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FIXED POINT THEOREMS VIA CONTRACTIVE FIXED POINT IN G-METRIC SPACES

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Abstract. The study of the existence of contractive fixed points for self-maps in metric spaces was carried out by Edelstein, Hoyle, Leader, and Reich. More recently, Kumara Swamy and Saravanan introduced a conjecture in the setting of G-metric spaces. In this paper, we establish several new fixed point theorems based on this conjecture and on contractive fixed points in G-metric spaces.

1. Introduction

Let h be a mapping on a metric space $(\mathcal{M}, d)$ and $\phi_0 \in \mathcal{M}$. The orbit $O_f(\phi_0)$ at ϕ_0 is the sequence of h-iterates $\langle \phi_0, h\phi_0, ..., h^n\phi_0, ...\rangle$. If every orbit $O_h(\phi_0)$ converges to a fixed point ρ of h, then ρ will become a contractive fixed point ρ of h. It is fact that convergent sequence in metric space has a

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unique limit, which indicates that a contractive fixed point is always a unique fixed point in a G-metric space.

In 2006, the G-metric was initiated by Mustafa et al. [9].

Definition 1.1. Let Y be a set which is nonempty and $G : Y \times Y \times Y \to [0, \infty)$ such that

(G1) $G(\tau, \nu, \varphi) \geq 0$ for all $\tau, \nu, \varphi \in Y$ with $G(\tau, \nu, \varphi) = 0$ if $\tau = \nu = \varphi$,
(G2) $G(\tau, \tau, \nu) > 0$ for all $\tau, \nu \in Y$ with $\tau \neq \nu$,
(G3) $G(\tau, \tau, \nu) \leq G(\tau, \nu, \varphi)$ for all $\tau, \nu \in Y$ with $\varphi \neq \nu$,
(G4) $G(\tau, \nu, \varphi) = G(\tau, \varphi, \nu) = G(\nu, \tau, \varphi) = G(\varphi, \tau, \nu) = G(\nu, \varphi, \tau)$ $= G(\varphi, \nu, \tau)$ for all $\tau, \nu, \varphi \in Y$,
(G5) $G(\tau, \nu, \varphi) \leq G(\tau, \varpi, \varpi) + G(\varpi, \nu, \varphi)$ for all $\tau, \nu, \varphi, \varpi \in Y$.

Then (Y, G) is called a G-metric space with G-metric G.

Note that (Y, G) indicates G-metric space throughout this paper. The following propositions are used to establish our proofs:

Proposition 1.2. ([11]) *Let (Y, G) be a G-metric space. Then*

$$G(\tau, \nu, \nu) \leq 2G(\tau, \tau, \nu) \text{ for } \tau, \nu \in Y. \tag{1.1}$$

Proposition 1.3. ([11]) *Let (Y, G) be a G-metric space.*

$$\text{If } G(\tau, \nu, \nu) = 0 \text{ for } \tau, \nu \in Y, \text{ then } \tau = \nu. \tag{1.2}$$

Mustafa et al. developed the following notions in (Y, G):

Definition 1.4. ([9]) Let (Y, G) be a G-metric space. A G-ball in Y is defined by $B_G(\tau, r) = \{\tau \in Y : G(\tau, \tau, \nu) < r\}$. It is straightforward to verify that the collection of all G-balls constitutes a basis for a topology, known as the G-metric topology $T(G)$ on Y.

Definition 1.5. ([9]) Let (Y, G) be a G-metric space. A sequence $\{\tau_n\}_{n=1}^{\infty}$ in a (Y, G) is said to be G-convergent to λ if it converges to λ with respect to the G-metric topology $T(G)$ on Y.

Lemma 1.6. ([9]) *Let (Y, G) be a G-metric space. Then the following statements are equivalent:*

(a) *$\{\tau_n\}_{n=1}^{\infty} \subset Y$ is G-convergent to an element $\rho \in Y$,*
(b) *$\lim_{n \to \infty} G(\tau_n, \tau_n, \rho) = 0$,*
(c) *$\lim_{n \to \infty} G(\tau_n, \rho, \rho) = 0$.*

Definition 1.7. Let (Y, G) be a G-metric space. $\{\tau_n\}_{n=1}^{\infty}$ is said to be G-Cauchy, if for every $\varepsilon > 0$, there is a positive integer N such that $G(\tau_n, \tau_\xi, \tau_\zeta) < \varepsilon$ for all $n, \xi, \zeta \geq N$.

Definition 1.8. ([9]) If every Cauchy sequence in Y is G-convergent in it, then (Y, G) is called G-complete.

Definition 1.9. ([12]) If g is a mapping on a G-metric space (Y, G) and $\phi_0 \in Y$, then orbit $O_g(\phi_0)$ at ϕ_0 is the sequence of g-iterates $\{\phi_0, g\phi_0, \ldots, g^n\phi_0, \ldots\}$.

Definition 1.10. ([12]) A fixed point p of g on Y is a G-contractive fixed point of it if the orbital sequence $\{y,\ gy,\ \ldots,\ g^n y,\ \ldots\}$ is G-convergent to p at each $y \in Y$.

Mohanta proved the result as follows:

Theorem 1.11. ([7]) *Let (Y, G) complete and h be a self-map on (Y, G) with*

$$\begin{aligned} &G(h\tau, h\vartheta, h\varphi) \\ &\leq \varpi \max\{G(\tau, h\vartheta, h\vartheta), G(\vartheta, h\tau, h\tau), G(\varphi, h\varphi, h\varphi), G(\vartheta, h\varphi, h\varphi), \\ &\qquad G(\varphi, h\vartheta, h\vartheta), G(\tau, h\tau, h\tau), G(\varphi, h\tau, h\tau), \\ &\qquad G(\tau, h\varphi, h\varphi), G(\vartheta, h\vartheta, h\vartheta)\} \end{aligned} \tag{1.3}$$

for all $\tau, \nu, \varphi \in Y$ and $0 < \varpi < 1/3$. Then, h has a fixed point ρ which is unique.

Vats et al. proved the following result:

Theorem 1.12. ([19]) *Let (Y, G) be a complete and h denote a self-map on Y with*

$$\begin{aligned} G(h\tau, h\vartheta, h\varphi) \leq \varpi \max\{&G(\tau, h\tau, h\tau) + G(\vartheta, h\vartheta, h\vartheta) + G(\varphi, h\varphi, h\varphi), \quad (1.4) \\ &G(\tau, h\vartheta, h\vartheta) + G(\vartheta, h\tau, h\tau) + G(\varphi, h\vartheta, h\vartheta), \\ &G(\tau, h\varphi, h\varphi) + G(\vartheta, h\varphi, h\varphi) + G(\varphi, h\tau, h\tau)\} \end{aligned}$$

for all $\tau, \nu, \varphi \in Y$ and $0 < \varpi < 1/4$. Then, h has a fixed point ρ which is unique.

In [15], Phaneendra with the first author, generalized Theorem 1.11 and Theorem 1.12 by employing wider inequality.

Given $\tau, \nu, \varphi \in Y$, define $S(\tau, \nu, \varphi) = \max T_h(\tau, \nu, \varphi)$, where

$$T_h(\tau, \nu, \varphi) = \{G(h^i\sigma, h^j\mu, h^k\epsilon) : 0 \leq i, j, k \leq 1; \sigma, \mu, \epsilon \in \{\tau, \nu, \varphi\}\}. \tag{1.5}$$

It may be noted that $T_h(\tau, \nu, \varphi)$ has 36 elements. With this notation, Phaneendra et. al. [15] proved the following theorem:

Theorem 1.13. ([15]) *Let h be a mapping on (Y, G) such that*

$$G(h\tau, h\vartheta, h\varphi) \leq \theta S(\tau, \nu, \varphi) \text{ for all } \tau, \nu, \varphi \in Y, \tag{1.6}$$

and $0 < \theta < 1/3$. *If* (Y, G) *is complete, then* h *has a fixed point* ρ *which is unique.*

2. Main results

If various nonnegative terms are inserted in $T_h(\tau, \nu, \varphi)$ and the choice of θ is extended, then it is very clear to impose one or more additional conditions to obtain a fixed point. For this purpose, the authors Kumara Swamy and Saravanan [4] made the conjecture recently. In this section we proved generalized fixed point theorems in G-metric spaces using the conjecture given below.

Conjecture 2.1. Let (Y, G) be complete and $h : Y \to Y$ satisfying the following inequality:

$$G(h\tau, h\vartheta, h\varphi) \le \mu \max T_h^{(s)}(\tau, \nu, \varphi) \quad \text{for all} \quad \tau, \nu, \varphi \in Y, s \ge 1, \tag{2.1}$$

where

$$T_h^{(s)}(\tau, \nu, \varphi) = \{G(h^i\sigma, h^j\mu, h^k\epsilon) : 0 \le i, j, k \le s; \sigma, \mu, \epsilon \in \{\tau, \nu, \varphi\}\} \tag{2.2}$$

and $0 < \mu < 1$. Given $\tau_0 \in Y$, suppose that $O_h(\tau_0)$ is bounded. Then h has a unique fixed point ρ.

Theorem 2.2. *Let* (Y, G) *be complete and* h*, a self-map on* Y *satisfying,*

$$\begin{aligned} G(h\tau, h\vartheta, h\varphi) \le \theta \max\{&G(\tau, h\vartheta, h\vartheta) + G(\vartheta, h\tau, h\tau) + G(\varphi, h\varphi, h\varphi), \\ &G(\vartheta, h\varphi, h\varphi) + G(\varphi, h\vartheta, h\vartheta) + G(\tau, h\tau, h\tau), \\ &G(\varphi, h\tau, h\tau) + G(\tau, h\varphi, h\varphi) + G(\vartheta, h\vartheta, h\vartheta)\}, \end{aligned} \tag{2.3}$$

where $0 < \theta < 1/3$. *Then* h *has a unique fixed point.*

Proof. For each $\phi_0 \in Y$, $O_h(\phi_0)$ is bounded. Suppose that $O_h(\tau_0)$ is unbounded if possible. Then there is a nonzero whole number n so that

$$G(\phi_1, \phi_n, \phi_n) > \mu \max C_n, \tag{2.4}$$

where

$$C_n = \{G(\phi_1, \phi_r, \phi_r) : 0 \le r \le n-1\}, \quad n \ge 2 \tag{2.5}$$

and

$$\mu = \frac{6\theta}{1 - 2\theta}. \tag{2.6}$$

Now from the inequality (2.3) with $\phi = \phi_0$, $\nu = \varphi = \phi_{n-1}$, we have

$$\begin{aligned} &G(h\phi_0, h\phi_{n-1}, h\phi_{n-1}) \\ &\le \theta \max\{G(\phi_0, h\phi_{n-1}, h\phi_{n-1}) + G(\phi_{n-1}, h\phi_0, h\phi_0) + G(\phi_{n-1}, h\phi_{n-1}, h\phi_{n-1}), \\ &\qquad G(\phi_{n-1}, h\phi_{n-1}, h\phi_{n-1}) + G(\phi_{n-1}, h\phi_{n-1}, h\phi_{n-1}) + G(\phi_0, h\phi_0, h\phi_0), \end{aligned}$$

$$G(\phi_{n-1}, h\phi_0, h\phi_0) + G(\phi_0, h\phi_{n-1}, h\phi_{n-1}) + G(\phi_{n-1}, h\phi_{n-1}, h\phi_{n-1})\}$$

or

$$G(h\phi_0, h\phi_{n-1}, h\phi_{n-1}) \leq \theta M, \tag{2.7}$$

where

$$\begin{aligned} M = \max\{&G(\phi_0, \phi_n, \phi_n) + G(\phi_{n-1}, \phi_1, \phi_1) + G(\phi_{n-1}, \phi_n, \phi_n), \\ &2G(\phi_{n-1}, \phi_n, \phi_n) + G(\phi_0, \phi_1, \phi_1), \\ &G(\phi_{n-1}, \phi_1, \phi_1) + G(\phi_0, \phi_n, \phi_n) + G(\phi_{n-1}, \phi_n, \phi_n)\}. \end{aligned} \tag{2.8}$$

Now two cases arise:

Case (a): Let $M = G(\phi_0, \phi_n, \phi_n) + G(\phi_{n-1}, \phi_1, \phi_1) + G(\phi_{n-1}, \phi_n, \phi_n)$. In view of (1.1), (2.4), (2.5) and (2.6), it follows from (2.8) that

$$\begin{aligned} G(\phi_1, \phi_n, \phi_n) &\leq \theta[G(\phi_0, \phi_n, \phi_n) + G(\phi_{n-1}, \phi_1, \phi_1) + G(\phi_{n-1}, \phi_n, \phi_n)] \\ &\leq \theta\{[G(\phi_0, \phi_1, \phi_1) + G(\phi_1, \phi_n, \phi_n)] + 2G(\phi_1, \phi_{n-1}, \phi_{n-1}) \\ &\qquad + [G(\phi_{n-1}, \phi_1, \phi_1) + G(\phi_1, \phi_n, \phi_n)]\} \\ &\leq \theta[2G(\phi_1, \phi_0, \phi_0) + 2G(\phi_1, \phi_n, \phi_n) + 4G(\phi_1, \phi_{n-1}, \phi_{n-1})] \end{aligned}$$

or

$$(1-2\theta)G(\phi_1, \phi_n, \phi_n) \leq k[2G(\phi_1, \phi_0, \phi_0) + 4G(\phi_1, \phi_{n-1}, \phi_{n-1})],$$

so that

$$G(\phi_1, \phi_n, \phi_n) \leq \frac{6\theta}{(1-2\theta)} \max C_n \leq \mu \max C_n < G(\phi_1, \phi_n, \phi_n),$$

which is a contradiction.

Case (b): Suppose that $M = 2G(\phi_{n-1}, \phi_n, \phi_n) + G(\phi_0, \phi_1, \phi_1)$. In view of (1.1), (2.4), (2.5) and (2.6), it follows from (2.8) that

$$\begin{aligned} G(\phi_1, \phi_n, \phi_n) &\leq \theta[2G(\phi_{n-1}, \phi_n, \phi_n) + G(\phi_0, \phi_1, \phi_1)] \\ &\leq \theta\{2[G(\phi_{n-1}, \phi_1, \phi_1) + G(\phi_1, \phi_n, \phi_n)] + 2G(\phi_1, \phi_0, \phi_0)\} \\ &\leq \theta[4G(\phi_1, \phi_{n-1}, \phi_{n-1}) + 2G(\phi_1, \phi_n, \phi_n) + 2G(\phi_1, \phi_0, \phi_0)] \end{aligned}$$

or

$$(1-2\theta)G(\phi_1, \phi_n, \phi_n) \leq \theta[2G(\phi_1, \phi_0, \phi_0) + 4G(\phi_1, \phi_{n-1}, \phi_{n-1})],$$

so that

$$G(\phi_1, \phi_n, \phi_n) \leq \frac{6\theta}{(1-2\theta)} \max C_n \leq \mu \max C_n < G(\phi_1, \phi_n, \phi_n),$$

which is again a contradiction. From the above two cases, it is clear that $O_h(\phi_0)$ is bounded and $\sup[O_h(\phi_0)] < \infty$. Hence the existence of the contractive fixed point is ensured by Conjecture 2.1. That is, ϕ_0 is a unique fixed point of h. □

Theorem 2.3. *Let (Y, G) be complete and h, a self-map on Y satisfying*

$$\begin{aligned} G(h\tau, h\vartheta, h\varphi) \leq \theta\{&G(\tau, h\tau, h\tau), G(\vartheta, h\vartheta, h\vartheta), G(\varphi, h\varphi, h\varphi), G(\tau, h\vartheta, h\vartheta),\\ &G(\vartheta, h\varphi, h\varphi), G(\varphi, h\tau, h\tau), G(\tau, h\varphi, h\varphi), G(\vartheta, h\tau, h\tau),\\ &G(\varphi, h\vartheta, h\vartheta), G(\tau, h\vartheta, h\varphi), G(\vartheta, h\varphi, h\tau), G(\varphi, h\tau, h\vartheta,),\\ &G(\tau, \vartheta, h\varphi), G(\vartheta, \varphi, h\tau), G(\varphi, \tau, h\vartheta), G(\tau, \vartheta, \varphi)\}, \end{aligned} \tag{2.9}$$

with $0 < \theta < 1/3$. Then h has a unique fixed point.

Proof. For each $\phi_0 \in Y$, $O_h(\phi_0)$ is bounded. Assume that $O_h(\tau_0)$ is not bounded. Then there is a nonzero whole number n such that

$$G(\phi_1, \phi_n, \phi_n) > \mu \max C_n, \tag{2.10}$$

where

$$C_n = \{G(\phi_1, \phi_r, \phi_r) : 0 \leq r \leq n-1\}, \quad n \geq 2 \tag{2.11}$$

and

$$\mu = \frac{6\theta}{(1-2\theta)}. \tag{2.12}$$

Now from the inequality (2.9) with $\phi = \phi_0$, $\nu = \varphi = \phi_{n-1}$, we have

$$\begin{aligned} G(h\phi_0, h\phi_{n-1}, h\phi_{n-1}) &= G(\phi_1, \phi_n, \phi_n)\\ &\leq \theta \max\{G(\phi_0, h\phi_0, h\phi_0), G(\phi_{n-1}, h\phi_{n-1}, h\phi_{n-1}),\\ &\qquad G(\phi_{n-1}, h\phi_{n-1}, h\phi_{n-1}), G(\phi_0, h\phi_{n-1}, h\phi_{n-1}),\\ &\qquad G(\phi_{n-1}, h\phi_{n-1}, h\phi_{n-1}), G(\phi_{n-1}, h\phi_0, h\phi_0),\\ &\qquad G(\phi_0, h\phi_{n-1}, h\phi_{n-1}), G(\phi_{n-1}, h\phi_0, h\phi_0),\\ &\qquad G(\phi_{n-1}, h\phi_{n-1}, h\phi_{n-1}), G(\phi_0, h\phi_{n-1}, h\phi_{n-1}),\\ &\qquad G(\phi_{n-1}, h\phi_{n-1}, h\phi_0), G(\phi_{n-1}, h\phi_0, h\phi_{n-1}),\\ &\qquad G(\phi_0, \phi_{n-1}, h\phi_{n-1}), G(\phi_{n-1}, \phi_{n-1}, h\phi_0),\\ &\qquad G(\phi_{n-1}, \phi_0, h\phi_{n-1}), G(\phi_0, \phi_{n-1}, \phi_{n-1})\}\\ &\leq \theta \max\{G(\phi_0, \phi_1, \phi_1), G(\phi_{n-1}, \phi_n, \phi_n),\\ &\qquad G(\phi_{n-1}, \phi_n, \phi_n), G(\phi_0, \phi_n, \phi_n), G(\phi_{n-1}, \phi_n, \phi_n),\\ &\qquad G(\phi_{n-1}, \phi_1, \phi_1), G(\phi_0, \phi_n, \phi_n), G(\phi_{n-1}, \phi_1, \phi_1),\\ &\qquad G(\phi_{n-1}, \phi_n, \phi_n), G(\phi_0, \phi_n, \phi_n), G(\phi_{n-1}, \phi_{n-1}, \phi_1), \end{aligned}$$

$$\begin{aligned}
&G(\phi_{n-1},\phi_n,\phi_1), G(\phi_{n-1},\phi_1,\phi_n), G(\phi_0,\phi_{n-1},\phi_n),\\
&G(\phi_{n-1},\phi_0,\phi_n), G(\phi_0,\phi_{n-1},\phi_{n-1})\}\\
\le{}& \theta M, \qquad (2.13)
\end{aligned}$$

where

$$\begin{aligned}
M = \max\{&G(\phi_0,\phi_1,\phi_1), G(\phi_{n-1},\phi_n,\phi_n), G(\phi_0,\phi_n,\phi_n), 2G(\phi_1,\phi_{n-1},\phi_{n-1}),\\
&G(\phi_{n-1},\phi_1,\phi_n), G(\phi_0,\phi_{n-1},\phi_n), G(\phi_0,\phi_{n-1},\phi_{n-1})\}. \qquad (2.14)
\end{aligned}$$

Now we present different cases:

Case (a): Suppose that $M = G(\phi_0,\phi_1,\phi_1)$. In view of (1.1), (2.10), (2.11) and (2.12), it follows from (2.14) that

$$G(\phi_1,\phi_n,\phi_n) \le \theta G(\phi_0,\phi_1,\phi_1) \le 2\theta G(\phi_1,\phi_0,\phi_0) \le \mu \max C_n < G(\phi_1,\phi_n,\phi_n),$$

which is a contradiction.

Case (b): If $M = G(\phi_{n-1},\phi_n,\phi_n)$. In view of (G5), (1.1), (2.10), (2.11) and (2.12), it follows from (2.14) that

$$\begin{aligned}
G(\phi_1,\phi_n,\phi_n) &\le \theta G(\phi_{n-1},\phi_n,\phi_n)\\
&\le \theta(G(\phi_{n-1},\phi_1,\phi_1) + G(\phi_1,\phi_n,\phi_n))\\
&\le \theta(2G(\phi_1,\phi_{n-1},\phi_{n-1}) + G(\phi_1,\phi_n,\phi_n))\\
&\le \frac{2\theta}{1-\theta} G(\phi_1,\phi_{n-1},\phi_{n-1}))\\
&\le \mu \max C_n\\
&< G(\phi_1,\phi_n,\phi_n),
\end{aligned}$$

which is a contradiction.

Case (c): Suppose that $M = G(\phi_0,\phi_n,\phi_n)$. In view of (G5), (1.1), (2.10), (2.11) and (2.12), it follows from (2.14) that

$$\begin{aligned}
G(\phi_1,\phi_n,\phi_n) &\le \theta G(\phi_0,\phi_n,\phi_n)\\
&\le \theta[G(\phi_0,\phi_1,\phi_1) + G(\phi_1,\phi_n,\phi_n)]\\
&\le \frac{2\theta}{1-\theta} G(\phi_1,\phi_0,\phi_0) \le \mu \max C_n < G(\phi_1,\phi_n,\phi_n),
\end{aligned}$$

which is a contradiction.

Case (d): Suppose that $M = 2G(\phi_1,\phi_{n-1},\phi_{n-1})$. In view of (1.1), (2.10), (2.11) and (2.12), it follows from (2.14) that

$$G(\phi_1,\phi_n,\phi_n) \le 2kG(\phi_1,\phi_{n-1},\phi_{n-1}) \le \mu \max C_n < G(\phi_1,\phi_n,\phi_n),$$

which is a contradiction.

Case (e): Suppose that $M = G(\phi_{n-1}, \phi_1, \phi_n)$. In view of (G5), (1.1), (2.10), (2.11) and (2.12), it follows from (2.14) that

$$\begin{aligned}G(\phi_1, \phi_n, \phi_n) &\leq \theta G(\phi_{n-1}, \phi_1, \phi_n)\\ &\leq \theta[G(\phi_{n-1}, \phi_1, \phi_1) + G(\phi_1, \phi_1, \phi_n)]\\ &\leq \theta[2G(\phi_1, \phi_{n-1}, \phi_{n-1}) + 2G(\phi_1, \phi_n, \phi_n)]\\ &\leq \frac{2\theta}{1-2\theta} G(\phi_1, \phi_{n-1}, \phi_{n-1}) \leq \mu \max C_n < G(\phi_1, \phi_n, \phi_n),\end{aligned}$$

which is a contradiction.

Case (f): Suppose that $M = G(\phi_0, \phi_{n-1}, \phi_n)$. In view of (G5), (1.1), (2.10), (2.11) and (2.12), it follows from (2.14) that

$$\begin{aligned}&G(\phi_1, \phi_n, \phi_n)\\ &\leq \theta[G(\phi_0, \phi_{n-1}, \phi_n)]\\ &\leq \theta[G(\phi_0, \phi_1, \phi_1) + G(\phi_1, \phi_n, \phi_n) + G(\phi_n, \phi_{n-1}, \phi_n)]\\ &\leq \theta[2G(\phi_1, \phi_0, \phi_0) + G(\phi_1, \phi_n, \phi_n) + G(\phi_n, \phi_n, \phi_1) + G(\phi_1, \phi_1, \phi_{n-1})]\\ &\leq \theta[2G(\phi_1, \phi_0, \phi_0) + 2G(\phi_1, \phi_n, \phi_n) + 2G(\phi_1, \phi_{n-1}, \phi_{n-1})],\end{aligned}$$

so that

$$G(\phi_1, \phi_n, \phi_n) \leq \frac{4\theta}{1-2\theta} \max C_n \leq \mu \max C_n < G(\phi_1, \phi_n, \phi_n),$$

which is a contradiction.

Case (g): Suppose that $M = G(\phi_0, \phi_{n-1}, \phi_{n-1})$. In view of (G5), (1.1), (2.10), (2.11) and (2.12), it follows from (2.14) that

$$\begin{aligned}G(\phi_1, \phi_n, \phi_n) &\leq \theta G(\phi_0, \phi_{n-1}, \phi_{n-1})\\ &\leq \theta[G(\phi_0, \phi_1, \phi_1) + G(\phi_1, \phi_{n-1}, \phi_{n-1})]\\ &\leq \theta[2G(\phi_1, \phi_0, \phi_0) + G(\phi_1, \phi_{n-1}, \phi_{n-1})]\\ &\leq 3\theta \max C_n \leq \mu \max C_n < G(x_1, x_n, x_n),\end{aligned}$$

which is a contradiction.

All these contradictions indicate that $O_h(\phi_0)$ is bounded and $\sup[O_h(\phi_0)] < \infty$. Then ϕ_0 is a contractive fixed point ensured by Conjecture 2.1. That is, ϕ_0 is a unique fixed point of h. □

3. CONCLUSIONS

In this paper, we have established two fixed point theorems in the setting of G-metric spaces by utilizing contractive fixed point conditions derived from a conjecture. The results presented generalize and extend several existing fixed point results in the literature and contribute to a deeper understanding of the structure of G-metric spaces.

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