f_0, f_2 - f_0, \cdots, f_n - f_0\|_\alpha \leq \frac{s}{r},$$

where s and r are positive integers with $s \geq 2$. By Theorem 4.7, it suffices to show that Ψ is an n-Lipschitz mapping with n-Lipschitz constant 1. Let

$$p_j = f_0 + \frac{j}{r} \cdot \frac{1}{\|f_1 - f_0, f_2 - f_0, \cdots, f_n - f_0\|_\alpha}(f_1 - f_0)$$

for each $j = 0, 1, \cdots, s$. Then

$$\begin{aligned}
&\|p_j - p_{j-1}, f_2 - p_{j-1}, \cdots, f_n - p_{j-1}\|_\alpha \\
&= \|p_j - p_{j-1}, f_2 - f_0, \cdots, f_n - f_0\|_\alpha \\
&= \|\frac{1}{r} \cdot \frac{1}{\|f_1 - f_0, f_2 - f_0, \cdots, f_n - f_0\|_\alpha}(f_1 - f_0), f_2 - f_0, \cdots, f_n - f_0\|_\alpha \\
&= \frac{1}{r} \cdot \frac{1}{\|f_1 - f_0, f_2 - f_0, \cdots, f_n - f_0\|_\alpha}\|f_1 - f_0, f_2 - f_0, \cdots, f_n - f_0\|_\alpha \\
&= \frac{1}{r}
\end{aligned}$$

for all $j = 1, 2, \cdots, s$. By Lemma 4.14, we have

$$\begin{aligned}
&\|\Psi(p_j) - \Psi(p_{j-1}), \Psi(f_2) - \Psi(f_0), \cdots, \Psi(f_n) - \Psi(f_0)\|_\beta \\
&= \|\Psi(p_j) - \Psi(p_{j-1}), \Psi(f_2) - \Psi(p_{j-1}), \cdots, \Psi(f_n) - \Psi(p_{j-1})\|_\beta \\
&= \frac{1}{r}
\end{aligned}$$

for all $j = 1, 2, \cdots, s$. Since $f_1 = p_{s-1} + \lambda(p_s - p_{s-1})$ for some $\lambda \in (0, 1]$. We obtain that

$$\Psi(f_1) = \Psi(p_{s-1}) + \eta(\Psi(p_s) - \Psi(p_{s-1}))$$

for some $\eta \in (0,1]$ by the hypothesis. Thus we have

$$\begin{aligned}
&\|\Psi(f_1) - \Psi(p_{s-1}), \Psi(f_2) - \Psi(f_0), \cdots, \Psi(f_n) - \Psi(f_0)\|_\beta \\
&= \|\eta(\Psi(p_s) - \Psi(p_{s-1})), \Psi(f_2) - \Psi(f_0), \cdots, \Psi(f_n) - \Psi(f_0)\|_\beta \\
&= \eta\|\Psi(p_s) - \Psi(p_{s-1}), \Psi(f_2) - \Psi(f_0), \cdots, \Psi(f_n) - \Psi(f_0)\|_\beta \\
&\leq \|\Psi(p_s) - \Psi(p_{s-1}), \Psi(f_2) - \Psi(f_0), \cdots, \Psi(f_n) - \Psi(f_0)\|_\beta.
\end{aligned}$$

Hence

$$\begin{aligned}
&\|\Psi(f_1) - \Psi(f_0), \Psi(f_2) - \Psi(f_0), \cdots, \Psi(f_n) - \Psi(f_0)\|_\beta \\
&\leq \|\Psi(p_1) - \Psi(p_0), \Psi(f_2) - \Psi(f_0), \cdots, \Psi(f_n) - \Psi(f_0)\|_\beta \\
&\quad + \|\Psi(p_2) - \Psi(p_1), \Psi(f_2) - \Psi(f_0), \cdots, \Psi(f_n) - \Psi(f_0)\|_\beta \\
&\quad + \cdots \\
&\quad + \|\Psi(p_{s-1}) - \Psi(p_{s-2}), \Psi(f_2) - \Psi(f_0), \cdots, \Psi(f_n) - \Psi(f_0)\|_\beta \\
&\quad + \|\Psi(f_1) - \Psi(p_{s-1}), \Psi(f_2) - \Psi(f_0), \cdots, \Psi(f_n) - \Psi(f_0)\|_\beta \\
&\leq \sum_{j=1}^{s} \|\Psi(p_j) - \Psi(p_{j-1}), \Psi(f_2) - \Psi(f_0), \cdots, \Psi(f_n) - \Psi(f_0)\|_\beta \\
&= \frac{s}{r}.
\end{aligned}$$

Therefore,

$$\|\Psi(f_1) - \Psi(f_0), \cdots, \Psi(f_n) - \Psi(f_0)\|_\beta \leq \|f_1 - f_0, f_2 - f_0, \cdots, f_n - f_0\|_\alpha$$

for all $f_0, f_1, \cdots, f_n \in \Im(X)$. This completes the proof. □

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