Nonlinear Functional Analysis and Applications
Vol. 31, No. 2 (2026), pp. 343-364
ISSN: 1229-1595(print), 2466-0973(online)
https://doi.org/10.22771/nfaa.2026.31.02.02
http://nfaa.kyungnam.ac.kr/journal-nfaa

A NEW TOPOLOGICAL APPROACH TO N-BIPOLAR SOFT GENERALIZED TOPOLOGICAL SPACES WITH APPLICATION IN DECISION MAKING

F. Y. Al-Quhali[1], **Saif Z. Hameed**[2], **Amira R. Abdel-Malek**[3], **Essam El-Seidy**[4] **and A. E. Radwan**[5]

[1]Department of Mathematics, Faculty of Education, Amran University, Amran, Yemen
Department of Mathematics, Faculty of Science, Ain Shams University, Cairo, Egypt
e-mail: `fyalquhali89@gmail.com`

[2]Department of Mathematics, College of Basic Education, Mustansiriyah University, Baghdad, Iraq
e-mail: `saif.zuhar.edbs@uomustansiriyah.edu.iq`

[3]Department of Mathematical and Natural Sciences, Faculty of Engineering, Egyption Russian University, Badr, Egypt
e-mail: `amira-ragab@era.edu.eg`

[4]Department of Mathematics, Faculty of Science, Ain Shams University, Cairo, Egypt
e-mail: `esam_elsedy@hotmail.com`

[5]Department of Mathematics, Faculty of Science, Ain Shams University, Cairo, Egypt
e-mail: `zezoradwan@yahoo.com`

Abstract. In this research article, we present a new topological approach to N-bipolar soft generalized topology (NB$\hat{G}$ST$_s$) denoted by $\hat{G}^N_\triangle$. Also, we introduce the N-bipolar soft $\hat{G}^N_\triangle$-open and the N-bipolar soft $\hat{G}^N_\triangle$-closed. Moreover, we define the N-bipolar soft $\hat{G}^N_\triangle$-interior, N-bipolar soft $\hat{G}^N_\triangle$-closure, N-bipolar soft $\hat{G}^N_\triangle$-boundary and N-bipolar soft $\hat{G}^N_\triangle$-exterior. Also, their characterizations and properties are studied. Finally, in the realm of decision-making problems, we show case the utilization of N-bipolar soft generalized topological spaces.

---

[0]Received August 14, 2025. Revised January 17, 2026. Accepted February 22, 2026.
[0]2020 Mathematics Subject Classification: 54A05, 54A40, 90B50.
[0]Keywords: N-bipolar soft set, N-bipolar soft generalized topology, N-bipolar soft $\hat{G}^N_\triangle$-interior, closure, boundary, N-bipolar soft $\hat{G}^N_\triangle$-exterior, decision making.

[0]Corresponding author: F. Y. Al-Quhali(`fyalquhali89@gmail.com`).

1. INTRODUCTION AND PRELIMINARIES

Molodtsov [10] used an acceptable parametrization. He initiated the introductory notion of soft set proposition in 1999 and presented the first result of the proposition. His work has attracted numerous researchers to study this concept. Topology is prominent in colorful branches of mathematics. Therefore, Shabir and Naz were the pioneers who introduced the concept of soft topological spaces [14]. Akdag and Ozkan [1, 2] presented the concepts of soft α-open, the soft b-open, and their respective continuous functions. The strongly soft b^*-separation axioms, N-bipolar soft connectedness, disconnectedness and strongly soft b^*-ideal are studied by Hameed and Saif et al. [8, 7, 12]. Shabir and Naz [16] introduced the concept of the bipolar soft set. In [6], in their study, Fatimah et al. introduced the concept of N-soft sets, which serves as an expanded model of the soft set. Mustafa [11] introduced and studied the N-bipolar soft set and N-bipolar soft topological spaces. Shabir and Naz [15], defined the concept of N-bipolar and elucidated its utilization in the context of decision-making. Also, some basic operations on the bipolar N-soft sets was described by Kamaci and Petchimuthu in [9]. The $N-$Bipolar Soft Continuous Mappings are introduced by Al-Quhali et al. [3].

Csaszar [4] introduced the generalized topology concept and studied some of its basic properties. Soft generalized topological spaces was defined in [17]. An innovative approach exploring the realm of bipolar soft generalized topological structures and their practical implementation in decision-making was presented in [13]. According to [5], the process of making decisions is built upon two perspectives, specifically the negative and positive aspects.

In the present work, we introduce N-bipolar soft generalized topology ($NBS\hat{G}T_S$) and we study the N-bipolar soft $\hat{G}^N_\Delta$-open and the N-bipolar soft $\hat{G}^N_\Delta$-closed. Moreover, we define the N-bipolar soft $\hat{G}^N_\Delta$-interior, N-bipolar soft $\hat{G}^N_\Delta$-closure, N-bipolar soft $\hat{G}^N_\Delta$-boundary and N-bipolar soft $\hat{G}^N_\Delta$-exterior and some of their properties are studied. Finally, we use our results on this level in decision-making problems.

In this study, consider $\check{D}$ as an initial universe and $P(\check{D})$ as the power set of $\check{D}$. Additionally, $\breve{E}$ (which is not equal to $\emptyset$) stands for the collection of parameters that are being considered and $\emptyset \neq \Delta \sqsubseteq \breve{E}$.

Definition 1.1. ([10]) (Ψ, Δ) is referred to be a soft set over $\check{D}$ if Ψ is a map from Δ to $P(\check{D})$.

Definition 1.2. ([6]) (Ψ, Δ, N) is an N-soft set on $\check{D}$ if $\Psi : \Delta \to 2^{(\check{D}\times\check{R})}$ for each $\nabla \in \Delta$ and $\eth \in \check{D}$, there exists a unique $(\eth, r_\nabla) \in \check{D} \times \check{R}$ such that $(\eth, r_\nabla) \in \Psi(\nabla)$, $r_\nabla \in \check{R}$, $\check{R} = \{0, 1, ..., N-1\}$ with $N = \{2, 3, ...\}$.

Definition 1.3. ([16]) (Ψ, γ, Δ) is a bipolar soft set on $\check{D}$ if $\Psi : \Delta \to 2^{\check{D}}$ and $\gamma : \neg\Delta \to 2^{\check{D}}$ with the property that for each $\nabla \in \Delta, \Psi(\nabla) \cap \gamma(\neg\nabla) = \emptyset$.

Definition 1.4. ([11]) $(\Psi, \gamma, \Delta, N)$ is an N-bipolar soft set (NBS-set) on $\check{D}$ if $\Psi : \Delta \to 2^{(\check{D}\times\check{R})}$ and $\gamma : \neg\Delta \to 2^{(\check{D}\times\check{R})}$ with the property that for each $\nabla \in \Delta$ and $\eth \in \check{D}$, there exists a unique $(\eth, r_\nabla)$, $(\eth, r_{\neg\nabla}) \in \check{D} \times \check{R}$ such that $(\eth, r_\nabla) \in \Psi(\nabla), (\eth, r_{\neg\nabla}) \in \gamma(\neg\nabla), r_\nabla \neq r_{\neg\nabla}$ and $0 < r_\nabla + r_{\neg\nabla} \leq N-1, r_\nabla, r_{\neg\nabla} \in \check{R}$, $\check{R} = \{0, 1, ..., N-1\}$, $N = \{2, 3, ...\}$.

Definition 1.5. ([11]) Let $(\Psi, \gamma, \Delta, N)$ be an NBS-set on $\check{D}$. The complement of $(\Psi, \gamma, \Delta, N)$, denoted as $(\Psi, \gamma, \Delta, N)^c$, can be as follows: $(\Psi, \gamma, \Delta, N)^c = (\Psi^c, \gamma^c, \Delta, N)$ with $\Psi^c(\nabla)(\eth) = \gamma(\neg\nabla)(\eth)$ and $\gamma^c(\neg\nabla)(\eth) = \Psi(\nabla)(\eth)$ for all $\nabla \in \Delta$ and $\eth \in \check{D}$.

Definition 1.6. ([11]) An NBS-set $(\Psi, \gamma, \Delta, N)$ on $\check{D}$ is referred to as an empty NBS-set, denoted as $\emptyset^N_\Delta = (\Psi_0, \gamma_{N-1}, \Delta, N)$, satisfying the condition for each $\nabla \in \Delta, \Psi_0(\nabla)(\eth) = 0$ and $\gamma_{N-1}(\neg\nabla)(\eth) = N-1$ for all $\eth \in \check{D}$.

Definition 1.7. ([11]) An NBS-set $(\Psi, \gamma, \Delta, N)$ on $\check{D}$ is referred to as a universal NBS-set, denoted as $\check{D}^N_\Delta = (\Psi_{N-1}, \gamma_0, \Delta, N)$, satisfying the condition for each $\nabla \in \Delta$, $\Psi_{N-1}(\nabla)(\eth) = N$-1 and $\gamma_0(\neg\nabla)(\eth) = 0$ for all $\eth \in \check{D}$.

Definition 1.8. ([11]) The N-bipolar soft power whole set $\varphi\omega(\Psi, \gamma, \Delta, N)$ of the NBS-set $(\Psi, \gamma, \Delta, N)$ is defined by

$$\varphi\omega(\Psi, \gamma, \Delta, N) = \{(\Psi, \gamma)_i : (\Psi, \gamma)_i \sqsubseteq (\Psi, \gamma, \Delta, N), i \in \mathbb{N}\}$$

such that $\Psi(\nabla)(\eth) = \Psi_i(\nabla)(\eth)$ and $\gamma(\neg\nabla)(\eth) = \gamma_i(\neg\nabla)(\eth); \nabla \in \Delta$ and $\eth \in \check{D}$, where $(\Psi, \gamma)_i = (\Psi_i, \gamma_i, \Delta, N)$ is NBS-subset of $(\Psi, \gamma, \Delta, N)$.

Definition 1.9. ([11]) Let $(\Psi, \gamma, \Delta, N)$ be an NBS-set on $\check{D}$. A collection of NBS-subsets of $(\Psi, \gamma, \Delta, N)$ is referred to as N-bipolar soft topology ($NBST_S$) on $(\Psi, \gamma, \Delta, N)$ denoted as $\Im^N_\Delta$, if the following conditions are satisfied.

(1) $\emptyset^N_\Delta, (\Psi, \gamma, \Delta, N) \in \Im^N_\Delta$.

(2) Arbitrary unions of members $\Im^N_\Delta$ of belong to $\Im^N_\Delta$.

(3) Finite intersections of members $\Im^N_\Delta$ of belong to $\Im^N_\Delta$.

The pair $((\Psi, \gamma, \Delta, N), \Im^N_\Delta)$ is said to be an N-bipolar soft topological space ($NBST_S$). Every member of $\Im^N_\Delta$ is referred to as an N-bipolar soft open set (NBS-open set). In addition, the complement of an N-bipolar soft open set is said to be an N-bipolar soft closed set (NBS-closed set).

Definition 1.10. ([11]) Let $((\Psi,\gamma,\Delta,N),\Im^N_\Delta)$ be an $NBST_S$ and $(\Psi,\gamma)_1 = (\Psi_1,\gamma_1,\Delta,N) \sqsubseteq (\Psi,\gamma,\Delta,N)$. Then the collection

$$\tilde{\Im}^N_{(\Psi,\gamma)_1} = \{(\Psi,\gamma)_i \cap (\Psi_1,\gamma_1,\Delta,N) : (\Psi,\gamma)_i \in \Im^N_\Delta\}$$

is referred to as an N-bipolar soft relative topology or N-bipolar soft subtopology on $(\Psi_1,\gamma_1,\Delta,N)$. The pair $((\Psi_1,\gamma_1,\Delta,N),\tilde{\Im}^N_{(\Psi,\gamma)_1})$ is referred to as an N-bipolar soft sub-space of $((\Psi,\gamma,\Delta,N),\Im^N_\Delta)$.

Definition 1.11. ([11]) Let $((\Psi,\gamma,\Delta,N),\Im^N_\Delta)$,$((\Psi,\gamma,\Delta,N),\eta^N_\Delta)$ be two $NBST_{SS}$ over (Ψ,γ,Δ,N), if $\Im^N_\Delta \sqsubseteq \eta^N_\Delta$ then η^N_Δ is said to be finer than $\Im^N_\Delta$.

Definition 1.12. ([11]) Let $((\Psi,\gamma,\Delta,N),\Im^N_\Delta)$ be an $NBST_S$. Every member of $\Im^N_\Delta$ can be expressed as a union of some elements from subfamily β of $\Im^N_\Delta$, and β is referred to as an N-bipolar soft basis for $\Im^N_\Delta$.

For more details, one may see [9], [11] and [15].

2. N-BIPOLAR SOFT GENERALIZED TOPOLOGY $(NBS\hat{G}T_s)$

One of the most important properties of generalized topology $(NBS\hat{G}T_S)$ is discussed and explored in this section. Also, we introduce the $NBS\hat{G}^N_\Delta$-open and the $NBS\hat{G}^N_\Delta$-closed.

Definition 2.1. Let $\hat{G}^N_\Delta$ be the family of NBS-subsets over (Ψ,γ,Δ,N), then $\hat{G}^N_\Delta$ is called an N-bipolar soft generalized topology $(NBS\hat{G}T_S)$ over (Ψ,γ,Δ,N) if it satisfies the following:
(1) $\emptyset^N_\Delta \in \hat{G}^N_\Delta$.
(2) If $(\Psi_i,\gamma_i,\Delta,N) \in \hat{G}^N_\Delta$ for all $i \in \zeta$, then $\cup_{i\in\zeta}(\Psi_i,\gamma_i,\Delta,N) \in \hat{G}^N_\Delta$.
Then $((\Psi,\gamma,\Delta,N),\hat{G}^N_\Delta)$is called an N-bipolar soft generalized topology $(NBS\hat{G}T_S)$ on (Ψ,γ,Δ,N).

Definition 2.2. Let $((\Psi,\gamma,\Delta,N),\hat{G}^N_\Delta)$ be an $NBS\hat{G}T_S$. The members of $\hat{G}^N_\Delta$ are called N-bipolar soft $\hat{G}^N_\Delta$-open sets ($NBS\hat{G}^N_\Delta$-open sets) in (Ψ,γ,Δ,N).

Definition 2.3. Let $((\Psi,\gamma,\Delta,N),\hat{G}^N_{\Delta 1})$ and $((\Psi,\gamma,\Delta,N),\hat{G}^N_{\Delta 2})$ be two $NBS\hat{G}T_{SS}$. Then:
(1) If $\hat{G}^N_{\Delta 1} \sqsubseteq \hat{G}^N_{\Delta 2}$ or $\hat{G}^N_{\Delta 2} \sqsubseteq \hat{G}^N_{\Delta 1}$, then $\hat{G}^N_{\Delta 1}$ is comparable with $\hat{G}^N_{\Delta 2}$.
(2) If $\hat{G}^N_{\Delta 1} \sqsubseteq \hat{G}^N_{\Delta 2}$ then $\hat{G}^N_{\Delta 2}$ is an N-bipolar soft finer (NBS-finer), then $\hat{G}^N_{\Delta 1}$.

Theorem 2.4. *Let $((\Psi,\gamma,\Delta,N),\hat{G}^N_\Delta)$ be an $NBS\hat{G}T_S$ and $\{\hat{G}^N_{\Delta\varepsilon}\}_{(\varepsilon\in I)}$ be an indexed family of $NBS\hat{G}T_S$. Then $\cap_{(\varepsilon\in I)}\hat{G}^N_\Delta$ is an $NBS\hat{G}T_S$, where each $\hat{G}^N_{\Delta\varepsilon}$ is NBS-finer than $\cap_{(\varepsilon\in I)}\hat{G}^N_{\Delta\varepsilon}$ for each ε.*

Proof. Since each $\hat{G}^{N}_{\Delta\varepsilon}, \varepsilon \in I$ is an $NBS\hat{G}T_S$ over $(\Psi, \gamma, \Delta, N)$, the NBS-set $\emptyset^{N}_{\Delta} \in \hat{G}^{N}_{\Delta\varepsilon}, \varepsilon \in I$ and hence $\emptyset^{N}_{\Delta} \in \cap_{(\varepsilon\in I)}\hat{G}^{N}_{\Delta\varepsilon}$. Let $\{(\Psi_k, \gamma_k, \Delta, N) : k \in J\}$ be a family of NBS-sets in $\cap_{(\varepsilon\in I)}\hat{G}^{N}_{\Delta\varepsilon}$. Then each $(\Psi_k, \gamma_k, \Delta, N) \in \hat{G}^{N}_{\Delta\varepsilon}$. But $\hat{G}^{N}_{\Delta\varepsilon}$ being $NBS\hat{G}T_S$ is closed under arbitrary NBS-unions. Therefore, $\cup_{k\in J}(\Psi_k, \gamma_k, \Delta, N) \in \cap_{(\varepsilon\in I)}\hat{G}^{N}_{\Delta\varepsilon}$. Hence, $\cap_{(\varepsilon\in I)}\hat{G}^{N}_{\Delta\varepsilon}$ is an $NBS\hat{G}T_S$ define on $(\Psi, \gamma, \Delta, N)$. Clearly each $\hat{G}^{N}_{\Delta\varepsilon}, \varepsilon \in I$, is NBS-finer than $\cap_{(\varepsilon\in I)}\hat{G}^{N}_{\Delta\varepsilon}$ for each ε. □

Remark 2.5. Let $((\Psi, \gamma, \Delta, N), \hat{G}^{N}_{\Delta 1})$ and $((\Psi, \gamma, \Delta, N), \hat{G}^{N}_{\Delta 2})$ be two $NBS\hat{G}T_{SS}$. Then $((\Psi, \gamma, \Delta, N), \hat{G}^{N}_{\Delta 1} \cup \hat{G}^{N}_{\Delta 2})$ may not be an $NBS\hat{G}T_S$ as evidenced by the following example.

Example 2.6. Consider a set of houses under consideration denoted as
$\check{D} = \{\eth_1, \eth_2\}$ and $\Delta = \{\nabla_1 = \text{Morbled}, \nabla_2 = \text{Contemporary}\}$,
$\neg\Delta = \{\neg\nabla_1 = \text{Wooden}, \neg\nabla_2 = \text{Conventional}\}$.

Consider a $5BS$-set to describe the design of houses in the following manner:
$(\Psi, \gamma, \Delta, 5) = \{(\langle\nabla_1, \{(\eth_1, 2), (\eth_2, 1)\}\rangle, \langle\neg\nabla_1, \{(\eth_1, 1), (\eth_2, 2)\}\rangle),$
$(\langle\nabla_2, \{(\eth_1, 3), (\eth_2, 1)\}\rangle, \langle\neg\nabla_2, \{(\eth_1, 1), (\eth_2, 2)\}\rangle)\}$
$\hat{G}^{N}_{\Delta 1} = \{\emptyset^{N}_{\Delta}, (\Psi, \gamma)_1\}, \hat{G}^{N}_{\Delta 2} = \{\emptyset^{N}_{\Delta}, (\Psi, \gamma)_2\}$,
where $(\Psi, \gamma)_1$,$(\Psi, \gamma)_2$ are $5BS$-subsets on $5BS$- set $(\Psi, \gamma, \Delta, 5)$ defined as follows:
$(\Psi, \gamma)_1 = \{(\langle\nabla_1, \{(\eth_1, 2)\}\rangle, \langle\neg\nabla_1, \{(\eth_1, 1)\}\rangle),$
$(\langle\nabla_2, \{(\eth_1, 3)\}\rangle, \langle\neg\nabla_2, \{(\eth_1, 1), (\eth_2, 2)\}\rangle)\}$,
$(\Psi, \gamma)_2 = \{(\langle\nabla_1, \{(\eth_1, 2), (\eth_2, 1)\}\rangle, \langle\neg\nabla_1, \{(\eth_1, 1), (\eth_2, 2)\}\rangle),$
$(\langle\nabla_2, \{(\eth_1, 3)\}\rangle, \langle\neg\nabla_2, \{(\eth_1, 1)\}\rangle)\}$.
Then $((\Psi, \gamma, \Delta, 5), \hat{G}^{N}_{\Delta 1})$ and $((\Psi, \gamma, \Delta, 5), \hat{G}^{N}_{\Delta 2})$ is $5BS\hat{G}T_{SS}$. But we note that $\hat{G}^{N}_{\Delta 1} \cup \hat{G}^{N}_{\Delta 2}$ is not $5BS\hat{G}T_S$, because $(\Psi, \gamma)_1 \cup (\Psi, \gamma)_2$ from $5BS\hat{G}T_S$ of $(\Psi, \gamma, \Delta, 5)$.

Theorem 2.7. *Consider a family of NBS-sets ϑ defined on $(\Psi, \gamma, \Delta, N)$. Then there exists a unique $NBS\hat{G}T_S$ $\hat{G}^{N}_{\Delta}$which is the smallest $NBS\hat{G}T_S$ containing ϑ.*

Proof. Let a collection of all $NBS\hat{G}T_S$ on $(\Psi, \gamma, \Delta, N)$ include ϑ with certainty, and the members intersection of this collection be $\hat{G}^{N}_{\Delta}$. Using Theorem 2.5, $\hat{G}^{N}_{\Delta}$is the smallest $NBS\hat{G}T_S$ containing ϑ, because any $NBS\hat{G}T_S$ will be a member of the collection of $NBS\hat{G}T_S$, and thus bipolar soft finer than its intersections $\hat{G}^{N}_{\Delta}$. Uniqueness of $\hat{G}^{N}_{\Delta}$ is trivial. □

Definition 2.8. Let $((\Psi, \gamma, \Delta, N), \hat{G}^{N}_{\Delta})$ be an $NBS\hat{G}T_S$. If every member of $\hat{G}^{N}_{\Delta}$can be expressed as a union of some elements of a subfamily β^{N}_{Δ}, then β^{N}_{Δ} is called an N-bipolar soft basis for $\hat{G}^{N}_{\Delta}$.

Definition 2.9. Let $((\Psi,\gamma,\Delta,N),\hat{G}^N_\Delta)$ be an $NBS\hat{G}T_S$ and $(\Psi,\gamma)_1=(\Psi_1,\gamma_1,\Delta,N)\sqsubseteq(\Psi,\gamma,\Delta,N)$. Then the collection

$$\hat{G}^N_{(\Psi,\gamma)_1}=\{(\Psi,\gamma)_\varepsilon\cap(\Psi_1,\gamma_1,\Delta,N):(\Psi,\gamma)_\varepsilon\in\hat{G}^N_\Delta\}$$

is referred to as an N-bipolar soft generalized sub-topology ($NBS\hat{G}ST_S$) on (Ψ,γ,Δ,N). The pair $((\Psi_1,\gamma_1,\Delta,N),\hat{G}^N_{(\Psi,\gamma)_1})$ is referred to as an N-bipolar soft generalized sub-topology ($NBS\hat{G}ST_S$) of $((\Psi,\gamma,\Delta,N),\hat{G}^N_\Delta)$.

Theorem 2.10. *Let* $((\Psi,\gamma,\Delta,N),\hat{G}^N_\Delta)$ *be an* $NBS\hat{G}T_S$ *and* $(\Psi_1,\gamma_1,\Delta,N)\sqsubseteq(\Psi,\gamma,\Delta,N)$. *Then an* $NBS\hat{G}ST_S$ *on* $(\Psi_1,\gamma_1,\Delta,N)$ *is an* $NBS\hat{G}T_S$.

Proof. Since $\emptyset^N_\Delta\in\hat{G}^N_\Delta$, then $\emptyset^N_\Delta\cap(\Psi_1,\gamma_1,\Delta,N)=\emptyset^N_\Delta\in\hat{G}^N_{(\Psi,\gamma)_1}$. Suppose that $\{(\Psi_2,\gamma_2,\Delta,N)_\varepsilon\}_{\varepsilon\in I}\in\hat{G}^N_{(\Psi,\gamma)_1}$. Since each $(\Psi_2,\gamma_2,\Delta,N)_\varepsilon=(\Psi,\gamma)_{2\varepsilon}\cap(\Psi_1,\gamma_1,\Delta,N)$ where $(\Psi,\gamma)_\varepsilon\in\hat{G}^N_\Delta$. Now, consider

$$\begin{aligned}\cup_{\varepsilon\in I}(\Psi_2,\gamma_2,\Delta,N)_\varepsilon&=\cup_{\varepsilon\in I}((\Psi_2,\gamma_2,\Delta,N)_\varepsilon\cap(\Psi_1,\gamma_1,\Delta,N))\\&=(\cup_{\varepsilon\in I}(\Psi_2,\gamma_2,\Delta,N)_\varepsilon)\cap(\Psi_1,\gamma_1,\Delta,N)\\&\in\hat{G}^N_{(\Psi,\gamma)_1},\end{aligned}$$

since $\hat{G}^N_\Delta$ is closed under arbitrary NBS-unions. □

Theorem 2.11. *Let* $((\Psi,\gamma,\Delta,N),\hat{G}^N_\Delta)$ *be an* $NBS\hat{G}T_S$. *If* β^N_Δ *is an* N*-bipolar soft basis for* $\hat{G}^N_\Delta$, *then the collection*

$$\beta^N_{(\Psi,\gamma)_1}=\{(\Psi_2,\gamma_2,\Delta,N)_\varepsilon\cap(\Psi_1,\gamma_1,\Delta,N):(\Psi_2,\gamma_2,\Delta,N)_{\varepsilon\in\beta^N_\Delta},\varepsilon\in I\}$$

is an N*-bipolar soft basis for* $NBS\hat{G}ST_S$ *on* $(\Psi_1,\gamma_1,\Delta,N)$.

Proof. Let $(\Psi_3,\gamma_3,\Delta,N)$ be any element of the $NBS\hat{G}ST_S$ on $(\Psi_1,\gamma_1,\Delta,N)$. Then

$$(\Psi_3,\gamma_3,\Delta,N)=(\Psi_4,\gamma_4,\Delta,N)\cap(\Psi_1,\gamma_1,\Delta,N),$$

where $(\Psi_4,\gamma_4,\Delta,N)\in\hat{G}^N_\Delta$. Then we have $(\Psi_4,\gamma_4,\Delta,N)$ can be expressed as the NBS-union of some element of β^N_Δ, that is,

$$(\Psi_4,\gamma_4,\Delta,N)=\cup_{(\Psi_2,\gamma_2,\Delta,N)_\varepsilon\in\beta^N_\Delta)}(\Psi_2,\gamma_2,\Delta,N)_\varepsilon.$$

Therefore,

$$\begin{aligned}(\Psi_3,\gamma_3,\Delta,N)&=(\cup_{(\Psi_2,\gamma_2,\Delta,N)_\varepsilon\in\beta^N_\Delta}(\Psi_2,\gamma_2,\Delta,N)_\varepsilon)\cap(\Psi_1,\gamma_1,\Delta,N)\\&=\cup_{(\Psi_2,\gamma_2,\Delta,N)_\varepsilon\in\beta^N_\Delta}((\Psi_2,\gamma_2,\Delta,N)_\varepsilon\cap(\Psi_1,\gamma_1,\Delta,N)).\end{aligned}$$

Thus, each element of the $NBS\hat{G}ST_S$ on $(\Psi_1, \gamma_1, \Delta, N)$ is the NBS-union of some element of $\beta^N_{(\Psi,\gamma)_1}$. Hence, $\beta^N_{(\Psi,\gamma)_1}$ is an N-bipolar soft basis for $NBS\hat{G}ST_S$ on $(\Psi_1, \gamma_1, \Delta, N)$. □

Definition 2.12. Let $((\Psi, \gamma, \Delta, N), \hat{G}^N_\Delta)$ be an $NBS\hat{G}T_S$ and $(\Psi_1, \gamma_1, \Delta, N)$, $(\Psi_2, \gamma_2, \Delta, N)$ be two NBS-subset of $(\Psi, \gamma, \Delta, N)$ such that $(\Psi_1, \gamma_1, \Delta, N) \sqsubseteq (\Psi_2, \gamma_2, \Delta, N)$. An NBS-set $(\Psi_2, \gamma_2, \Delta, N)$ is referred to as an N-bipolar soft $\hat{G}^N_\Delta$-neighborhood ($NBS\hat{G}^N_\Delta$-neighborhood) of $(\Psi_1, \gamma_1, \Delta, N)$, if there exists an $NBS\hat{G}^N_\Delta$-open sets $(\Psi_3, \gamma_3, \Delta, N)$ such that

$$(\Psi_1, \gamma_1, \Delta, N) \sqsubseteq (\Psi_3, \gamma_3, \Delta, N) \sqsubseteq (\Psi_2, \gamma_2, \Delta, N).$$

Definition 2.13. Let $((\Psi, \gamma, \Delta, N), \hat{G}^N_\Delta)$ be an $NBS\hat{G}T_S$ and $(\Psi_1, \gamma_1, \Delta, N)$, $(\Psi_2, \gamma_2, \Delta, N)$ be two NBS-subset of $(\Psi, \gamma, \Delta, N)$ such that $(\Psi_1, \gamma_1, \Delta, N) \sqsubseteq (\Psi_2, \gamma_2, \Delta, N)$. Then $(\Psi_1, \gamma_1, \Delta, N)$ is referred to as an N-bipolar soft $\hat{G}^N_\Delta$-interior ($NBS\hat{G}^N_\Delta$-interior) of $(\Psi_2, \gamma_2, \Delta, N)$, if $(\Psi_2, \gamma_2, \Delta, N)$ is the N-bipolar soft $\hat{G}^N_\Delta$-neighborhood of $(\Psi_1, \gamma_1, \Delta, N)$. In other words, the union of all $NBS\hat{G}^N_\Delta$-open subsets of $(\Psi_2, \gamma_2, \Delta, N)$ is called an $NBS\hat{G}^N_\Delta$-interior of $(\Psi_2, \gamma_2, \Delta, N)$ and it is denoted by $NBS\hat{G}int\hat{G}^N_\Delta\ (\Psi_1, \gamma_1, \Delta, N)$.

Theorem 2.14. *Let $((\Psi, \gamma, \Delta, N)$,$\hat{G}^N_\Delta)$ be an $NBS\hat{G}T_S$ and $(\Psi_1, \gamma_1, \Delta, N)$, $(\Psi_2, \gamma_2, \Delta, N)$ be two NBS-subsets of $(\Psi, \gamma, \Delta, N)$. Then*

1. $NBS\hat{G}int\hat{G}^N_\Delta\ (\Psi_1, \gamma_1, \Delta, N)$ *is a largest* $NBS\hat{G}^N_\Delta$*-open set contained in* $(\Psi_1, \gamma_1, \Delta, N)$*;*
2. $NBS\hat{G}int\hat{G}^N_\Delta(\Psi_1, \gamma_1, \Delta, N) \sqsubseteq (\Psi_1, \gamma_1, \Delta, N)$*;*
3. $(\Psi_1, \gamma_1, \Delta, N)$ *is an* $NBS\hat{G}^N_\Delta$*-open set* $NBS\hat{G}int\hat{G}^N_\Delta(\Psi_1, \gamma_1, \Delta, N) = (\Psi_1, \gamma_1, \Delta, N)$*;*
4. $NBS\hat{G}int\hat{G}^N_\Delta(NBS\hat{G}int\hat{G}^N_\Delta(\Psi_1, \gamma_1, \Delta, N)) = (\Psi_1, \gamma_1, \Delta, N)$*;*
5. *If* $(\Psi_1, \gamma_1, \Delta, N) \sqsubseteq (\Psi_2, \gamma_2, \Delta, N)$*, then* $NBS\hat{G}int\hat{G}^N_\Delta(\Psi_1, \gamma_1, \Delta, N) \sqsubseteq NBS\hat{G}int\hat{G}^N_\Delta(\Psi_2, \gamma_2, \Delta, N)$*;*
6. $NBS\hat{G}int\hat{G}^N_\Delta(\Psi_1, \gamma_1, \Delta, N) \cap NBS\hat{G}int\hat{G}^N_\Delta(\Psi_2, \gamma_2, \Delta, N)$ $= NBS\hat{G}int\hat{G}^N_\Delta((\Psi_1, \gamma_1, \Delta, N) \cap (\Psi_2, \gamma_2, \Delta, N))$*;*
7. $NBS\hat{G}int\hat{G}^N_\Delta(\Psi_1, \gamma_1, \Delta, N) \cup NBS\hat{G}int\hat{G}^N_\Delta(\Psi_2, \gamma_2, \Delta, N)$ $\sqsubseteq NBS\hat{G}int\hat{G}^N_\Delta((\Psi_1, \gamma_1, \Delta, N) \cup (\Psi_2, \gamma_2, \Delta, N))$.

Proof. The proof follows analogously to [11], with necessary modifications due to N-bipolar structure. □

Definition 2.15. Let $((\Psi, \gamma, \Delta, N), \hat{G}^N_\Delta)$ be an $NBS\hat{G}T_S$ and $(\Psi_1, \gamma_1, \Delta, N)$ be an NBS-subset of $(\Psi, \gamma, \Delta, N)$. Then $(\Psi_1, \gamma_1, \Delta, N)$ is called N-bipolar soft $\hat{G}^N_\Delta$-closed ($NBS\hat{G}^N_\Delta$-closed) if its NBS-complement $(\Psi_1, \gamma_1, \Delta, N)^c$ is $NBS\hat{G}^N_\Delta$-open.

Theorem 2.16. *Let $((\Psi, \gamma, \Delta, N), \hat{G}^N_\Delta)$ be an $NBS\hat{G}T_S$. Then the following properties hold:*

1. *$\hat{G}^N_\Delta$ is $NBS\hat{G}^N_\Delta$-closed.*
2. *Arbitrary NBS-intersections of the $NBS\hat{G}^N_\Delta$-closed sets are $NBS\hat{G}^N_\Delta$-closed.*

Proof. (1) Since the relative null NBS-set $\emptyset^N_\Delta$ is the complement of the absolute N-bipolar soft set $\hat{G}^N_\Delta$, and $\emptyset^N_\Delta \in \hat{G}^N_\Delta$. Thus, $\hat{G}^N_\Delta$ is $NBS\hat{G}^N_\Delta$-closed.
(2) Let $\{(\Psi_j, \gamma_j, \Delta, N)\}_{j\in J}$ be a given collection of $NBS\hat{G}^N_\Delta$-closed sets. Now, $(\cap_{j\in J}(\Psi_j, \gamma_j, \Delta, N))^c = \cup_{j\in J}(\Psi_j, \gamma_j, \Delta, N)^c$. Since $(\Psi_j, \gamma_j, \Delta, N)$ is an $NBS\hat{G}^N_\Delta$-closed for each $j \in J$. If we consider $(\Psi_j, \gamma_j, \Delta, N)^c$ is $NBS\hat{G}^N_\Delta$-open sets, then $\cup_{j\in J}(\Psi_j, \gamma_j, \Delta, N)^c$ is also $NBS\hat{G}^N_\Delta$-open. As a result, $(\cap_{j\in J}(\Psi_j, \gamma_j, \Delta, N))^c$ is also $NBS\hat{G}^N_\Delta$-open which means $\cap_{j\in J}(\Psi_j, \gamma_j, \Delta, N)$ $NBS\hat{G}^N_\Delta$-closed. □

Definition 2.17. Let $((\Psi, \gamma, \Delta, N), \hat{G}^N_\Delta)$ be an $NBS\hat{G}T_S$ and $(\Psi_1, \gamma_1, \Delta, N)$ be an NBS-subset of $(\Psi, \gamma, \Delta, N)$. Then the N-bipolar soft $\hat{G}^N_\Delta$-closure ($NBS\hat{G}^N_\Delta$-closure) of $(\Psi_1, \gamma_1, \Delta, N)$, denoted by $NBS\hat{G}cl_{\hat{G}^N_\Delta}(\Psi_1, \gamma_1, \Delta, N)$, is the NBS-intersection of all $NBS\hat{G}^N_\Delta$-closed sets containing $(\Psi_1, \gamma_1, \Delta, N)$.

Theorem 2.18. *Let $((\Psi, \gamma, \Delta, N), \hat{G}^N_\Delta)$ be an $NBS\hat{G}T_S$ and $(\Psi_1, \gamma_1, \Delta, N)$, $(\Psi_2, \gamma_2, \Delta, N)$ be two NBS-subsets of $(\Psi, \gamma, \Delta, N)$. Then*

1. *$NBS\hat{G}cl_{\hat{G}^N_\Delta}(\Psi_1, \gamma_1, \Delta, N)$ is the smallest $NBS\hat{G}^N_\Delta$-closed set containing $(\Psi_1, \gamma_1, \Delta, N)$.*
2. *$(\Psi_1, \gamma_1, \Delta, N) \sqsubseteq NBS\hat{G}cl_{\hat{G}^N_\Delta}(\Psi_1, \gamma_1, \Delta, N)$;*
3. *$(\Psi_1, \gamma_1, \Delta, N)$ is an $NBS\hat{G}^N_\Delta$-closed set*
   $NBS\hat{G}cl_{\hat{G}^N_\Delta}(\Psi_1, \gamma_1, \Delta, N) = (\Psi_1, \gamma_1, \Delta, N)$;
4. *$NBS\hat{G}cl_{\hat{G}^N_\Delta}(NBS\hat{G}cl_{\hat{G}^N_\Delta}(\Psi_1, \gamma_1, \Delta, N)) = NBS\hat{G}cl_{\hat{G}^N_\Delta}(\Psi_1, \gamma_1, \Delta, N)$;*
5. *If $(\Psi_1, \gamma_1, \Delta, N) \sqsubseteq (\Psi_2, \gamma_2, \Delta, N)$, then $NBS\hat{G}cl_{\hat{G}^N_\Delta}(\Psi_1, \gamma_1, \Delta, N) \sqsubseteq NBS\hat{G}cl_{\hat{G}^N_\Delta}(\Psi_2, \gamma_2, \Delta, N)$;*
6. *$NBS\hat{G}cl_{\hat{G}^N_\Delta}((\Psi_1, \gamma_1, \Delta, N) \cap (\Psi_2, \gamma_2, \Delta, N))$*
   $\sqsubseteq NBS\hat{G}cl_{\hat{G}^N_\Delta}(\Psi_1, \gamma_1, \Delta, N) \cap NBS\hat{G}cl_{\hat{G}^N_\Delta}(\Psi_2, \gamma_2, \Delta, N)$;
7. *$NBS\hat{G}cl_{\hat{G}^N_\Delta}(\Psi_1, \gamma_1, \Delta, N) \cup NBS\hat{G}cl_{\hat{G}^N_\Delta}(\Psi_2, \gamma_2, \Delta, N)$*
   $\sqsubseteq NBS\hat{G}cl_{\hat{G}^N_\Delta}((\Psi_1, \gamma_1, \Delta, N) \cup (\Psi_2, \gamma_2, \Delta, N))$.

Proof. It is easily to prove, by using reference [11] up to this level. □

Theorem 2.19. *Let* $((\Psi,\gamma,\Delta,N),\hat{G}^N_\Delta)$ *be an* $NBS\hat{G}T_S$ *and* $(\Psi_1,\gamma_1,\Delta,N)$ *be* NBS*-subset of* (Ψ,γ,Δ,N)*. Then*

$$NBS\hat{G}int_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)\sqsubseteq(\Psi_1,\gamma_1,\Delta,N)\sqsubseteq NBS\hat{G}cl_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N).$$

Proof. $(\Psi_1,\gamma_1,\Delta,N)=\cup\{(\Psi,\gamma)_\varepsilon:(\Psi,\gamma)_\varepsilon\in\hat{G}^N_\Delta;(\Psi,\gamma)_\varepsilon\sqsubseteq(\Psi_1,\gamma_1,\Delta,N);\varepsilon\in I\sqsubseteq\mathbb{N}\}$. So, $\Psi_\varepsilon(\nabla)(\eth)\leq\Psi_1(\nabla)(\eth)$ and $\gamma_\varepsilon(\nabla)(\eth)\geq\gamma_1(\nabla)(\eth)$ for each $\nabla\in\Delta$ and $\eth\in\check{D}$. Therefore,

$$\cup_{\varepsilon\in I}(\Psi,\gamma)_\varepsilon\sqsubseteq(\Psi_1,\gamma_1,\Delta,N)$$

and thus

$$NBS\hat{G}int_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)\sqsubseteq(\Psi_1,\gamma_1,\Delta,N).$$

□

Theorem 2.20. *Let* $((\Psi,\gamma,\Delta,N),\hat{G}^N_\Delta)$ *be an* $NBS\hat{G}T_S$ *and* $(\Psi_1,\gamma_1,\Delta,N)$, $(\Psi_2,\gamma_2,\Delta,N)$ *be two NBS-subsets of* (Ψ,γ,Δ,N)*. Then*

(1) $NBS\hat{G}cl_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)^c=(NBS\hat{G}int_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N))^c$;

(2) $NBS\hat{G}int_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)^c=(NBS\hat{G}cl_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N))^c$;

(3) $NBS\hat{G}cl_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)^c=(NBS\hat{G}int_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)^c)^c$;

(4) $NBS\hat{G}int_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)^c=(NBS\hat{G}cl_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)^c)^c$;

(5) $NBS\hat{G}int_{\hat{G}^N_\Delta}((\Psi_1,\gamma_1,\Delta,N)\backslash(\Psi_2,\gamma_2,\Delta,N))$
$=NBS\hat{G}int_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)\backslash NBS\hat{G}int_{\hat{G}^N_\Delta}(\Psi_2,\gamma_2,\Delta,N)$.

Proof. (1) Since $(\Psi_1,\gamma_1,\Delta,N)^c=(\cup\{(\Psi,\gamma)_\varepsilon:(\Psi,\gamma)_\varepsilon\in\hat{G}^N_\Delta;(\Psi,\gamma)_\varepsilon\sqsubseteq(\Psi_1,\gamma_1,\Delta,N);\varepsilon\in I\sqsubseteq\mathbb{N}\})^c=\cap\{(\Psi,\gamma)^c_\varepsilon:(\Psi,\gamma)^c_\varepsilon\in\hat{G}^{Nc}_\Delta;(\Psi,\gamma)^c_\varepsilon\sqsupseteq(\Psi_1,\gamma_1,\Delta,N)^c;\varepsilon\in I\sqsubseteq\mathbb{N}\}=NBS\hat{G}cl_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)^c$.

(2) to (4) the proof is similar to that of Part (1).

(5) From Theorem 2.14 (6), since

$$\begin{aligned}
&NBS\hat{G}int_{\hat{G}^N_\Delta}((\Psi_1,\gamma_1,\Delta,N)\backslash(\Psi_2,\gamma_2,\Delta,N))\\
&\quad=NBS\hat{G}int_{\hat{G}^N_\Delta}((\Psi_1,\gamma_1,\Delta,N)\cap(\Psi_2,\gamma_2,\Delta,N)^c)\\
&\quad\sqsubseteq NBS\hat{G}int_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)\cap NBS\hat{G}int_{\hat{G}^N_\Delta}(\Psi_2,\gamma_2,\Delta,N)^c\\
&\quad=NBS\hat{G}int_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)\cap(NBS\hat{G}cl_{\hat{G}^N_\Delta}(\Psi_2,\gamma_2,\Delta,N))^c\\
&\quad\sqsubseteq NBS\hat{G}int_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)\cap NBS\hat{G}int_{\hat{G}^N_\Delta}(\Psi_2,\gamma_2,\Delta,N)^c\\
&\quad=NBS\hat{G}int_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)\backslash NBS\hat{G}int_{\hat{G}^N_\Delta}(\Psi_2,\gamma_2,\Delta,N).
\end{aligned}$$

□

Definition 2.21. Let $((\Psi,\gamma,\Delta,N),\hat{G}^N_\Delta)$ be an $NBS\hat{G}T_S$ and $(\Psi_1,\gamma_1,\Delta,N)$ be an NBS-subset of (Ψ,γ,Δ,N). Then the N-bipolar soft $\hat{G}^N_\Delta$-boundary ($NBS\hat{G}^N_\Delta$-boundary) of $(\Psi_1,\gamma_1,\Delta,N)$, denoted by $b_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)$, is define by

$$b_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N) = NBS\hat{G}cl_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N) \cap NBS\hat{G}cl_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)^c.$$

Lemma 2.22. *It is clear that* $b_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N) = b_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)^c$.

Theorem 2.23. *Let* $((\Psi,\gamma,\Delta,N),\hat{G}^N_\Delta)$ *be an* $NBS\hat{G}T_S$ *and* $(\Psi_1,\gamma_1,\Delta,N)$ *be an* NBS*-subset of* (Ψ,γ,Δ,N). *Then*

(1) $b_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N) \sqsubseteq (\Psi_1,\gamma_1,\Delta,N)$;
(2) $(\Psi_1,\gamma_1,\Delta,N) \cup b_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N) \sqsubseteq NBS\hat{G}cl_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)$;
(3) $NBS\hat{G}int_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N) \sqsubseteq (\Psi_1,\gamma_1,\Delta,N)\backslash b_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)$;
(4) $b_{\hat{G}^N_\Delta}(NBS\hat{G}int_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)) \sqsubseteq b_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)$;
(5) $b_{\hat{G}^N_\Delta}(NBS\hat{G}cl_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)) \sqsubseteq b_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)$.

Proof. (1) Since

$$b_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N) = NBS\hat{G}cl_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N) \cap NBS\hat{G}cl_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)^c,$$

we have $b_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N) \sqsubseteq (\Psi_1,\gamma_1,\Delta,N)$.

(2) We have

$$\begin{aligned}
&(\Psi_1,\gamma_1,\Delta,N) \cup b_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)\\
&\quad= (\Psi_1,\gamma_1,\Delta,N) \cup (NBS\hat{G}cl_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)\\
&\qquad \cap NBS\hat{G}cl_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)^c)\\
&\quad= ((\Psi_1,\gamma_1,\Delta,N) \cup NBS\hat{G}cl_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N))\\
&\qquad \cap ((\Psi_1,\gamma_1,\Delta,N) \cup NBS\hat{G}cl_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)^c)\\
&\quad= NBS\hat{G}cl_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)\\
&\qquad \cap ((\Psi_1,\gamma_1,\Delta,N) \cup NBS\hat{G}cl_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)^c)\\
&\quad\sqsubseteq NBS\hat{G}cl_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N).
\end{aligned}$$

(3) We have

$$\begin{aligned}
&(\Psi_1, \gamma_1, \Delta, N)\backslash b_{\hat{G}^N_\Delta}(\Psi_1, \gamma_1, \Delta, N)\\
&\quad = (\Psi_1, \gamma_1, \Delta, N) \cap b_{\hat{G}^N_\Delta}(\Psi_1, \gamma_1, \Delta, N)^c\\
&\quad = (\Psi_1, \gamma_1, \Delta, N) \cap (NBS\hat{G}cl_{\hat{G}^N_\Delta}(\Psi_1, \gamma_1, \Delta, N)\\
&\qquad \cap NBS\hat{G}cl_{\hat{G}^N_\Delta}(\Psi_1, \gamma_1, \Delta, N)^c)^c\\
&\quad = (\Psi_1, \gamma_1, \Delta, N)\\
&\qquad \cap (NBS\hat{G}int_{\hat{G}^N_\Delta}(\Psi_1, \gamma_1, \Delta, N)^c \cup NBS\hat{G}int_{\hat{G}^N_\Delta}(\Psi_1, \gamma_1, \Delta, N))\\
&\quad = ((\Psi_1, \gamma_1, \Delta, N)\\
&\qquad \cap NBS\hat{G}int_{\hat{G}^N_\Delta}(\Psi_1, \gamma_1, \Delta, N)^c) \cup ((\Psi_1, \gamma_1, \Delta, N)\\
&\qquad \cap NBS\hat{G}int_{\hat{G}^N_\Delta}(\Psi_1, \gamma_1, \Delta, N))\\
&\quad = \emptyset^N_\Delta \cup NBS\hat{G}int_{\hat{G}^N_\Delta}(\Psi_1, \gamma_1, \Delta, N) \sqsupseteq NBS\hat{G}int_{\hat{G}^N_\Delta}(\Psi_1, \gamma_1, \Delta, N).
\end{aligned}$$

(4) We have

$$\begin{aligned}
&b_{\hat{G}^N_\Delta}(NBS\hat{G}int_{\hat{G}^N_\Delta}(\Psi_1, \gamma_1, \Delta, N))\\
&\quad = NBS\hat{G}cl_{\hat{G}^N_\Delta}(NBS\hat{G}int_{\hat{G}^N_\Delta}(\Psi_1, \gamma_1, \Delta, N))\\
&\qquad \cap NBS\hat{G}cl_{\hat{G}^N_\Delta}(NBS\hat{G}int_{\hat{G}^N_\Delta}(\Psi_1, \gamma_1, \Delta, N))^c\\
&\quad = NBS\hat{G}cl_{\hat{G}^N_\Delta}(NBS\hat{G}int_{\hat{G}^N_\Delta}(\Psi_1, \gamma_1, \Delta, N))\\
&\qquad \cap NBS\hat{G}cl_{\hat{G}^N_\Delta}(NBS\hat{G}cl_{\hat{G}^N_\Delta}(\Psi_1, \gamma_1, \Delta, N))^c\\
&\quad \sqsubseteq NBS\hat{G}cl_{\hat{G}^N_\Delta}(\Psi_1, \gamma_1, \Delta, N) \cap NBS\hat{G}cl_{\hat{G}^N_\Delta}(\Psi_1, \gamma_1, \Delta, N)^c\\
&\quad = b_{\hat{G}^N_\Delta}(\Psi_1, \gamma_1, \Delta, N).
\end{aligned}$$

(5) We have

$$\begin{aligned}
&b_{\hat{G}^N_\Delta}(NBS\hat{G}cl_{\hat{G}^N_\Delta}(\Psi_1, \gamma_1, \Delta, N))\\
&\quad = NBS\hat{G}cl_{\hat{G}^N_\Delta}(NBS\hat{G}cl_{\hat{G}^N_\Delta}(\Psi_1, \gamma_1, \Delta, N))\\
&\qquad \cap NBS\hat{G}cl_{\hat{G}^N_\Delta}(NBS\hat{G}cl_{\hat{G}^N_\Delta}(\Psi_1, \gamma_1, \Delta, N))^c\\
&\quad \sqsubseteq NBS\hat{G}cl_{\hat{G}^N_\Delta}(\Psi_1, \gamma_1, \Delta, N) \cap NBS\hat{G}cl_{\hat{G}^N_\Delta}(\Psi_1, \gamma_1, \Delta, N)^c\\
&\quad = b_{\hat{G}^N_\Delta}(\Psi_1, \gamma_1, \Delta, N).
\end{aligned}$$

□

In general, the equality between parts (2), (3), (4) and (5) in Theorem 2.23 is not held, as illustrated by the following example.

Example 2.24. Let $\check{D} = \{\eth_1, \eth_2\}$ be a set of houses under consideration and

$$\Delta = \{\nabla_1 = Morbled, \nabla_2 = Contemporary\},$$

and

$$\neg\Delta = \{\neg\nabla_1 = Wooden, \neg\nabla_2 = Conventional\}.$$

Consider a $5BS$-set that describe the house designs:
$(\Psi, \gamma, \Delta, 5) = \{(\langle\nabla_1, \{(\eth_1, 3), (\eth_2, 1)\}\rangle, \langle\neg\nabla_1, \{(\eth_1, 1), (\eth_2, 2)\}\rangle),$
$(\langle\nabla_2, \{(\eth_1, 2), (\eth_2, 3)\}\rangle, \langle\neg\nabla_2, \{(\eth_1, 1), (\eth_2, 2)\}\rangle)\}$
$\hat{G}^5_\Delta = \{\emptyset^5_\Delta, (\Psi, \gamma)_1, (\Psi, \gamma)_2\}$, where $(\Psi, \gamma)_1$,$(\Psi, \gamma)_2$ are $5BS$-subsets on $5BS$-set $(\Psi, \gamma, \Delta, 5)$ defined as follows:
$(\Psi, \gamma)_1 = \{(\langle\nabla_1, \{(\eth_1, 3), (\eth_2, 1)\}\rangle, \langle\neg\nabla_1, \{(\eth_1, 1), (\eth_2, 2)\}\rangle),$
$(\langle\nabla_2, \{(\eth_1, 2)\}\rangle, \langle\neg\nabla_2, \{(\eth_1, 1), (\eth_2, 2)\}\rangle)\},$
$(\Psi, \gamma)_2 = \{(\langle\nabla_1, \{(\eth_1, 3), (\eth_2, 1)\}\rangle, \langle\neg\nabla_1, \{(\eth_1, 1), (\eth_2, 2)\}\rangle),$
$(\langle\nabla_2, \{(\eth_1, 2)\}\rangle, \langle\neg\nabla_2, \{(\eth_1, 1)\}\rangle)\}.$
Then $((\Psi, \gamma, \Delta, 5), \hat{G}^5_\Delta)$ is 5-bipolar soft generalized topology spaces.
Let $(\Psi_1, \gamma_1, \Delta, 5) \sqsubseteq (\Psi, \gamma, \Delta, 5)$ such that

$$(\Psi_1, \gamma_1, \Delta, 5) = \{(\langle\nabla_1, \{(\eth_2, 1)\}\rangle, \langle\neg\nabla_1, \{(\eth_2, 2)\}\rangle)\}.$$

Then 5-bipolar soft closed super sets of $(\Psi_1, \gamma_1, \Delta, 5)$ are
$(\Psi, \gamma)^c_1 = \{(\langle\nabla_1, \{(\eth_1, 1), (\eth_2, 2)\}\rangle, \langle\neg\nabla_1, \{(\eth_1, 3), (\eth_2, 1)\}\rangle),$
$(\langle\nabla_2, \{(\eth_1, 1), (\eth_2, 2)\}\rangle, \langle\neg\nabla_2, \{(\eth_1, 2)\}\rangle)\}$, and
$(\Psi, \gamma)^c_2 = \{(\langle\nabla_1, \{(\eth_1, 1), (\eth_2, 2)\}\rangle, \langle\neg\nabla_1, \{(\eth_1, 3), (\eth_2, 1)\}\rangle),$
$(\langle\nabla_2, \{(\eth_1, 1)\}\rangle, \langle\neg\nabla_2, \{(\eth_1, 2)\}\rangle)\}.$
Now,

$$\begin{aligned} 5BS\hat{G}cl_{\hat{G}^5_\Delta}(\Psi_1, \gamma_1, \Delta, 5) &= 5BS\hat{G}cl_{\hat{G}^5_\Delta}(\Psi_1, \gamma_1, \Delta, 5)^c \\ &= \{(\langle\nabla_1, \{(\eth_1, 1), (\eth_2, 2)\}\rangle, \langle\neg\nabla_1, \{(\eth_1, 3), (\eth_2, 1)\}\rangle), \\ &\quad (\langle\nabla_2, \{(\eth_1, 1)\rangle\}, \langle\neg\nabla_2, \{(\eth_1, 2)\}\rangle)\}, \end{aligned}$$

$$\begin{aligned} b_{\hat{G}^5_\Delta}(\Psi_1, \gamma_1, \Delta, 5) &= 5BS\hat{G}cl_{\hat{G}^5_\Delta}(\Psi_1, \gamma_1, \Delta, 5) \cap 5BS\hat{G}cl_{\hat{G}^5_\Delta}(\Psi_1, \gamma_1, \Delta, 5)^c \\ &= \{(\langle\nabla_1, \{(\eth_1, 1), (\eth_2, 2)\}\rangle, \langle\neg\nabla_1, \{(\eth_1, 3), (\eth_2, 1)\}\rangle), \\ &\quad (\langle\nabla_2, \{(\eth_1, 1)\}\rangle, \langle\neg\nabla_2, \{(\eth_1, 2)\}\rangle)\}, \end{aligned}$$

$$\begin{aligned} &(\Psi_1, \gamma_1, \Delta, 5) \cup b_{\hat{G}^5_\Delta}(\Psi_1, \gamma_1, \Delta, 5) \\ &\quad = \{(\langle< \nabla_1, \{(\eth_1, 1), (\eth_2, 2)\}\rangle, \langle\neg\nabla_1, \{(\eth_1, 3), (\eth_2, 2)\}\rangle), \\ &\quad (\langle\nabla_2, \{(\eth_1, 1)\}\rangle, \langle\neg\nabla_2, \{(\eth_1, 2)\}\rangle)\}. \end{aligned}$$

Therefore, $5BS\hat{G}cl_{\hat{G}^5_\Delta}(\Psi_1,\gamma_1,\Delta,5) \not\sqsubseteq (\Psi_1,\gamma_1,\Delta,5)\cup b_{\hat{G}^5_\Delta}(\Psi_1,\gamma_1,\Delta,5)$.
Since $5BS\hat{G}int_{\hat{G}^5_\Delta}(\Psi_1,\gamma_1,\Delta,5)=\emptyset^5_\Delta$, then

$$(\Psi_1,\gamma_1,\Delta,5)\backslash b_{\hat{G}^5_\Delta}(\Psi_2,\gamma_2,\Delta,5) \not\sqsubseteq 5BS\hat{G}int_{\hat{G}^5_\Delta}(\Psi_1,\gamma_1,\Delta,5),$$

$$b_{\hat{G}^5_\Delta}(\Psi_2,\gamma_2,\Delta,5) \not\sqsubseteq b_{\hat{G}^5_\Delta}(5BS\hat{G}int_{\hat{G}^5_\Delta}(\Psi_1,\gamma_1,\Delta,5)),$$

$$\begin{aligned} & b_{\hat{G}^5_\Delta}(5BS\hat{G}cl_{\hat{G}^5_\Delta}(\Psi_1,\gamma_1,\Delta,5)) \\ & \quad =\{(\langle\nabla_1,\{(\eth_1,1),(\eth_2,1)\}\rangle,\langle\neg\nabla_1,\{(\eth_1,3),(\eth_2,2)\}\rangle), \\ & \quad\quad \langle\nabla_2,\{(\eth_1,1)\}\rangle,\langle\neg\nabla_2,\{(\eth_1,2)\}\rangle)\}. \end{aligned}$$

This implies that $b_{\hat{G}^5_\Delta}(\Psi_2,\gamma_2,\Delta,5) \not\sqsubseteq b_{\hat{G}^5_\Delta}(5BS\hat{G}cl_{\hat{G}^5_\Delta}(\Psi_1,\gamma_1,\Delta,5))$.

Theorem 2.25. *Let* $((\Psi,\gamma,\Delta,N),\hat{G}^N_\Delta$ *) be an* $NBS\hat{G}T_S$ *and* $(\Psi_1,\gamma_1,\Delta,N)$ *be an* NBS*-subset of* (Ψ,γ,Δ,N)*. Then*

$$b_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)\cap NBS\hat{G}int_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)=\emptyset^N_\Delta.$$

Proof.

$$\begin{aligned} & b_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)\cap NBS\hat{G}int_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N) \\ & \quad =(NBS\hat{G}cl_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)\backslash NBS\hat{G}int_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)) \\ & \quad\quad \cap NBS\hat{G}int_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N) \\ & \quad =NBS\hat{G}cl_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)\cap(NBS\hat{G}int_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N))^c \\ & \quad\quad \cap NBS\hat{G}int_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N) \\ & \quad =NBS\hat{G}cl_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)\cap(\Psi_1,\gamma_1,\Delta,N)=\emptyset^N_\Delta. \end{aligned}$$

□

Theorem 2.26. *Let* $((\Psi,\gamma,\Delta,N),\hat{G}^N_\Delta)$ *be an* $NBS\hat{G}T_S$ *and* $(\Psi_1,\gamma_1,\Delta,N)$ *be an* NBS*-subset of* (Ψ,γ,Δ,N)*. Then the following properties hold:*

(1) *If* $(\Psi_1,\gamma_1,\Delta,N)\in\hat{G}^N_\Delta$*, then* $(\Psi_1,\gamma_1,\Delta,N)\cap b_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)=\emptyset^N_\Delta$.

(2) *If* $(\Psi_1,\gamma_1,\Delta,N)$ *is an* $NBS\hat{G}^N_\Delta$*-closed, then*

$$b_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)\sqsubseteq(\Psi_1,\gamma_1,\Delta,N).$$

Proof. (1) Suppose that $(\Psi_1, \gamma_1, \Delta, N) \in \hat{G}^N_\Delta$. Then by Theorem 2.14 (3), we have $(\Psi_1, \gamma_1, \Delta, N) = NBS\hat{G}int_{\hat{G}^N_\Delta}(\Psi_1, \gamma_1, \Delta, N)$. Since $NBS\hat{G}int_{\hat{G}^N_\Delta}$ $(\Psi_1, \gamma_1, \Delta, N) \sqsubseteq (b_{\hat{G}^N_\Delta}(\Psi_1, \gamma_1, \Delta, N))^c$, it implies that

$$(\Psi_1, \gamma_1, \Delta, N) \sqsubseteq (b_{\hat{G}^N_\Delta}\Psi_1, \gamma_1, \Delta, N))^c.$$

Therefore, $(\Psi_1, \gamma_1, \Delta, N) \cap b_{\hat{G}^N_\Delta}\ (\Psi_1, \gamma_1, \Delta, N) = \emptyset^N_\Delta$.

(2) By Theorem 2.23 (1), we have

$$b_{\hat{G}^N_\Delta}(\Psi_1, \gamma_1, \Delta, N) \sqsubseteq NBS\hat{G}cl_{\hat{G}^N_\Delta}(\Psi_1, \gamma_1, \Delta, N).$$

Since $(\Psi_1, \gamma_1, \Delta, N)$ is an $NBS\hat{G}^N_\Delta$-closed set, then

$$b_{\hat{G}^N_\Delta}(\Psi_1, \gamma_1, \Delta, N) \sqsubseteq (\Psi_1, \gamma_1, \Delta, N).$$

□

Theorem 2.27. *Let* $((\Psi, \gamma, \Delta, N), \hat{G}^N_\Delta)$ *be an* $NBS\hat{G}T_S$ *and* $(\Psi_1, \gamma_1, \Delta, N)$ *be an* NBS*-subset of* $(\Psi, \gamma, \Delta, N)$. *If* $(\Psi_1, \gamma_1, \Delta, N)$ *is both an* $NBS\hat{G}^N_\Delta$*-open and* $NBS\hat{G}^N_\Delta$*-closed. Then* $b_{\hat{G}^N_\Delta}(\Psi_1, \gamma_1, \Delta, N) = \emptyset^N_\Delta$.

Proof. Assume that $(\Psi_1, \gamma_1, \Delta, N)$ is an $NBS\hat{G}^N_\Delta$-open and $NBS\hat{G}^N_\Delta$-closed. Then,

$$\begin{aligned} b_{\hat{G}^N_\Delta}(\Psi_1, \gamma_1, \Delta, N) &= NBS\hat{G}cl_{\hat{G}^N_\Delta}(\Psi_1, \gamma_1, \Delta, N) \cap NBS\hat{G}cl_{\hat{G}^N_\Delta}(\Psi_1, \gamma_1, \Delta, N)^c \\ &= NBS\hat{G}cl_{\hat{G}^N_\Delta}(\Psi_1, \gamma_1, \Delta, N) \cap (NBS\hat{G}int_{\hat{G}^N_\Delta}(\Psi_1, \gamma_1, \Delta, N))^c \\ &= (\Psi_1, \gamma_1, \Delta, N) \cap (\Psi_1, \gamma_1, \Delta, N)^c \\ &= \emptyset^N_\Delta. \end{aligned}$$

□

Definition 2.28. Let $((\Psi, \gamma, \Delta, N), \hat{G}^N_\Delta)$ be an $NBS\hat{G}T_S$ and $(\Psi_1, \gamma_1, \Delta, N)$ be an NBS-subset of $(\Psi, \gamma, \Delta, N)$. Then the $NBS\hat{G}^N_\Delta$-exterior of $(\Psi_1, \gamma_1, \Delta, N)$, denoted by $ext_{\hat{G}^N_\Delta}\ (\Psi_1, \gamma_1, \Delta, N)$, is an $NBS\hat{G}^N_\Delta$-interior of the an $NBS\hat{G}^N_\Delta$-complement of $(\Psi_1, \gamma_1, \Delta, N)$. In the other word, $ext_{\hat{G}^N_\Delta}\ (\Psi_1, \gamma_1, \Delta, N) = NBS\hat{G}int_{\hat{G}^N_\Delta}(\Psi_1, \gamma_1, \Delta, N)^c$.

Theorem 2.29. *Let* $((\Psi, \gamma, \Delta, N), \hat{G}^N_\Delta)$ *be an* $NBS\hat{G}T_S$ *and* $(\Psi_1, \gamma_1, \Delta, N)$ *be an* NBS*-subset of* $(\Psi, \gamma, \Delta, N)$. *Then*

(1) $ext_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)$ *is a largest* $NBS\hat{G}^N_\Delta$*-open set containing in* $(\Psi_1,\gamma_1,\Delta,N)^c$;

(2) $ext_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)=(NBS\hat{G}cl_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N))^c$;

(3) $ext_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)^c=NBS\hat{G}int_{\hat{G}^N_\Delta}(NBS\hat{G}int_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)^c)^c$ $=NBS\hat{G}int_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)$;

(4) $NBS\hat{G}int_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)\cap(\Psi_1,\gamma_1,\Delta,N)=\emptyset^N_\Delta$.

Proof. It is easy to prove the theorem. □

Theorem 2.30. *Let* $((\Psi,\gamma,\Delta,N),\hat{G}^N_\Delta)$ *be an* $NBS\hat{G}T_S$ *and* $(\Psi_1,\gamma_1,\Delta,N)$, $(\Psi_2,\gamma_2,\Delta,N)$ *be two NBS-subsets of* (Ψ,γ,Δ,N). *Then*

(1) $ext_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)\sqsubseteq(\Psi_1,\gamma_1,\Delta,N)^c$;

(2) $ext_{\hat{G}^N_\Delta}(ext_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N))^c=ext_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)$;

(3) *If* $(\Psi_1,\gamma_1,\Delta,N)\sqsubseteq(\Psi_2,\gamma_2,\Delta,N)$, *then* $ext_{\hat{G}^N_\Delta}(\Psi_2,\gamma_2,\Delta,N)\sqsubseteq ext_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)$;

(4) $NBS\hat{G}int_{\hat{G}^N_\Delta}(\Psi_2,\gamma_2,\Delta,N)\sqsubseteq ext_{\hat{G}^N_\Delta}(ext_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N))$;

(5) $ext_{\hat{G}^N_\Delta}((\Psi_1,\gamma_1,\Delta,N)\cap(\Psi_2,\gamma_2,\Delta,N))$ $\sqsupseteq ext_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)\cup ext_{\hat{G}^N_\Delta}(\Psi_2,\gamma_2,\Delta,N)$;

(6) $ext_{\hat{G}^N_\Delta}((\Psi_1,\gamma_1,\Delta,N)\cup(\Psi_2,\gamma_2,\Delta,N))$ $\sqsubseteq ext_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)\cap ext_{\hat{G}^N_\Delta}(\Psi_2,\gamma_2,\Delta,N)$.

Proof. It is easy to prove the theorem, so it is omitted. □

Theorem 2.31. *Let* $((\Psi,\gamma,\Delta,N),\hat{G}^N_\Delta)$ *be an* $NBS\hat{G}T_S$ *and* $(\Psi_1,\gamma_1,\Delta,N)$ *be an NBS-subset of* (Ψ,γ,Δ,N). *Then*

(1) $(b_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N))^c=NBS\hat{G}int_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)\cup ext_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)$.

(2) $b_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)\cup NBS\hat{G}int_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)\cup ext_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)$ $\sqsubseteq(\Psi,\gamma,\Delta,N)$.

Proof. (1) We have

$$\begin{aligned}(b_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N))^c&=NBS\hat{G}int_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)\\&\quad\cup NBS\hat{G}int_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)^c\\&=NBS\hat{G}int_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)\\&\quad\cup ext_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N).\end{aligned}$$

(2) From (1), we have

$$\begin{aligned}(b_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N))^c = {} & NBS\hat{G}int_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)\\ & \cup\, ext_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N).\end{aligned}$$

Therefore,

$$\begin{aligned}& b_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N) \cup NBS\hat{G}int_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N) \cup ext_{\hat{G}^N_\Delta}(\Psi_1,\gamma_1,\Delta,N)\\ & \quad \sqsubseteq (\Psi,\gamma,\Delta,N).\end{aligned}$$

□

3. DECISION MAKING VIA N-BIPOLAR SOFT GENERALIZED TOPOLOGY

Multi-criteria group decision making is a useful study topic with substantial theoretical and practical underpinnings. It addresses the task of categorizing or prioritizing the options by considering the viewpoints provided by diverse professionals regarding multiple criteria. This section focuses on evaluating the effectiveness of $NBS\hat{G}T_S$ in group decision making involving multiple criteria. We examine two decision experts, Δ_1 and Δ_2, and their associated two $NBS\hat{G}^N_\Delta$-open sets, $\mathbb{Q}_{\Delta_1}$and $\mathbb{Q}_{\Delta_2}$, in the context of $NBS\hat{G}T_S$. Rather than a single NBS-set (single opinion), $NBS\hat{G}T_S$ offers a more generalized group decision making involving multiple criteria. Two Algorithms, 1 and 2, are shown here for group decision making involving multiple criteria, based on $NBS\hat{G}T_S$.

We observe that the results extracted from both methods are identical, which supports the usefulness of the suggested techniques.

Algorithm 3.1. 1. Input $\check{D}=\{\eth_1,\eth_2,\ldots,\eth_p\}$ and $\Delta=\{\nabla_1,\nabla_2,\ldots,\nabla_q\}$.

2. Input the NBS-set (Ψ,γ,Δ,N) such that for each $\eth_i\in\check{D};\nabla_j\in\Delta$, there exist $r_{ij},\acute{r}_{ij}\in\check{R}$ where $N=\{2,3,\cdots\}$ and $\check{R}=\{0,1,\cdots,N-1\}$. After that, provide it in tabular form.
3. Input $\mathbb{Q}_{\Delta_1}$and $\mathbb{Q}_{\Delta_2}$ which are two NBS-subsets of (Ψ,γ,Δ,N).
4. Construct the N-bipolar soft generalized topology $\hat{G}^N_\Delta$ where $\mathbb{Q}_{\Delta_1}$ and $\mathbb{Q}_{\Delta_2}$ are N-bipolar soft sets in $\hat{G}^N_\Delta$.
5. Calculate the total NBS-set of all $NBS\hat{G}^N_\Delta$-open sets by the formula,
   $\mathbb{Q}^*_\Delta=[\frac{\mathbb{Q}^*_{\Delta_1}(\eth_i)}{\eth_i}:\eth_i\in\check{D}]$, where
   $\mathbb{Q}^*_\Delta=\mathbb{Q}^{+*}_\Delta(\eth_i)-\mathbb{Q}^{-*}_\Delta(\eth_i)$,$\mathbb{Q}^{+*}_\Delta(\eth_i)=\sum_j r_{ij}$ and $\mathbb{Q}^{-*}_\Delta(\eth_i)=\sum_j \acute{r}_{ij}$.
6. Add $\mathbb{Q}_{\Delta_1}$ and $\mathbb{Q}_{\Delta_2}$ to find decision NBS-set.
7. The optimal selection is the highest grading provided by max $\mathbb{Q}^*_{\Delta_1\oplus\Delta_2}(\eth_i)$ from step 6.

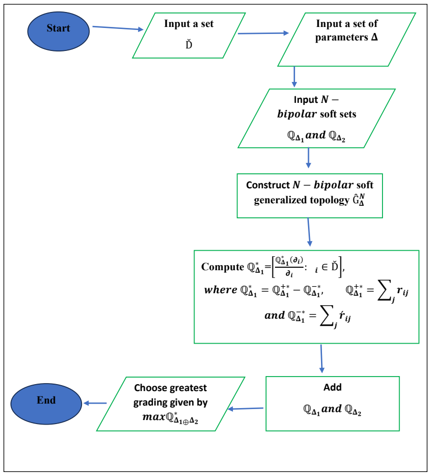

Figure 1: Algorithm 1

1.pdf

Example 3.2. Let $\check{D} = \{\eth_1, \eth_2, \eth_3, \eth_4, \eth_5\}$ be a set of houses under consideration, $\Delta = \{\nabla_1, \nabla_2, \nabla_3, \nabla_4, \nabla_5\}, \neg\Delta = \{\neg\nabla_1, \neg\nabla_2, \neg\nabla_3, \neg\nabla_4, \neg\nabla_5\}$. Consider a $5BS$-set which describes the house designs:

$$
\begin{aligned}
(\Psi, \gamma, \Delta, 5) = \{ & (\langle \nabla_1, \{(\eth_1, 3), (\eth_2, 4), (\eth_3, 3), (\eth_4, 3), (\eth_5, 2)\} \rangle , \\
& \langle \neg\nabla_1, \{(\eth_1, 1), (\eth_2, 0), (\eth_3, 0), (\eth_4, 1), (\eth_5, 1)\} \rangle), \\
& (\langle \nabla_2, \{(\eth_1, 3), (\eth_2, 0), (\eth_3, 2), (\eth_4, 2), (\eth_5, 1)\} \rangle , \\
& \langle \neg\nabla_2, \{(\eth_1, 0), (\eth_2, 4), (\eth_3, 1), (\eth_4, 0), (\eth_5, 0)\} \rangle), \\
& (\langle \nabla_3, \{(\eth_1, 2), (\eth_2, 1), (\eth_3, 3), (\eth_4, 4), (\eth_5, 2)\} \rangle , \\
& \langle \neg\nabla_3, \{(\eth_1, 1), (\eth_2, 3), (\eth_3, 1), (\eth_4, 0), (\eth_5, 1)\} \rangle), \\
& (\langle \nabla_4, \{(\eth_1, 0), (\eth_2, 4), (\eth_3, 1), (\eth_4, 1), (\eth_5, 3)\} \rangle , \\
& \langle \neg\nabla_4, \{(\eth_1, 4), (\eth_2, 0), (\eth_3, 2), (\eth_4, 0), (\eth_5, 1)\} \rangle), \\
& (\langle \nabla_5, \{(\eth_1, 4), (\eth_2, 2), (\eth_3, 1), (\eth_4, 2), (\eth_5, 3)\} \rangle , \\
& \langle \neg\nabla_5, \{(\eth_1, 0), (\eth_2, 1), (\eth_3, 2), (\eth_4, 1), (\eth_5, 1)\} \rangle)\}.
\end{aligned}
$$

Let $\Delta_1 = \{\nabla_1, \nabla_2, \nabla_3\}, \Delta_2 = \{\nabla_1, \nabla_2\} \subseteq \Delta$. Consider the $5BS$-subsets $\mathbb{Q}_{\Delta_1} = (\Psi_1, \gamma_1, \Delta_1, 5)$ and $\mathbb{Q}_{\Delta_2} = (\Psi_2, \gamma_2, \Delta_2, 5)$ described in Tables 1 and 2 respectively.

$\mathbb{Q}_{\Delta_1}$	$(\nabla_1, \neg\nabla_1)$	$(\nabla_2, \neg\nabla_2)$	$(\nabla_3, \neg\nabla_3)$
$\eth_1$	$(3,1)$	$(4,0)$	$(1,2)$
$\eth_2$	$(4,0)$	$(0,4)$	$(0,4)$
$\eth_3$	$(3,1)$	$(4,0)$	$(3,1)$
$\eth_4$	$(3,1)$	$(0,4)$	$(4,0)$
$\eth_5$	$(0,4)$	$(1,0)$	$(0,4)$

Table1 $5-$ bipolar soft set $\mathbb{Q}_{\Delta_1}$

$\mathbb{Q}_{\Delta_2}$	$(\nabla_1, \neg\nabla_1)$	$(\nabla_2, \neg\nabla_2)$
$\eth_1$	$(3,1)$	$(3,0)$
$\eth_2$	$(0,4)$	$(0,4)$
$\eth_3$	$(3,1)$	$(0,4)$
$\eth_4$	$(3,1)$	$(2,0)$
$\eth_5$	$(0,4)$	$(0,4)$

Table2 $5-$ bipolar soft set $\mathbb{Q}_{\Delta_2}$

$\hat{G}^5_\Delta = \{\emptyset^N_\Delta, \mathbb{Q}_{\Delta_1}, \mathbb{Q}_{\Delta_2}\}$ is a 5-bipolar soft generalized topology $\hat{G}^5_\Delta$ on $(\Psi, \gamma, \Delta, 5)$. Therefore, we calculate the aggregate $5BS$-sets of all $5BS\hat{G}^5_\Delta$-open sets in the following manner:

$\mathbb{Q}^*_{\Delta_1} = \left[\frac{5}{\eth_1}, \frac{-4}{\eth_2}, \frac{8}{\eth_3}, \frac{2}{\eth_4}, \frac{-7}{\eth_5}\right]$, $\mathbb{Q}^*_{\Delta_2} = \left[\frac{5}{\eth_1}, \frac{-8}{\eth_2}, \frac{-2}{\eth_3}, \frac{4}{\eth_4}, \frac{-8}{\eth_5}\right]$

$\mathbb{Q}^*_{\Delta_1 \oplus \Delta_2}(\eth_i) = \mathbb{Q}^*_{\Delta_1}(\eth_i) + \mathbb{Q}^*_{\Delta_2}(\eth_i)$ for all $\eth_i \in \check{D}$.

So, $\mathbb{Q}^*_{\Delta_1}(\eth_i) + \mathbb{Q}^*_{\Delta_2}(\eth_i) = \left[\frac{10}{\eth_1}, \frac{-12}{\eth_2}, \frac{6}{\eth_3}, \frac{6}{\eth_4}, \frac{-15}{\eth_5}\right]$. Since$max\mathbb{Q}^*_{\Delta_1 \oplus \Delta_2}(\eth_i) = 10$, the employee $\eth_1$ is selected and the ranking decision is $\eth_1 > \eth_3 = \eth_4 > \eth_2 > \eth_5$. Thus, the example demonstrates how 5-bipolar soft generalized topology can be effectively applied to multi-criteria decision-making problems by systematically aggregating positive and negative information.

Algorithm 3.3. 1. Input $\check{D} = \{\eth_1, \eth_2, \dots, \eth_p\}$ and $\Delta = \{\nabla_1, \nabla_2, \dots, \nabla_q\}$

2. Input the NBS-set $(\Psi, \gamma, \Delta, N)$ such that for each $\eth_i \in \check{D}; \nabla_j \in \Delta, \exists r_{ij}, \acute{r}_{ij} \in \check{R}$ where $N = \{2, 3, ...\}$ and $\check{R} = \{0, 1, ..., N-1\}$. After that, provide it in tabular form.
3. Input $\mathbb{Q}_{\Delta_1}$and $\mathbb{Q}_{\Delta_2}$ which are two NBS-subsets of $(\Psi, \gamma, \Delta, N)$.
4. Construct the N-bipolar soft generalized topology $\hat{G}^N_\Delta$ where $\mathbb{Q}_{\Delta_1}$ and $\mathbb{Q}_{\Delta_2}$ are N-bipolar soft sets in $\hat{G}^N_\Delta$.
5. Calculate the total NBS-set of all $NBS\hat{G}^N_\Delta$-open sets by the formula,

$$c\mathbb{Q}^{+}_{\Delta} = \left[\frac{c\mathbb{Q}^{+}_{\Delta_1}(\nabla_j)}{\nabla_j} : \nabla_j \in \Delta\right] \text{ and } c\mathbb{Q}^{-}_{\Delta} = \left[\frac{c\mathbb{Q}^{-}_{\Delta_1}(\nabla_j)}{\neg\nabla_j} : \neg\nabla_j \in \neg\Delta\right],$$

where $\mathbb{Q}^{+}_{\Delta}(\nabla_j) = \sum_j r_{ij}$ and $\mathbb{Q}^{-}_{\Delta}\ (\nabla_j) = \sum_j \acute{r}_{ij}$.

6. Locate the positive total NBS-sets by the formula $M_{\mathbb{Q}^{+*}_{\Delta_1}} = M_{\mathbb{Q}^{+}_{\Delta_1}} + M^{T}_{c\mathbb{Q}^{+}_{\Delta_1}}$, where $M_{\mathbb{Q}^{+*}_{\Delta_1}}$, $M_{\mathbb{Q}^{+}_{\Delta_1}}$ and $M^{T}_{c\mathbb{Q}^{+}_{\Delta_1}}$ are representation matrices of $\mathbb{Q}^{+*}_{\Delta_1}$, $\mathbb{Q}^{+}_{\Delta_1}$ and $c\mathbb{Q}^{+}_{\Delta_1}$ respectively. Also $M^{T}_{c\mathbb{Q}^{+}_{\Delta_1}}$ is transpose of matrix $M_{c\mathbb{Q}^{+}_{\Delta_1}}$.
7. Locate the negative total NBS-sets by the formula $M_{\mathbb{Q}^{-*}_{\Delta_1}} = M_{\mathbb{Q}^{-}_{\Delta_1}} + M^{T}_{c\mathbb{Q}^{-}_{\Delta_1}}$, where $M_{\mathbb{Q}^{-*}_{\Delta_1}}$, $M_{\mathbb{Q}^{-}_{\Delta_1}}$ and $M^{T}_{c\mathbb{Q}^{-}_{\Delta_1}}$ are representation matrices of $\mathbb{Q}^{-*}_{\Delta_1}$, $\mathbb{Q}^{-}_{\Delta_1}$ and $c\mathbb{Q}^{-}_{\Delta_1}$ respectively. Also $M^{T}_{c\mathbb{Q}^{-}_{\Delta_1}}$ is transpose of matrix $M_{c\mathbb{Q}^{-}_{\Delta_1}}$.
8. Locate the total NBS-sets by the formula $M_{\mathbb{Q}^{*}_{\Delta_1}} = M_{\mathbb{Q}^{+*}_{\Delta_1}} - M_{\mathbb{Q}^{-*}_{\Delta_1}}$.
9. Add $\mathbb{Q}^{*}_{\Delta_1}$ and $\mathbb{Q}^{*}_{\Delta_2}$ to find decision NBS-set.
10. The optimal selection is the highest grading provided by $max\mathbb{Q}^{*}_{\Delta_1\oplus\Delta_2}(\eth_i)$ from step 9.

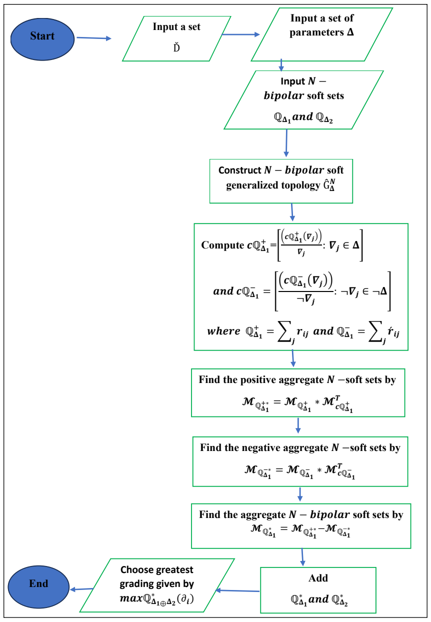

Figure 2: Algorithm 2

2.pdf

Example 3.4. Consider the $5BS$-set $(\Psi, \gamma, \Delta, 5)$, $\mathbb{Q}_{\Delta_1} = (\Psi_1, \gamma_1, \Delta_1, 5)$ and $\mathbb{Q}_{\Delta_2} = (\Psi_2, \gamma_2, \Delta_2, 5)$ given in Example 3.1.

Once again, $\hat{G}^N_\Delta = \{\emptyset^N_\Delta, \mathbb{Q}_{\Delta_1}, \mathbb{Q}_{\Delta_2})\}$ is a $5-$bipolar soft generalized topology $\hat{G}^N_\Delta$ as

5. $c\mathbb{Q}^+_{\Delta_1} = \left[\frac{13}{\nabla_1}, \frac{9}{\nabla_2}, \frac{8}{\nabla_3}\right]$ and $c\mathbb{Q}^-_{\Delta_1} = \left[\frac{7}{\neg\nabla_1}, \frac{8}{\neg\nabla_2}, \frac{11}{\neg\nabla_3}\right]$.
   Also $c\mathbb{Q}^+_{\Delta_2} = \left[\frac{9}{\nabla_1}, \frac{5}{\nabla_2}\right]$ and $c\mathbb{Q}^-_{\Delta_2} = \left[\frac{11}{\neg\nabla_1}, \frac{12}{\neg\nabla_2}\right]$.

6. $$M_{\mathbb{Q}^{+*}_{\Delta_1}} = \begin{pmatrix} 3 & 4 & 1 \\ 4 & 0 & 0 \\ 3 & 4 & 3 \\ 3 & 0 & 4 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 13 \\ 9 \\ 8 \end{pmatrix} = \begin{pmatrix} 83 \\ 52 \\ 99 \\ 63 \\ 9 \end{pmatrix}$$
   and $$M_{\mathbb{Q}^{+*}_{\Delta_2}} = \begin{pmatrix} 3 & 3 \\ 0 & 0 \\ 3 & 0 \\ 3 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 9 \\ 5 \end{pmatrix} = \begin{pmatrix} 42 \\ 0 \\ 27 \\ 37 \\ 0 \end{pmatrix}$$

7. $$M_{\mathbb{Q}^{-*}_{\Delta_1}} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 4 & 4 \\ 1 & 0 & 1 \\ 1 & 4 & 0 \\ 4 & 0 & 4 \end{pmatrix} \begin{pmatrix} 7 \\ 8 \\ 11 \end{pmatrix} = \begin{pmatrix} 29 \\ 76 \\ 18 \\ 39 \\ 72 \end{pmatrix} \text{ and}$$
   $$M_{\mathbb{Q}^{-*}_{\Delta_2}} = \begin{pmatrix} 1 & 0 \\ 4 & 4 \\ 1 & 4 \\ 1 & 0 \\ 4 & 4 \end{pmatrix} \begin{pmatrix} 11 \\ 12 \end{pmatrix} = \begin{pmatrix} 11 \\ 92 \\ 59 \\ 11 \\ 92 \end{pmatrix}$$

8. $$M_{\mathbb{Q}^*_{\Delta_1}} = M_{\mathbb{Q}^{+*}_{\Delta_1}} - M_{\mathbb{Q}^{-*}_{\Delta_1}} = \begin{pmatrix} 83 \\ 52 \\ 99 \\ 63 \\ 9 \end{pmatrix} - \begin{pmatrix} 29 \\ 76 \\ 18 \\ 39 \\ 72 \end{pmatrix} = \begin{pmatrix} 54 \\ -24 \\ 81 \\ 24 \\ -63 \end{pmatrix}$$
   and that means $\mathbb{Q}^*_{\Delta_1} = \left[\frac{54}{\eth_1}, \frac{-24}{\eth_2}, \frac{81}{\eth_3}, \frac{24}{\eth_4}, \frac{-63}{\eth_5}\right]$.
   $$M_{\mathbb{Q}^*_{\Delta_2}} = M_{\mathbb{Q}^{+*}_{\Delta_2}} - M_{\mathbb{Q}^{-*}_{\Delta_2}} = \begin{pmatrix} 32 \\ 0 \\ 27 \\ 37 \\ 0 \end{pmatrix} - \begin{pmatrix} 11 \\ 92 \\ 59 \\ 11 \\ 92 \end{pmatrix} = \begin{pmatrix} 31 \\ -92 \\ -32 \\ 26 \\ -92 \end{pmatrix} \text{ and}$$
   that means $\mathbb{Q}^*_{\Delta_2} = \left[\frac{31}{\eth_1}, \frac{-92}{\eth_2}, \frac{-32}{\eth_3}, \frac{26}{\eth_4}, \frac{-92}{\eth_5}\right]$.

$$\mathbb{Q}^*_{\Delta_1 \oplus \Delta_2}(\eth_i) = \mathbb{Q}^*_{\Delta_1}(\eth_i) + \mathbb{Q}^*_{\Delta_2}(\eth_i) \text{ for all } \eth_i \in \check{D}.$$

So,

$$\mathbb{Q}^*_{\Delta_1}(\eth_i) + \mathbb{Q}^*_{\Delta_2}(\eth_i) = \left[\frac{85}{\eth_1}, \frac{-116}{\eth_2}, \frac{49}{\eth_3}, \frac{50}{\eth_4}, \frac{-155}{\eth_5}\right].$$

Since $max\mathbb{Q}^*_{\Delta_1 \oplus \Delta_2} = 83$, the employee $\eth_1$ is chosen and the ranking decision is $\eth_1 > \eth_4 > \eth_3 > \eth_2 > \eth_5$.

Thus, the example demonstrates the effectiveness of the matrix aggregation approach in enhancing multi-criteria decision-making under 5-bipolar soft generalized topological structures.

4. CONCLUSION

In this article, we present the concept of $NBS\hat{G}T_S$ which is defined on an initial NBS-set and gave basic definitions and theorems of concept $NBS\hat{G}^N_\Delta$-interior, $NBS\hat{G}^N_\Delta$-closure, $NBS\hat{G}^N_\Delta$-boundary and $NBS\hat{G}^N_\Delta$-exterior. We are optimistic that the discoveries presented in this research paper will serve as a catalyst for researchers to augment and advance the study of $NBS\hat{G}T_S$. We have presented two algorithms for multi-criteria group decision-making, employing NBS-sets and $NBS\hat{G}T_S$. The flow of Algorithms 1 and Algorithm 2 can be observed in figures 1 and 2 respectively.

REFERENCES

[1] M. Akdag and A. Ozkan, *Soft b-open sets and soft b-continuous functions,* Math Sci, **8**(127) (2014), 1–9.

[2] M. Akdag and A. Ozkan, *Soft α-open sets and soft α-continuous functions,* Abstr. Anal. Appl., (2014), 1–7, https://doi.org/10.1155/2014/891341.

[3] F.Y. Al-Quhali, A.R. Abdel-Malek, E. El-Seidy and A.E. Radwan, *On $N-$Bipolar Soft Continuous Mappings with Application in Omicron Disease,* Appl. Math. Inf. Sci., **19**(3) (2025), 635–646.

[4] A. Csaszar, *Generalized toplogy, generalized continuity,* Acta Math. Hunga., **96** (2002), 351–357.

[5] D. Dubois and H. Prade, *An introduction to bipolar representations of information and preference,* Int. J. Intell. Syst., **23**(8) (2008), 866–877.

[6] F. Fatimah, D. Rosadi, R. Hakim and J. Alcantud, *N-soft sets and their decision-making algorithms,* Soft Comput., **22** (2018), 3829–3842.

[7] S.Z. Hameed, H.I. Mustafa, A. E. Radwan and E. El-Seidy, *N-Bipolar Soft cnnectedness, disconnectedness and compactness spaces with application in infinite game,* Nonlinear Funct. Anal. Appl., **30**(1) (2025), 25–42.

[8] S.Z. Hameed, A.E. Radwan and E. El-Seidy, *On Soft Strongly b^*-Separation Axioms,* Iraqi J. Sci., **65**(6) (2024), 3259–3269.

[9] H. Kamaci and S. Petchimuthu, *Bipolar N-soft set theory with applications,* Soft Comput., **24** (2020), 16727–16743.

[10] D. Molodtsov, *Soft set theory First results,* Comput. Math. Appl., **57**(4-5) (1999), 19–31.

[11] H.I. Mustafa, *Generalized Bipolar-Soft Sets, Generalized Bipolar-Soft Topology and their Decision Making*, Filomat, **35**(13), (2021), 4587–4611.

[12] A.E. Radwan, E. El-Seidy and S.Z. Hameed, *On Soft Strongly* b^**-Closed Via Soft Ideal,* Commu. Appl. Nonlinear Anal., **31**(2) (2024), 284–293.

[13] H. Saleh, B. Asaad and R. Mohammed, *Bipolar soft generalized topological structures and their application in decision making,* Euro. J. Pure Appl. Math., **15**(2) (2022), 646–671.

[14] Shabir and M. Naz, *On soft topological spaces,* Comput. Math. Appl., **61**(7) (2011), 1786–1799.

[15] M. Shabir and J. Fatima, N*-Bipolar Soft Sets and Their Application in Decision Making,* (2021). DOI:10.21203/rs.3.rs-755020/v1.

[16] M. Shabir and M. Naz, *On bipolar soft sets,* Cornell University, arXiv preprint arXiv:1303.1344, (2013). doi.org/10.48550/arXiv.1303.1344.

[17] J. Thomas and S.J. Johna, *On soft generalized topological spaces,* J. New Results in Sci., **4**(1304-7981) (2014), 1–15.