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COMMON FIXED POINT THEOREM FOR R-WEAKLY COMMUTING MAPS IN $b-$FUZZY METRIC SPACES

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Abstract. In this paper, we consider complete $b-$fuzzy metric space and prove common fixed point theorem for R-weakly commuting maps in this spaces. Our results generalize the recent result many other known results.

1. Introduction

The concept of fuzzy sets was introduced initially by Zadeh [15] in 1965. Since then, to use this concept in topology and analysis, many authors have expansively developed the theory of fuzzy sets and application. George and Veeramani [5], Kramosil and Michalek [7] have introduced the concept of fuzzy topological spaces induced by fuzzy metric which have very important applications in quantum particle physics, particularly in connections with both string and E-infinity theory which were given and studied by El Naschie [1, 2, 3, 4]. Many authors [6, 8, 10, 12, 13] have proved fixed point theorem in fuzzy (probabilistic) metric spaces.

Definition 1.1. A binary operation $* : [0,1] \times [0,1] \longrightarrow [0,1]$ is a continuous t-norm if it satisfies the following conditions:

(1) $*$ is associative and commutative,
(2) $*$ is continuous,

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(3) $a * 1 = a$ for all $a \in [0,1]$,
(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0,1]$.

Two typical examples of a continuous t-norm are $a * b = ab$ and $a * b = \min(a,b)$.

Definition 1.2. A 3-tuple $(X, M, *)$ is called a *fuzzy metric space* if X is an arbitrary (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0,\infty)$, satisfying the following conditions for each $x, y, z \in X$ and $t, s > 0$,

(1) $M(x, y, t) > 0$,
(2) $M(x, y, t) = 1$ if and only if $x = y$,
(3) $M(x, y, t) = M(y, x, t)$,
(4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(5) $M(x, y, .) : (0, \infty) \longrightarrow [0, 1]$ is continuous.

Definition 1.3. A 3-tuple $(X, M, *)$ is called a $b-$fuzzy metric space if X is an arbitrary (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0,\infty)$, satisfying the following conditions for each $x, y, z \in X$, $t, s > 0$ and $b \geq 1$ be a given real number,

(1) $M(x, y, t) > 0$,
(2) $M(x, y, t) = 1$ if and only if $x = y$,
(3) $M(x, y, t) = M(y, x, t)$,
(4) $M(x, y, \frac{t}{b}) * M(y, z, \frac{s}{b}) \leq M(x, z, t + s)$,
(5) $M(x, y, .) : (0, \infty) \longrightarrow [0, 1]$ is continuous.

It should be noted that, the class of $b-$fuzzy metric spaces is effectively larger than that of fuzzy metric spaces, since a $b-$fuzzy metric is a fuzzy metric when $b = 1$.

We present an example shows that a $b-$fuzzy metric on X need not be a fuzzy metric on X.

Example 1.4. Let $M(x, y, t) = e^{\frac{-|x-y|^p}{t}}$, where $p > 1$ is a real number. We show that M is a $b-$fuzzy metric with $b = 2^{p-1}$.

Obviously conditions (1), (2), (3) and (5) of Definition 1.3 are satisfied.

If $1 < p < \infty$, then the convexity of the function $f(x) = x^p$ $(x > 0)$ implies

$$\left(\frac{a+c}{2}\right)^p \leq \frac{1}{2}\left(a^p + c^p\right),$$

and hence, $(a + c)^p \leq 2^{p-1}(a^p + c^p)$ holds. Therefore,

$$\begin{aligned} \frac{|x-y|^p}{t+s} &\leq 2^{p-1}\frac{|x-z|^p}{t+s}+2^{p-1}\frac{|z-y|^p}{t+s} \\ &\leq 2^{p-1}\frac{|x-z|^p}{t}+2^{p-1}\frac{|z-y|^p}{s} \\ &= \frac{|x-z|^p}{t/2^{p-1}}+\frac{|z-y|^p}{s/2^{p-1}}. \end{aligned}$$

Thus, for each $x, y, z \in X$, we obtain

$$\begin{aligned} M(x,y,t+s) &= e^{\frac{-|x-y|^p}{t+s}} \\ &\geq M(x,z,\frac{t}{2^{p-1}}) * M(z,y,\frac{s}{2^{p-1}}), \end{aligned}$$

where $a * b = a.b$. So condition (4) of Definition 1.3 is hold and M is a $b-$fuzzy metric.

It should be noted that in preceding example, for $p = 2$ it is easy to see that $(X, M, *)$ is not a fuzzy metric space.

Example 1.5. Let $M(x,y,t) = e^{\frac{-d(x,y)}{t}}$ or $M(x,y,t) = \frac{t}{t+d(x,y)}$, where d is a b-metric on X and $a * b = a.b$. Then it is easy to show that M is a $b-$fuzzy metric.

Obviously conditions (1), (2), (3) and (5) of Definition 1.3 are satisfied. For each $x, y, z \in X$ we obtain

$$\begin{aligned} M(x,y,t+s) &= e^{\frac{-d(x,y)}{t+s}} \\ &\geq e^{-b\frac{d(x,z)+d(z,y)}{t+s}} \\ &= e^{-b\frac{d(x,z)}{t+s}}.e^{-b\frac{d(z,y)}{t+s}} \\ &\geq e^{\frac{-d(x,z)}{t/b}}.e^{\frac{-d(z,y)}{s/b}} \\ &= M(x,z,\frac{t}{b}) * M(z,y,\frac{s}{b}). \end{aligned}$$

So condition (4) of Definition 1.3 is hold and M is a $b-$fuzzy metric.

Before stating and proving our results, we present some definition and proposition in $b-$metric space.

Definition 1.6. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function. Then f is called $b-$nondecreasing, if $x > by$ this implies $f(x) \geq f(y)$ for each $x, y \in \mathbb{R}$.

Lemma 1.7. ([11]) *Let $(X, M, *)$ be a $b-$fuzzy metric space. Then $M(x, y, t)$ is $b-$nondecreasing with respect to t, for all x, y in X. Also,*

$$M(x,y,b^n t) \geq M(x,y,t), \quad \forall n \in \mathbb{N}.$$

Let $(X, M, *)$ be a $b-$fuzzy metric space. For $t > 0$, the open ball $B(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

We recall the notions of convergence and completeness in a $b-$fuzzy metric space. Let $(X, M, *)$ be a $b-$fuzzy metric space. Let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exists $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subset A$. Then τ is a topology on X (induced by the $b-$fuzzy metric M). A sequence $\{x_n\}$ in X converges to x if and only if $M(x_n, x, t) \to 1$ as $n \to \infty$, for each $t > 0$. It is called a Cauchy sequence if for each $0 < \varepsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for each $n, m \geq n_0$. The $b-$fuzzy metric space $(X, M, *)$ is said to be complete if every Cauchy sequence is convergent. A subset A of X is said to be F-bounded if there exists $t > 0$ and $0 < r < 1$ such that $M(x, y, t) > 1 - r$ for all $x, y \in A$.

Lemma 1.8. ([11]) *In a $b-$fuzzy metric space $(X, M, *)$ the following assertions hold:*

(i) *If sequence $\{x_n\}$ in X converges to x, then x is unique.*
(ii) *If sequence $\{x_n\}$ in X is converges to x, then sequence $\{x_n\}$ is a Cauchy sequence.*

In $b-$fuzzy metric space we have the following propositions.

Proposition 1.9. ([11] Prop. 1.8) *Let $(X, M, *)$ be a $b-$fuzzy metric space and suppose that $\{x_n\}$ and $\{y_n\}$ are b-convergent to x, y respectively then we have*

$$M(x, y, \frac{t}{b^2}) \leq \limsup_{n \longrightarrow \infty} M(x_n, y_n, t) \leq M(x, y, b^2 t),$$

$$M(x, y, \frac{t}{b^2}) \leq \liminf_{n \longrightarrow \infty} M(x_n, y_n, t) \leq M(x, y, b^2 t).$$

Proposition 1.10. *Let $(X, M, *)$ be a $b-$fuzzy metric space and suppose that $\{x_n\}$ is b-convergent to x then we have*

$$M(x, y, \frac{t}{b}) \leq \limsup_{n \longrightarrow \infty} M(x_n, y, t) \leq M(x, y, bt),$$

$$M(x, y, \frac{t}{b}) \leq \liminf_{n \longrightarrow \infty} M(x_n, y, t) \leq M(x, y, bt).$$

Proof. By condition (4) of Definition 1.3 we have:

$$M(x, y, t) \geq M(x, x_n, \frac{\delta}{b}) * M(x_n, y, \frac{t - \delta}{b}),$$

taking the upper limit as $n \to \infty$ we get

$$\begin{aligned} M(x,y,t) &\geq \limsup_{n\longrightarrow\infty} M(x,x_n,\frac{\delta}{b}) * \limsup_{n\longrightarrow\infty} M(x_n,y,\frac{t-\delta}{b}) \\ &= \limsup_{n\longrightarrow\infty} M(x_n,y,\frac{t-\delta}{b}), \end{aligned}$$

as $\delta \longrightarrow 0$ we have

$$M(x,y,bt) \geq \limsup_{n\longrightarrow\infty} M(x_n,y,t).$$

Also, by condition (4) of Definition 1.3 we have:

$$M(x_n,y,t) \geq M(x_n,x,\frac{\delta}{b}) * M(x,y,\frac{t-\delta}{b}),$$

taking the upper limit as $n \to \infty$ we get

$$\limsup_{n\longrightarrow\infty} M(x_n,y,t) \geq M(x,y,\frac{t-\delta}{b}),$$

as $\delta \longrightarrow 0$ we have

$$\limsup_{n\longrightarrow\infty} M(x_n,y,t) \geq M(x,y,\frac{t}{b}).$$

It follows that

$$M(x,y,\frac{t}{b}) \leq \limsup_{n\longrightarrow\infty} M(x_n,y,t) \leq M(x,y,bt).$$

Similarly, we can show that

$$M(x,y,\frac{t}{b}) \leq \liminf_{n\longrightarrow\infty} M(x_n,y,t) \leq M(x,y,bt).$$

□

Remark 1.11. In general, a $b-$fuzzy metric is not continuous.

2. The main results

Definition 2.1. ([9], Definition 1.2) Let (X,d) be a metric space and $F : X \longrightarrow X$ be a map. F is called sequentially convergent if $\{y_n\}$ is convergent provided $\{Fy_n\}$ is convergent.

We start our work by proving the following crucial theorem.

Definition 2.2. Let f and g be maps from a $b-$fuzzy metric space $(X,M,*)$ into itself. The maps f and g are said to be weakly commuting if

$$M(fgx,gfx,t) \geq M(fx,gx,t)$$

for each $x \in X$ and $t > 0$.

Definition 2.3. Let f and g be maps from a $b-$fuzzy metric space $(X, M, *)$ into itself. The maps f and g are said to be R-weakly commuting if there exists some positive real number R such that

$$M(fgx, gfx, t) \geq M(fx, gx, t/R)$$

for each $x \in X$ and $t > 0$.

Weak commutativity implies R-weak commutativity in $b-$fuzzy metric space. However, R-weak commutativity implies weak commutativity only when $R \leq 1$.

Example 2.4. Let $X = \mathbb{R}$. Let $a * b = ab$ for all $a, b \in [0, 1]$ and let M be the $b-$fuzzy set on $X \times X \times]0, +\infty[$ defined as follows:

$$M(x, y, t) = e^{\frac{-(x-y)^2}{t}},$$

for all $t \in \mathbb{R}^+$. Then $(X, M, *)$ is a $b-$fuzzy metric space. Define $f(x) = 2x - 1$ and $g(x) = x^2$. Then

$$\begin{aligned} M(fgx, gfx, t) &= e^{\frac{-4(x-1)^4}{t}} \\ &= e^{\frac{-(x-1)^4}{t/4}} = M(fx, gx, t/4) \\ &< e^{\frac{-(x-1)^4}{t}} = M(fx, gx, t). \end{aligned}$$

Therefore, for $R = 4$, f and g are R-weakly commuting. But f and g are not weakly commuting since exponential function is strictly increasing.

Theorem 2.5. *Let F, f and g be maps from a complete $b-$fuzzy metric space $(X, M, *)$ into itself. Let f and g be R-weakly commuting self-mappings of X satisfying the following conditions:*

(a) $f(X) \subseteq g(X)$*;*
(b) f *or* g *is continuous;*
(c) $M(Ffx, Ffy, t) \geq \gamma(M(Fgx, Fgy, b^4 t))$*, where* $\gamma : [0, 1] \to [0, 1]$ *is a continuous function such that* $\gamma(a) > a$ *for each* $a \in]0, 1[$.

Also, if F is one to one, continuous and sequentially convergent. Then we have

(i) f *and* g *have a unique common fixed point* $a \in X$.
(ii) *If* $Ff = fF$ *and* $Fg = gF$*, then* F*,* f *and* g *have a unique common fixed point.*

Proof. Let x_0 be an arbitrary point in X. By (a), choose a point x_1 in X such that $fx_0 = gx_1$. In general choose x_{n+1} such that $fx_n = gx_{n+1}$ and

$y_n = Ffx_n = Fgx_{n+1}$. Then, for $t > 0$,

$$\begin{aligned} M(y_n, y_{n+1}, t) &= M(Ffx_n, Ffx_{n+1}, t) \\ &\geq \gamma(M(Fgx_n, Fgx_{n+1}, b^4 t)) = \gamma(M(Ffx_{n-1}, Ffx_n, b^4 t)) \\ &\geq M(Ffx_{n-1}, Ffx_n, b^4 t) \\ &\geq M(Ffx_{n-1}, Ffx_n, t). \end{aligned}$$

Thus $\{M(Ffx_n, Ffx_{n+1}, t); n \geq 0\}$ is increasing sequence in $[0,1]$. Therefore, tends to a limit $a(t) \leq 1$. We claim that $a(t) = 1$. For if $a(t) < 1$ on making $n \longrightarrow \infty$ in the above inequality we get $a(t) \geq \gamma(a(t)) > a(t)$, a contradiction. Hence $a(t) = 1$, i.e.,

$$\lim_n M(Ffx_n, Ffx_{n+1}, t) = 1.$$

If we define

$$c_n(t) = M(Ffx_n, Ffx_{n+1}, t), \tag{2.1}$$

then $\lim_{n\to\infty} c_n(t) = 1$. Now, we prove that $\{y_n = Ffx_n\}$ is a Cauchy sequence in $f(X)$ for $n = 1, 2, 3, \cdots$. Suppose that $\{y_n\}$ is not a Cauchy sequence in $f(X)$. Then there is an $\epsilon \in]0,1[$ such that for each integer k, there exist integers $m(k)$ and $n(k)$ with $m(k) > n(k) \geq k$ such that

$$d_k(t) = M(y_{n(k)}, y_{m(k)}, t) \leq 1 - \epsilon \quad \text{for} \quad k = 1, 2, \cdots. \tag{2.2}$$

We may assume that

$$M(y_{n(k)}, y_{m(k)-1}, t) > 1 - \epsilon, \tag{2.3}$$

by choosing $m(k)$ be the smallest number exceeding $n(k)$ for which (2.2) holds. Using (2.1), we have

$$\begin{aligned} 1 - \epsilon \geq d_k(t) &\geq M\left(y_{n(k)}, y_{m(k)-1}, \frac{t}{2b}\right) * M\left(y_{m(k)-1}, y_{m(k)}, \frac{t}{2b}\right) \\ &\geq c_k\left(\frac{t}{2b}\right) * (1 - \epsilon) \end{aligned} \tag{2.4}$$

Hence, $d_k(t) \longrightarrow 1 - \epsilon$ for every $t > 0$ as $k \longrightarrow \infty$.

Also notice

$$\begin{aligned} &d_k(t) = M(y_{n(k)}, y_{m(k)}, t) \\ &\geq M\left(y_{n(k)}, y_{n(k)+1}, \frac{t}{3b}\right) * M\left(y_{n(k)+1}, y_{m(k)+1}, \frac{t}{3b}\right) * M\left(y_{m(k)+1}, y_{m(k)}, \frac{t}{3b}\right) \\ &\geq c_k\left(\frac{t}{3b}\right) * \gamma\left(M\left(y_{n(k)}, y_{m(k)}, \frac{tb^3}{3}\right)\right) * c_k\left(\frac{t}{3b}\right) \\ &= c_k\left(\frac{t}{3b}\right) * \gamma\left(d_k\left(\frac{tb^3}{3}\right)\right) * c_k\left(\frac{t}{3b}\right). \end{aligned}$$

Thus, as $k \longrightarrow \infty$ in the above inequality we have

$$1-\epsilon \geq \gamma(1-\epsilon) > 1-\epsilon$$

which is a contradiction. Thus, $\{Ffx_n\}_n$ is Cauchy and by the completeness of X, $\{Ffx_n\}_n$ converges to z in X. Also $\{Fgx_n\}_n$ converges to z in X. Since F is sequentially convergent, $\{fx_n\}$ and $\{gx_n\}$ converges to some $a \in X$ and also from the continuity of F, $\{Ffx_n\}$ converges to Fa. That is, since $\{y_n\}$ converges to z, then $y_n = Ffx_n = Fgx_{n+1} \longrightarrow Fa = z$. Let us suppose that the mapping f is continuous. Then $\lim_n ffx_n = fa$ and $\lim_n fgx_n = fa$. Further we have since f and g are R-weakly commuting

$$M(fgx_n, gfx_n, t) \geq M(fx_n, gx_n, t/R).$$

Taking the lower limit as $n \to \infty$in the above inequality

$$\begin{aligned} M(fa, \liminf_{n\longrightarrow\infty} gfx_n, b^2t) \geq & \liminf_{n\longrightarrow\infty} M(fgx_n, gfx_n, t) \\ & \geq \liminf_{n\longrightarrow\infty} M(fx_n, gx_n, \frac{t}{R}) \geq M(a, a, \frac{t}{Rb^2}) = 1. \end{aligned}$$

Similarly,

$$M(fa, \limsup_{n\longrightarrow\infty} gfx_n, b^2t) = 1,$$

hence we get $\lim_n gfx_n = fa$. We now prove that $a = fa$. Suppose $a \neq fa$, since F is one to one we get $Ffa \neq Fa = z$. then $M(Fa, Ffa, t) < 1$. By (c)

$$\begin{aligned} M(Ffa, Fa, b^2t) \geq & \liminf_{n\longrightarrow\infty} M(Fffx_n, Ffx_n, t) \\ & \geq \gamma(\liminf_{n\longrightarrow\infty} M(Fgfx_n, Fgx_n, b^4t)) \\ & \geq \gamma(M(Ffa, Fa, b^2t)) \\ & > M(Ffa, Fa, b^2t), \end{aligned}$$

a contradiction. Therefore, $Ffa = Fa$, this implies that $fa = a$. Since $f(X) \subseteq g(X)$ we can find a_1 in X such that $a = fa = ga_1$. Now,

$$M(Fffx_n, Ffa_1, t) \geq \gamma(M(Fgfx_n, Fga_1, b^4t)).$$

Taking limit inf as $n \to \infty$ we get

$$\begin{aligned} M(Ffa, Ffa_1, bt) \geq & \liminf_{n\longrightarrow\infty} M(Fffx_n, Ffa_1, t) \\ & \geq \gamma(\liminf_{n\longrightarrow\infty} M(Fgfx_n, Fga_1, b^4t)) \\ & \geq \gamma(M(Ffa, Fga_1, b^3t)) \\ & = \gamma(M(Ffa, Ffa, b^3t)) = 1, \end{aligned}$$

since $\gamma(1) = 1$, which implies that $Ffa = Ffa_1$. Since F is one to one, then $a = fa = fa_1 = ga_1$. Also for any $t > 0$,

$$M(fa, ga, t) = M(fga_1, gfa_1, t) \geq M(fa_1, ga_1, t/R) = 1$$

which again implies that $fa = ga$. Thus a is a common fixed point of f and g.

Now to prove uniqueness let if possible $a' \neq a$ be another common fixed point of f and g, hence $Fa \neq Fa'$. Then there exists $t > 0$ such that $M(Fa, Fa', t) < 1$, and

$$\begin{aligned} M(Fa, Fa', t) &= M(Ffa, Ffa', t) \\ &\geq \gamma(M(Fga, Fga', b^4t)) = \gamma(M(Fa, Fa', b^4t)) \\ &> M(Fa, Fa', b^4t) \geq M(Fa, Fa', t) \end{aligned}$$

which is contradiction. Therefore, $Fa = Fa'$, since F is one to one this implies that $a = a'$ is a unique common fixed point of f and g. Now, we need only prove f, g and F have a unique common fixed point. Let a be the unique fixed point of f. Suppose to the contrary that $Fa \neq a$. Since F is one to one, $F^2a \neq Fa$. Then

$$\begin{aligned} M(Fa, F^2a, t) &= M(Ffa, FFfa, t) = M(Ffa, FfFa, t) \\ &\geq \gamma(M(Fga, FgFa, b^4t) \\ &= \gamma(M(Fga, FFga, b^4t) \\ &= \gamma(M(Fa, F^2a, b^4t) \\ &> M(Fa, F^2a, t) \end{aligned}$$

which is contradiction. Therefore, $Fa = F^2a$ implies that $Fa = a$. This proves that a is a unique common fixed point of f, g and F. □

Corollary 2.6. *Let f and g be maps from a complete $b-$fuzzy metric space $(X, M, *)$ into itself. Let f and g be R-weakly commuting self-mappings of X satisfying the following conditions:*

(a) *$f(X) \subseteq g(X)$;*
(b) *f or g is continuous;*
(c) *$M(fx, fy, t) \geq \gamma(M(gx, gy, b^4t))$, where $\gamma : [0, 1] \to [0, 1]$ is a continuous function such that $\gamma(a) > a$ for each $a \in]0, 1[$.*

Then f and g have a unique common fixed point $a \in X$.

Proof. If we take F as identity map on X, then Theorem 2.5 follows that f and g have a unique common fixed point. □

Corollary 2.7. *Let F, f and g be maps from a complete $b-$fuzzy metric space $(X, M, *)$ into itself. Let Ff and Fg be R-weakly commuting self-mappings of X satisfying the following conditions:*

(a) $Ff(X) \subseteq Fg(X)$;
(b) Ff *or* Fg *is continuous;*
(c) $M(Ffx, Ffy, t) \geq \gamma(M(Fgx, Fgy, b^4 t))$, *where* $\gamma : [0,1] \to [0,1]$ *is a continuous function such that* $\gamma(a) > a$ *for each* $a \in]0,1[$.

If $Ff = fF$ *and* $Fg = gF$, *then* F, f *and* g *have a unique common fixed point.*

Proof. By Corollary 2.6 follows that Ff and Fg have a unique common fixed point $a \in X$, i.e. $Ffa = Fga = a$. Now, we show that $Fa = a$.

$$\begin{aligned} M(Fa, a, t) &= M(FFfa, Ffa, t) = M(FfFa, Ffa, t) \\ &\geq \gamma(M(FgFa, Fga, b^4 t) \\ &= \gamma(M(FFga, Fga, b^4 t) \\ &= \gamma(M(Fa, a, b^4 t) \\ &> M(Fa, a, t) \end{aligned}$$

which is contradiction. Therefore, $Fa = a$. Hence, $Ffa = Fga = a = Fa$, also $fa = fFa = Ffa = a$ and $ga = gFa = Fga = a$ it follows that $fa = ga = a$. □

Now we give an example to support our Theorem 2.5.

Example 2.8. Consider Example 1.4 in which $X = [0,1]$. Let $a * c = ac$ for all $a, c \in [0,1]$ and let M be the $b-$fuzzy set on $X \times X \times]0, +\infty[$ defined as follows:

$$M(x, y, t) = e^{\frac{-(x-y)^2}{t}},$$

for all $t \in \mathbb{R}^+$. Then $(X, M, *)$ is a $b-$fuzzy metric space for $b = 2$. Define $f(x) = \frac{x}{12}$, $g(x) = \frac{x}{2}$ and $F(x) = \frac{x}{2}$. It is evident that $f(X) \subseteq g(X)$, f is continuous. Define $\gamma : (0,1) \to (0,1)$ by $\gamma(a) = \sqrt{a}$, for $0 < a < 1$. Since

$$(\frac{x}{24} - \frac{y}{24})^2 \leq (\frac{1}{24})^2 (x-y)^2 \leq \frac{1}{2} . \frac{1}{16}(x-y)^2 = \frac{1}{2}(\frac{x}{4} - \frac{y}{4})^2,$$

hence it follows that

$$\begin{aligned} M(Ffx, Ffy, t) &= e^{\frac{-(\frac{x}{24} - \frac{y}{24})^2}{t}} \\ &\geq e^{\frac{-(\frac{x}{4} - \frac{y}{4})^2}{2t}} = \gamma(M(Fgx, Fgy, b^4 t)) \end{aligned}$$

for all x, y in X, f and g are R-weakly commuting. Thus all the conditions of last theorem are satisfied and 0 is a common fixed point of f and g.

Corollary 2.9. *Let $(X, M, *)$ be a complete fuzzy metric space and let Ff and Fg be R-weakly commuting self-mappings of X satisfying the following conditions:*

(a) $Ff(X) \subseteq Fg(X)$;
(b) Ff *or* Fg *is continuous;*
(c) $M(Ffx, Ffy, t) \geq \gamma(M(Fgx, Fgy, t))$, *where* $\gamma : [0, 1] \to [0, 1]$ *is a continuous function such that* $\gamma(a) > a$ *for each* $a \in]0, 1[$.

Also, if $Ff = fF$ *and* $Fg = gF$, *then* F, f *and* g *have a unique common fixed point.*

Proof. If we take $b = 1$, then Corollary 2.7 follows that F, f and g have a unique common fixed point. □

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