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A CAPUTO–FABRIZIO FRACTIONAL MODEL OF NICHOLSON'S BLOWFLIES: EXISTENCE, UNIQUENESS, AND STABILITY ANALYSIS

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Abstract. This paper introduces a novel formulation of the Nicholson's blowflies model using the Caputo–Fabrizio fractional derivative, which features a non-singular exponential kernel. This approach enhances the classical model by capturing memory effects and hereditary characteristics inherent in population dynamics while preserving analytical tractability and compatibility with standard initial conditions. Using fixed point theory, we establish sufficient conditions for the existence, uniqueness, and Ulam–Hyers-type stability of solutions to the proposed fractional-order system. Additionally, we extend the framework to more realistic biological scenarios involving harvesting, impulsive effects, and stochastic influences. The theoretical results are supported by numerical simulations that illustrate the dynamical behavior of the model and confirm its robustness under perturbations.

1. Introduction

Fractional calculus has emerged as a powerful tool for modeling complex systems that exhibit memory, hereditary behavior, or nonlocal interactions.

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Unlike classical integer-order derivatives, fractional derivatives provide a more flexible framework by incorporating past states of the system into current dynamics, making them especially suitable for biological and ecological models where time delays, memory, and feedback loops play critical roles [16, 21]. In particular, the use of fractional operators with non-singular kernelssuch as the Caputo–Fabrizio (CF) derivativehas attracted significant attention in recent years [1, 9]. This operator addresses some of the analytical challenges associated with traditional fractional derivatives, namely the presence of singular kernels at the origin. With its smooth, exponential kernel, the CF derivative enables models to better reflect processes with fading memory and facilitates the imposition of standard initial conditions, thereby improving both the theoretical and numerical treatment of real-world biological phenomena [5, 12, 15, 23].

One of the most studied population models incorporating time delay is the Nicholsons blowflies model, originally proposed by Nicholson in 1954 to describe the oscillatory behavior observed in laboratory experiments on blowfly populations. The oscillations were attributed to a time lag between egg laying and maturation, leading to a delay differential equation characterizing adult population size. Since then, the Nicholson model has become a canonical example in the study of delay differential equations, with rich dynamics including periodicity, chaos, persistence, and stability. A comprehensive survey of the model's development, including its qualitative properties and generalizations, is provided by Berezansky et al. [6]. Subsequent contributions have explored the model's behavior under various conditionssuch as density-dependent mortality [8, 27], stochastic perturbations [19, 28], spatial structure and patch dynamics [14, 18, 26, 29], and almost periodic or bounded solutions [7, 22]. In a parallel way, attention has also been given to the discrete-time version of the Nicholson model, particularly under various perturbations, offering new insights into its dynamical behavior in non-continuous frameworks [4, 11, 10].

In recent years, researchers have extended the Nicholson model into the fractional domain to incorporate memory effects more explicitly. El-Sayed et al. [13] and Ladjimi et al. [17] introduced fractional-order versions of the model using the classical Caputo and Riemann–Liouville derivatives. These studies, which represented the only existing studies that have analyzed the Nicholson model within a fractional calculus framework, revealed that fractional formulations could enhance the realism and descriptive power of population models by better capturing the hereditary nature of biological processes. However, most of this work has relied on fractional derivatives with singular kernels, which pose challenges in analysis and computation, especially in the presence of discontinuities or initial singularities. To our knowledge, there is no existing research that has employed the CF fractional derivativefeaturing a non-singular exponential kernelto the Nicholsons blowflies model. This gap is significant, as the CF derivative provides analytical advantages such as well-behaved initial conditions, improved numerical stability, and the ability to describe fading memory effects more naturally.

Motivated by these benefits and the biological importance of the Nicholson model, this paper introduces a novel formulation of the model involving the υ^{th}-order CF fractional derivative of the form

$$\begin{cases} \left({}^{CF}D_0^{\upsilon} y\right)(\varkappa) = Py(\varkappa-\zeta)\exp\left(-\gamma y(\varkappa-\zeta)\right) - \delta y(\varkappa),\ \varkappa \in [0,\mathfrak{T}],\ \mathfrak{T}>0, \\ y(\varkappa) = \phi(\varkappa), \quad \varkappa \in [-\zeta, 0], \quad 0<\upsilon<1. \end{cases} \tag{1.1}$$

In this context, ϕ is a continuous, nonnegative function, specifically $\phi \in C\left([-\zeta,0],\mathbb{R}^+\right)$, with the condition $\phi(0)>0$. The variable $y(\varkappa)$ denotes the size of the adult population. The constant $P>0$ represents the maximum per capita daily egg production rate. The parameter $\frac{1}{\gamma}>0$ characterizes the population size at which reproductive output is maximized. The daily per capita adult mortality rate is given by $\delta>0$, and $\zeta>0$ denotes the generation time, defined as the period from birth to the onset of maturity.

This study's core contributions are detailed as follows. It introduces a new fractional-order formulation of Nicholsons blowflies model using the Caputo–Fabrizio derivative, offering a singularity-free, biologically consistent way to incorporate memory effects. A rigorous analytical findings are established on qualitative characteristics of solutions using fixed point theory. The approach is further extended to more realistic biological scenarios, such as models with harvesting and impulsive effects, demonstrating its flexibility and robustness. Numerical simulations validate the theoretical findings, showing stable and meaningful dynamics. These contributions provide a solid foundation for future research on fractional-order biological systems with memory and delay.

The remainder of the paper is organized as follows. In Section 2, we provide the necessary preliminaries, including the definition and properties of the Caputo–Fabrizio fractional derivative. Section 3 is devoted to establishing sufficient conditions for the existence and uniqueness of solutions to the proposed fractional Nicholson model using fixed point theory. In Section 4, we investigate various forms of Ulam–Hyers (UH) stability, demonstrating the model's robustness under perturbations. Section 5 presents an extended discussion, where we explore the applicability of the analytical framework to more general versions of the model involving harvesting terms, impulsive effects and stochastic perturbations. In Section 6, we provide numerical simulations to illustrate and validate the theoretical results. The study concludes by highlighting the principal outcomes and proposing avenues for future work.

2. Some preliminaries

In this section, we present the definition of the Caputo–Fabrizio fractional derivative, along with several fundamental properties that will be instrumental in deriving the main results of the paper.

Definition 2.1. ([1]) Let $y \in H^1(0, \mathcal{T})$, $\mathcal{T} > 0$, and $\upsilon \in (0,1)$. The υ^{th}-order CF fractional derivative is defined by

$$\left({}^{CF}D_0^{\upsilon} y\right)(\varkappa) = \frac{(2-\upsilon)\mathcal{B}(\upsilon)}{2(1-\upsilon)} \int_0^{\varkappa} y'(s) \exp\left[-\upsilon\left(\frac{\varkappa - s}{1-\upsilon}\right)\right] ds, \quad 0 \le \varkappa \le \mathfrak{T}.$$

The corresponding CF fractional integral is given by

$$\left({}^{CF}I_0^{\upsilon} y\right)(\varkappa) = \frac{2(1-\upsilon)}{(2-\upsilon)\mathcal{B}(\upsilon)} y(\varkappa) + \frac{2\upsilon}{(2-\upsilon)\mathcal{B}(\upsilon)} \int_0^{\varkappa} y(s) ds, \quad 0 \le \varkappa \le \mathfrak{T}.$$

Remark 2.2. The function $\mathcal{B}(\nu)$ is a normalization factor associated with the Caputo Fabrizio fractional derivative and is only required to satisfy $\mathcal{B}(\nu) > 0$ together with suitable consistency conditions such as $\mathcal{B}(0) = \mathcal{B}(1) = 1$. In many applied and illustrative studies, it is customary to take $\mathcal{B}(\nu) = 1$ for simplicity, since this choice merely rescales the operator and does not affect the qualitative analytical results. In the numerical examples presented in this paper, we adopt this standard simplification.

Theorem 2.3. ([1]) *Let $\upsilon \in (0,1)$, $y_0 \in \mathbb{R}$, and $h : [0, \mathfrak{T}] \to \mathbb{R}$. Then, the unique solution of the following initial value problem:*

$$\begin{cases} \left({}^{CF}D_0^{\upsilon} y\right)(\varkappa) = h(\varkappa), \quad 0 \le \varkappa \le \mathfrak{T}, \\ y(0) = y_0, \end{cases}$$

is given by

$$y(\varkappa) = y_0 + \frac{2(1-\upsilon)}{(2-\upsilon)\mathcal{B}(\upsilon)}[h(\varkappa) - h(0)] + \frac{2\upsilon}{(2-\upsilon)\mathcal{B}(\upsilon)} \int_0^{\varkappa} h(s) ds, \quad 0 \le \varkappa \le \mathfrak{T}.$$

Theorem 2.4. ([2]) *Any contraction mapping S of a closed subset $\mathfrak{C}$ of a Banach space $\mathcal{X}$ into itself has a unique fixed point.*

Theorem 2.5. ([2]) *Let $\mathfrak{M}$ be a losed, convex, non-empty subset of a Banach space $\mathcal{X}$. Suppose S_1 and S_2 are maps from $\mathcal{X}$ into $\mathcal{X}$ such that*

(i) *$S_1 x + S_2 y \in \mathfrak{M}$ for all x, $y \in \mathfrak{M}$;*
(ii) *S_1 is continuous as well as compact;*
(iii) *S_2 is contraction.*

Then, there exists a $z \in \mathfrak{M}$ with $S_1 z + S_2 z = z$.

Denote by

$$f(\varkappa, y(\varkappa)) = Py(\varkappa - \zeta)\exp\left(-\gamma y(\varkappa - \zeta)\right) - \delta y(\varkappa), \quad \varkappa \in [0, \mathfrak{T}].$$

Let ε be a positive real number and $\psi : [-\zeta, \mathfrak{T}] \to \mathbb{R}^+$ be a continuous function. We consider the following inequalities:

$$\left|\left({}^{CF}D_0^{\upsilon} y\right)(\varkappa) - f(\varkappa, y(\varkappa))\right| \leq \varepsilon, \quad \varkappa \in [0, \mathfrak{T}], \tag{2.1}$$

$$\left|\left({}^{CF}D_0^{\upsilon} y\right)(\varkappa) - f(\varkappa, y(\varkappa))\right| \leq \psi(\varkappa), \quad \varkappa \in [0, \mathfrak{T}], \tag{2.2}$$

$$\left|\left({}^{CF}D_0^{\upsilon} y\right)(\varkappa) - f(\varkappa, y(\varkappa))\right| \leq \varepsilon\psi(\varkappa), \quad \varkappa \in [0, \mathfrak{T}]. \tag{2.3}$$

Now, we state all the relevant stability definitions.

Definition 2.6. ([20, 24]) (1.1) is UH stable if, there is a $c_f \in \mathbb{R}^+$ so that for every $\varepsilon \in \mathbb{R}^+$ and, for every solution $v \in C^1([-\zeta, \mathfrak{T}], \mathbb{R})$ of (2.1), there is a solution $y \in C^1([-\zeta, \mathfrak{T}], \mathbb{R})$ of (1.1) such that

$$|v(\varkappa) - y(\varkappa)| \leq c_f \varepsilon, \quad \varkappa \in [-\zeta, \mathfrak{T}].$$

Definition 2.7. ([20, 24]) (1.1) is generalized UH stable if, there is a mapping $\theta_f \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\theta_f(0) = 0$ so that for every solution $v \in C^1([-\zeta, \mathfrak{T}], \mathbb{R})$ of (2.1), there is a solution $y \in C^1([-\zeta, \mathfrak{T}], \mathbb{R})$ of (1.1) such that

$$|v(\varkappa) - y(\varkappa)| \leq \theta_f(\varepsilon), \quad \varkappa \in [-\zeta, \mathfrak{T}].$$

Definition 2.8. ([20, 24]) (1.1) is Ulam–Hyers–Rassias (UHR) stable with respect to a mapping ψ, if there is a $c_{f,\psi} \in \mathbb{R}^+$ so that for every $\varepsilon \in \mathbb{R}^+$ and, for every solution $v \in C^1([-\zeta, \mathfrak{T}], \mathbb{R})$ of (2.1), there is a solution $y \in C^1([-\zeta, \mathfrak{T}], \mathbb{R})$ of (1.1) such that

$$|v(\varkappa) - y(\varkappa)| \leq c_{f,\psi}\varepsilon\psi(\varkappa), \quad \varkappa \in [-\zeta, \mathfrak{T}].$$

Definition 2.9. ([20, 24]) (1.1) is generalized UHR stable with respect to a mapping ψ, if there is a $c_{f,\psi} \in \mathbb{R}^+$ so that for every solution $v \in C^1([-\zeta, \mathfrak{T}], \mathbb{R})$ of (2.1), there is a solution $y \in C^1([-\zeta, \mathfrak{T}], \mathbb{R})$ of (1.1) such that

$$|v(\varkappa) - y(\varkappa)| \leq c_{f,\psi}\psi(\varkappa), \quad \varkappa \in [-\zeta, \mathfrak{T}].$$

Remark 2.10. Obviously,

(1) Definition 2.7 follows from Definition 2.6;
(2) Definition 2.9 follows from Definition 2.8;
(3) Definition 2.6 follows from Definition 2.8 for $\psi(\varkappa) \equiv 1$.

Remark 2.11. A mapping $v \in C^1([-\zeta, \mathfrak{T}], \mathbb{R})$ is a solution of (2.1) if, there is a mapping $g \in C^1([-\zeta, \mathfrak{T}], \mathbb{R})$ (depending on v) such that

(1) $|g(\varkappa)| \leq \varepsilon$ for $\varkappa \in [0, \mathfrak{T}]$;

(2) $\left({}^{CF}D_0^{\upsilon} v\right)(\varkappa) = f(\varkappa, v(\varkappa)) + g(\varkappa)$, $\varkappa \in [0, \mathfrak{T}]$.

Similar observations follow regarding the solutions of (2.2) and (2.3).

Remark 2.12. If $v \in C^1([-\zeta, \mathfrak{T}], \mathbb{R})$ is a solution of (2.1), then it is a solution of

$$\begin{aligned} &\Big| v(\varkappa) - v(0) - \frac{2(1-\upsilon)}{(2-\upsilon)\mathcal{B}(\upsilon)} \Big[f(\varkappa, v(\varkappa)) - f(0, v(0)) \Big] \\ &\quad - \frac{2\upsilon}{(2-\upsilon)\mathcal{B}(\upsilon)} \int_0^{\varkappa} f(s, v(s)) ds \Big| \\ &\leq \frac{2\varepsilon}{(2-\upsilon)\mathcal{B}(\upsilon)} \left[2(1-\upsilon) + \upsilon \mathfrak{T} \right], \quad \varkappa \in [0, \mathfrak{T}]. \end{aligned} \tag{2.4}$$

By Remark 2.11 and Theorem 2.3, we have that

$$\begin{aligned} v(\varkappa) = v(0) &+ \frac{2(1-\upsilon)}{(2-\upsilon)\mathcal{B}(\upsilon)} \Big[f(\varkappa, v(\varkappa)) + g(\varkappa) - f(0, v(0)) - g(0) \Big] \\ &+ \frac{2\upsilon}{(2-\upsilon)\mathcal{B}(\upsilon)} \int_0^{\varkappa} \left[f(s, v(s)) + g(s) \right] ds, \quad 0 \leq \varkappa \leq \mathfrak{T}. \end{aligned}$$

For each $0 \leq \varkappa \leq \mathfrak{T}$, we have

$$\begin{aligned} &\Big| v(\varkappa) - v(0) - \frac{2(1-\upsilon)}{(2-\upsilon)\mathcal{B}(\upsilon)} \Big[f(\varkappa, v(\varkappa)) - f(0, v(0)) \Big] \\ &\quad - \frac{2\upsilon}{(2-\upsilon)\mathcal{B}(\upsilon)} \int_0^{\varkappa} f(s, v(s)) ds \Big| \\ &= \Big| \frac{2(1-\upsilon)}{(2-\upsilon)\mathcal{B}(\upsilon)} \left[g(\varkappa) - g(0) \right] + \frac{2\upsilon}{(2-\upsilon)\mathcal{B}(\upsilon)} \int_0^{\varkappa} g(s) ds \Big| \\ &\leq \frac{2(1-\upsilon)}{(2-\upsilon)\mathcal{B}(\upsilon)} \Big| g(\varkappa) - g(0) \Big| + \frac{2\upsilon}{(2-\upsilon)\mathcal{B}(\upsilon)} \int_0^{\varkappa} |g(s)| \, ds \\ &\leq \frac{4\varepsilon(1-\upsilon)}{(2-\upsilon)\mathcal{B}(\upsilon)} + \frac{2\varepsilon\upsilon\varkappa}{(2-\upsilon)\mathcal{B}(\upsilon)} \\ &\leq \frac{4\varepsilon(1-\upsilon)}{(2-\upsilon)\mathcal{B}(\upsilon)} + \frac{2\varepsilon\upsilon\mathfrak{T}}{(2-\upsilon)\mathcal{B}(\upsilon)} \\ &= \frac{2\varepsilon}{(2-\upsilon)\mathcal{B}(\upsilon)} \left[2(1-\upsilon) + \upsilon\mathfrak{T} \right]. \end{aligned}$$

Further, similar observations follow regarding the solutions of (2.2) and (2.3).

3. Existence and uniqueness of solutions

Let $\mathcal{X}$ be the collection of all bounded and continuous real-valued functions defined on $[-\zeta, \mathfrak{T}]$. Clearly, $\mathcal{X}$ is a Banach space equipped with the norm

$$\|y\| = \max_{\varkappa \in [-\zeta, \mathfrak{T}]} |y(\varkappa)|,$$

for any $y \in \mathcal{X}$. Consider the set

$$\mathcal{M} = \{y \in \mathcal{X} : \|y\| \leq r\}.$$

Clearly, $\mathcal{M}$ is a closed, convex, non-empty subset of $\mathcal{X}$. By a solution y of (1.1) we mean that it satisfies the relation

$$\begin{aligned} y(\varkappa) &= \phi(0) + \frac{2(1-\upsilon)}{(2-\upsilon)\mathcal{B}(\upsilon)} \\ &\quad \times \Big[P y(\varkappa - \zeta) \exp(-\gamma y(\varkappa - \zeta)) - \delta y(\varkappa) - P\phi(-\zeta) \exp(-\gamma \phi(-\zeta)) + \delta \phi(0) \Big] \\ &\quad + \frac{2\upsilon}{(2-\upsilon)\mathcal{B}(\upsilon)} \int_0^{\varkappa} [P y(s-\zeta) \exp(-\gamma y(s-\zeta)) - \delta y(s)]\, ds, \quad 0 \leq \varkappa \leq \mathfrak{T}, \end{aligned}$$

and $y(\varkappa) = \phi(\varkappa)$ for $\varkappa \in [-\zeta, 0]$. Now, we establish sufficient conditions on existence of solutions for the model (1.1) using Theorem 2.5. Define the operators F, $\mathfrak{T}_1$ and $\mathfrak{T}_2$ by

$$\begin{aligned} \big(Fy\big)(\varkappa) &= \phi(0) + \frac{2(1-\upsilon)}{(2-\upsilon)\mathcal{B}(\upsilon)} \\ &\quad \times \Big[P y(\varkappa - \zeta) \exp(-\gamma y(\varkappa - \zeta)) - \delta y(\varkappa) - P\phi(-\zeta) \exp(-\gamma \phi(-\zeta)) + \delta \phi(0) \Big] \\ &\quad + \frac{2\upsilon}{(2-\upsilon)\mathcal{B}(\upsilon)} \int_0^{\varkappa} [P y(s-\zeta) \exp(-\gamma y(s-\zeta)) - \delta y(s)]\, ds, \quad 0 \leq \varkappa \leq \mathfrak{T}, \end{aligned}$$

$$\big(\mathfrak{T}_1 y\big)(\varkappa) = \frac{2\upsilon}{(2-\upsilon)\mathcal{B}(\upsilon)} \int_0^{\varkappa} [P y(s-\zeta) \exp(-\gamma y(s-\zeta)) - \delta y(s)]\, ds, 0 \leq \varkappa \leq \mathfrak{T},$$

and

$$\begin{aligned} \big(\mathfrak{T}_2 y\big)(\varkappa) &= \phi(0) + \frac{2(1-\upsilon)}{(2-\upsilon)\mathcal{B}(\upsilon)} \\ &\quad \times \Big[P y(\varkappa - \zeta) \exp(-\gamma y(\varkappa - \zeta)) - \delta y(\varkappa) - P\phi(-\zeta) \exp(-\gamma \phi(-\zeta)) + \delta \phi(0) \Big], \end{aligned}$$

$0 \leq \varkappa \leq \mathfrak{T}$.

Clearly, one can observe that F, $\mathfrak{T}_1$, $\mathfrak{T}_2$ maps $\mathcal{M}$ into $\mathcal{X}$.

Theorem 3.1. *Choose r such that*

$$\phi(0)+\frac{2(1-\upsilon)}{(2-\upsilon)\mathcal{B}(\upsilon)}\Big[P\phi(-\zeta)\exp(-\gamma\phi(-\zeta))+\delta\phi(0)\Big] + \frac{2r}{(2-\upsilon)\mathcal{B}(\upsilon)}\left(\frac{P}{\gamma e}+\delta\right)[(1-\upsilon)+\upsilon\mathfrak{T}]\le r \tag{3.1}$$

and

$$\frac{2(1-\upsilon)\,(P+\delta)}{(2-\upsilon)\mathcal{B}(\upsilon)}<1. \tag{3.2}$$

Then, the model (1.1) *has a solution in the set* $\mathcal{M}$*.*

Proof. Proof will be provided in a number of steps.

Step 1: $\mathfrak{T}_1$ is a continuous map.

Consider a sequence $\{y_n\}$ in $\mathcal{M}$ converging to y in $\mathcal{M}$. It is enough to show that the sequence $\{\mathfrak{T}_1 y_n\}$ in $\mathcal{X}$ converges to $\mathfrak{T}_1 y$ in $\mathcal{X}$. For this, take

$$\begin{aligned}
&\big|\big(\mathfrak{T}_1y_n\big)(\varkappa)-\big(\mathfrak{T}_1y\big)(\varkappa)\big|\\
&\quad\le\frac{2\upsilon}{(2-\upsilon)\mathcal{B}(\upsilon)}\int_0^{\varkappa}\Big[P\,|y_n(s-\zeta)\exp(-\gamma y_n(s-\zeta))\\
&\qquad -y(s-\zeta)\exp(-\gamma y(s-\zeta))|+\delta\,|y_n(s)-y(s)|\Big]ds\\
&\quad\le\frac{2\upsilon}{(2-\upsilon)\mathcal{B}(\upsilon)}\int_0^{\varkappa}\left[P\,|y_n(s-\zeta)-y(s-\zeta)|+\delta\,|y_n(s)-y(s)|\right]ds\\
&\quad\le\frac{2\upsilon}{(2-\upsilon)\mathcal{B}(\upsilon)}\left[P\|y_n-y\|+\delta\|y_n-y\|\right]\varkappa\\
&\quad\le\frac{2\upsilon\mathfrak{T}\,(P+\delta)}{(2-\upsilon)\mathcal{B}(\upsilon)}\|y_n-y\|,
\end{aligned}$$

implying that

$$\|\big(\mathfrak{T}_1y_n\big)-\big(\mathfrak{T}_1y\big)\|\le\frac{2\upsilon\mathfrak{T}\,(P+\delta)}{(2-\upsilon)\mathcal{B}(\upsilon)}\|y_n-y\|,$$

and hence $\mathfrak{T}_1y_n\to\mathfrak{T}_1y$ whenever $y_n\to y$. Thus, $\mathfrak{T}_1$ is a continuous map.

Step 2: $\mathfrak{T}_1$ is uniformly bounded.

It is enough to show that for each $y \in \mathcal{M}$, $\|(\mathfrak{T}_1 y)\| \leq l$, where l is a positive constant. To see this, let $y \in \mathcal{M}$ and consider

$$\begin{aligned}
\left|(\mathfrak{T}_1 y)(\varkappa)\right| &\leq \frac{2\upsilon}{(2-\upsilon)\mathcal{B}(\upsilon)} \int_0^{\varkappa} |Py(s-\zeta)\exp(-\gamma y(s-\zeta)) - \delta y(s)|\, ds \\
&\leq \frac{2\upsilon}{(2-\upsilon)\mathcal{B}(\upsilon)} \int_0^{\varkappa} \left[P\,|y(s-\zeta)\exp(-\gamma y(s-\zeta))| + \delta\,|y(s)|\right] ds \\
&\leq \frac{2\upsilon}{(2-\upsilon)\mathcal{B}(\upsilon)} \left[\frac{P}{\gamma e}\|y\| + \delta\|y\|\right]\varkappa \\
&\leq \frac{2r\upsilon\mathfrak{T}\left(\frac{P}{\gamma e} + \delta\right)}{(2-\upsilon)\mathcal{B}(\upsilon)},
\end{aligned}$$

implying that $\|(\mathfrak{T}_1 y)\| \leq l$, where

$$l = \frac{2r\upsilon\mathfrak{T}\left(\frac{P}{\gamma e} + \delta\right)}{(2-\upsilon)\mathcal{B}(\upsilon)}.$$

Thus, $\mathfrak{T}_1$ is uniformly bounded.

Step 3: $\mathfrak{T}_1$ is equicontinuous.

To see this, let $y \in \mathcal{M}$, $0 \leq \varkappa_1 < \varkappa_2 \leq \mathfrak{T}$, and consider

$$\begin{aligned}
&\left|(\mathfrak{T}_1 y)(\varkappa_2) - (\mathfrak{T}_1 y)(\varkappa_1)\right| \\
&\quad\leq \frac{2\upsilon}{(2-\upsilon)\mathcal{B}(\upsilon)} \Big| \int_0^{\varkappa_2} \left[Py(s-\zeta)\exp(-\gamma y(s-\zeta)) - \delta y(s)\right] ds \\
&\qquad - \int_0^{\varkappa_1} \left[Py(s-\zeta)\exp(-\gamma y(s-\zeta)) - \delta y(s)\right] ds \Big| \\
&\quad\leq \frac{2\upsilon}{(2-\upsilon)\mathcal{B}(\upsilon)} \int_{\varkappa_1}^{\varkappa_2} \left[P\,|y(s-\zeta)\exp(-\gamma y(s-\zeta))| + \delta\,|y(s)|\right] ds \\
&\quad\leq \frac{2\upsilon}{(2-\upsilon)\mathcal{B}(\upsilon)} \left[\frac{P}{\gamma e}\|y\| + \delta\|y\|\right](\varkappa_2 - \varkappa_1) \\
&\quad\leq \frac{2r\upsilon\left(\frac{P}{\gamma e} + \delta\right)}{(2-\upsilon)\mathcal{B}(\upsilon)}(\varkappa_2 - \varkappa_1).
\end{aligned}$$

In view of the uniform continuity of the function $(\varkappa_2 - \varkappa_1)$ on the interval $[0, \mathfrak{T}]$, we observe that $\mathfrak{T}_1$ is equicontinuous. Thus, we conclude that $\mathfrak{T}_1$ is compact by the Arzela–Ascoli theorem. It follows from Steps 1-3 that the condition (ii) of Theorem 2.5 is satisfied.

Step 4: $\mathfrak{T}_2$ is a contraction mapping.

To see this, we take $y,\, v \in \mathcal{M},\, \varkappa \in [0, \mathfrak{T}]$ and consider

$$\begin{aligned}
&\big|\big(\mathfrak{T}_2 y\big)(\varkappa) - \big(\mathfrak{T}_2 v\big)(\varkappa)\big| \\
&\quad \leq \frac{2(1-\upsilon)}{(2-\upsilon)\mathcal{B}(\upsilon)}\Big[P\left|y(\varkappa-\zeta)\exp\left(-\gamma y(\varkappa-\zeta)\right) - v(\varkappa-\zeta)\exp\left(-\gamma v(\varkappa-\zeta)\right)\right| \\
&\qquad + \delta\left|y(\varkappa) - v(\varkappa)\right|\Big] \\
&\quad \leq \frac{2(1-\upsilon)}{(2-\upsilon)\mathcal{B}(\upsilon)}\Big[P\left|y(\varkappa-\zeta) - v(\varkappa-\zeta)\right| + \delta\left|y(\varkappa) - v(\varkappa)\right|\Big] \\
&\quad \leq \frac{2(1-\upsilon)}{(2-\upsilon)\mathcal{B}(\upsilon)}\Big[P\|y-v\| + \delta\|y-v\|\Big] \\
&\quad = \frac{2(1-\upsilon)\left(P+\delta\right)}{(2-\upsilon)\mathcal{B}(\upsilon)}\|y-v\|
\end{aligned}$$

implying that

$$\|\mathfrak{T}_2 y - \mathfrak{T}_2 v\| \leq \frac{2(1-\upsilon)\left(P+\delta\right)}{(2-\upsilon)\mathcal{B}(\upsilon)}\|y-v\|.$$

Since

$$\frac{2(1-\upsilon)\left(P+\delta\right)}{(2-\upsilon)\mathcal{B}(\upsilon)} < 1,$$

$\mathfrak{T}_2$ is a contraction.

Step 5: $\mathfrak{T}_1 v + \mathfrak{T}_2 y \in \mathcal{M}$ for all $y,\, v \in M$.
Consider

$$\begin{aligned}
&\big|\big(\mathfrak{T}_1 v\big)(\varkappa) + \big(\mathfrak{T}_2 y\big)(\varkappa)\big| \\
&\quad \leq \big|\big(\mathfrak{T}_1 v\big)(\varkappa)\big| + \big|\big(\mathfrak{T}_2 y\big)(\varkappa)\big| \\
&\quad \leq \phi(0) + \frac{2(1-\upsilon)}{(2-\upsilon)\mathcal{B}(\upsilon)}\Big|Py(\varkappa-\zeta)\exp\left(-\gamma y(\varkappa-\zeta)\right) - \delta y(\varkappa) \\
&\qquad - P\phi(-\zeta)\exp\left(-\gamma\phi(-\zeta)\right) + \delta\phi(0)\Big| \\
&\qquad + \frac{2\upsilon}{(2-\upsilon)\mathcal{B}(\upsilon)}\int_0^{\varkappa}\left|Pv(s-\zeta)\exp\left(-\gamma v(s-\zeta)\right) - \delta v(s)\right| ds \\
&\quad \leq \phi(0) + \frac{2(1-\upsilon)}{(2-\upsilon)\mathcal{B}(\upsilon)}\Big[P\left|y(\varkappa-\zeta)\exp\left(-\gamma y(\varkappa-\zeta)\right)\right| + \delta\left|y(\varkappa)\right| \\
&\qquad + P\phi(-\zeta)\exp\left(-\gamma\phi(-\zeta)\right) + \delta\phi(0)\Big] \\
&\qquad + \frac{2\upsilon}{(2-\upsilon)\mathcal{B}(\upsilon)}\int_0^{\varkappa}\left[P\left|v(s-\zeta)\exp\left(-\gamma v(s-\zeta)\right)\right| + \delta\left|v(s)\right|\right] ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \phi(0)+\frac{2(1-\upsilon)}{(2-\upsilon)\mathcal{B}(\upsilon)}\times\left[\frac{P}{\gamma e}\|y\|+\delta\|y\|\right.\\
&\quad\left.+P\phi(-\zeta)\exp(-\gamma\phi(-\zeta))+\delta\phi(0)\right]\\
&\quad+\frac{2\upsilon}{(2-\upsilon)\mathcal{B}(\upsilon)}\left[\frac{P}{\gamma e}\|v\|+\delta\|v\|\right]\varkappa\\
&\leq \phi(0)+\frac{2(1-\upsilon)}{(2-\upsilon)\mathcal{B}(\upsilon)}\Big[P\phi(-\zeta)\exp\left(-\gamma\phi(-\zeta)\right)+\delta\phi(0)\Big]\\
&\quad+\frac{2r}{(2-\upsilon)\mathcal{B}(\upsilon)}\left(\frac{P}{\gamma e}+\delta\right)[(1-\upsilon)+\upsilon\mathfrak{T}]\\
&\leq r
\end{aligned}
$$

implying that $\mathfrak{T}_1 v+\mathfrak{T}_2 y\in\mathcal{M}$. All conditions of Theorem 2.5 are satisfied. Thus, by Theorem 2.5, there exists $w\in\mathcal{M}$ such that $\mathfrak{T}_1 w+\mathfrak{T}_2 w=w$. Hence model (1.1) has a solution in $\mathcal{M}$. □

Compared to Theorem 3.1, the following corollary relaxes the assumptions by removing the explicit dependence on the initial history terms $\varphi(-\zeta)$ and $\delta\varphi(0)$, thus broadening the scope of admissible initial conditions.

Corollary 3.2. *Let $\varphi\in C([-\zeta,0],\mathbb{R}^+)$ with $\varphi(0)>0$, and choose $r>0$ such that*

$$
\varphi(0)+\frac{2r}{(2-\upsilon)\mathcal{B}(\upsilon)}\left(\frac{P}{\gamma e}+\delta\right)[(1-\upsilon)+\upsilon\mathfrak{T}]\leq r
$$

and

$$
\frac{2(P+\delta)}{(2-\upsilon)\mathcal{B}(\upsilon)}<1.
$$

Then, the model (1.1) *has at least one solution in the set $\mathcal{M}$.*

Now, we establish sufficient conditions on the existence of a unique solution for the model (1.1) using Theorem 2.4.

Theorem 3.3. *Choose r such that* (3.1) *and*

$$
\frac{2\,(P+\delta)\,[(1-\upsilon)+\upsilon\mathfrak{T}]}{(2-\upsilon)\mathcal{B}(\upsilon)}<1 \tag{3.3}
$$

hold. Then, the model (1.1) *has a unique solution in the set $\mathcal{M}$.*

Proof. We use the contraction mapping theorem to prove the result. Since $Fy = \mathfrak{T}_1 y + \mathfrak{T}_2 y$, it follows from the proof of the above Theorem that

$$\begin{aligned}\|(Fy)\| &\leq \phi(0) + \frac{2(1-\upsilon)}{(2-\upsilon)\mathcal{B}(\upsilon)}\Big[P\phi(-\zeta)\exp\left(-\gamma\phi(-\zeta)\right) + \delta\phi(0)\Big] \\ &\quad + \frac{2r}{(2-\upsilon)\mathcal{B}(\upsilon)}\left(\frac{P}{\gamma e} + \delta\right)[(1-\upsilon) + \upsilon\mathfrak{T}] \\ &\leq r\end{aligned}$$

for each $y \in \mathcal{M}$. This shows that F maps $\mathcal{M}$ into $\mathcal{M}$.

Next, we show that F is a contraction map. To see this, take y, $v \in \mathcal{M}$, $\varkappa \in [0, \mathfrak{T}]$ and consider

$$\begin{aligned}&\left|(Fy)(\varkappa) - (Fv)(\varkappa)\right| \\ &\leq \frac{2(1-\upsilon)}{(2-\upsilon)\mathcal{B}(\upsilon)}\Big[P\left|y(\varkappa-\zeta)\exp\left(-\gamma y(\varkappa-\zeta)\right) - v(\varkappa-\zeta)\exp\left(-\gamma v(\varkappa-\zeta)\right)\right| \\ &\quad + \delta\left|y(\varkappa) - v(\varkappa)\right|\Big] + \frac{2\upsilon}{(2-\upsilon)\mathcal{B}(\upsilon)} \\ &\quad \times \int_0^{\varkappa}\Big[P\left|y(s-\zeta)\exp\left(-\gamma y(s-\zeta)\right) - v(s-\zeta)\exp\left(-\gamma v(s-\zeta)\right)\right| \\ &\quad + \delta\left|y(s) - v(s)\right|\Big]ds \\ &\leq \frac{2(1-\upsilon)}{(2-\upsilon)\mathcal{B}(\upsilon)}\Big[P\left|y(\varkappa-\zeta) - v(\varkappa-\zeta)\right| + \delta\left|y(\varkappa) - v(\varkappa)\right|\Big] \\ &\quad + \frac{2\upsilon}{(2-\upsilon)\mathcal{B}(\upsilon)}\int_0^{\varkappa}\Big[P\left|y(s-\zeta) - v(s-\zeta)\right| + \delta\left|y(s) - v(s)\right|\Big]ds \\ &\leq \frac{2(1-\upsilon)}{(2-\upsilon)\mathcal{B}(\upsilon)}\Big[P\|y-v\| + \delta\|y-v\|\Big] \\ &\quad + \frac{2\upsilon}{(2-\upsilon)\mathcal{B}(\upsilon)}\Big[P\|y-v\| + \delta\|y-v\|\Big]\varkappa \\ &\leq \frac{2(1-\upsilon)\left(P+\delta\right)}{(2-\upsilon)\mathcal{B}(\upsilon)}\|y-v\| + \frac{2\upsilon\mathfrak{T}\left(P+\delta\right)}{(2-\upsilon)\mathcal{B}(\upsilon)}\|y-v\| \\ &= \frac{2\left(P+\delta\right)[(1-\upsilon)+\upsilon\mathfrak{T}]}{(2-\upsilon)\mathcal{B}(\upsilon)}\|y-v\|\end{aligned}$$

implying that

$$\|Fy - Fv\| \leq \frac{2\left(P+\delta\right)[(1-\upsilon)+\upsilon\mathfrak{T}]}{(2-\upsilon)\mathcal{B}(\upsilon)}\|y-v\|.$$

Since

$$\frac{2\left(P+\delta\right)[(1-\upsilon)+\upsilon\mathfrak{T}]}{(2-\upsilon)\mathcal{B}(\upsilon)} < 1,$$

F is a contraction. Thus, by Theorem 2.4, there exists $z \in \mathcal{M}$ such that $Fz = z$, which is a fixed point of F. Hence the model (1.1) has a unique solution in $\mathcal{M}$. □

Further, it is possible to introduce a result that weakens the growth requirement by removing the direct dependence on the past-state term $\varphi(-\zeta)$, while maintaining the key contraction condition needed to ensure uniqueness of the solution.

Corollary 3.4. *Let $\varphi \in C([-\zeta, 0], \mathbb{R}^+)$ with $\varphi(0) > 0$. Suppose there exists $r > 0$ such that*

$$\varphi(0) + \frac{2r}{(2-\upsilon)\mathcal{B}(\upsilon)} \left(\frac{P}{\gamma e} + \delta \right) [(1-\upsilon) + \upsilon \mathfrak{T}] \leq r$$

and

$$\frac{2(P+\delta)}{(2-\upsilon)\mathcal{B}(\upsilon)} [(1-\upsilon) + \upsilon \mathfrak{T}] < 1.$$

Then the model (1.1) *has a unique solution in the set $\mathcal{M}$.*

4. Ulam–Hyers stability of solutions

In this section, we study the Ulam–Hyers stability of (1.1).

Theorem 4.1. *Assume that* (3.1) *and* (3.3) *hold. Then, the model* (1.1) *is UH stable in $\mathcal{M}$.*

Proof. Let $v \in C^1([-\zeta, \mathfrak{T}], \mathbb{R})$ be a solution of the inequality (2.1). By Theorem 3.3, the following Cauchy problem has a unique solution:

$$\begin{cases} \left({}^{CF}D_0^{\upsilon} y\right)(\varkappa) = f(\varkappa, y(\varkappa)), & \varkappa \in [0, \mathfrak{T}], \quad \mathfrak{T} > 0, \\ y(\varkappa) = v(\varkappa), & \varkappa \in [-\zeta, 0], \quad 0 < \upsilon < 1. \end{cases}$$

Then, we have

$$\begin{aligned} y(\varkappa) = v(0) &+ \frac{2(1-\upsilon)}{(2-\upsilon)\mathcal{B}(\upsilon)} \Big[f(\varkappa, y(\varkappa)) - f(0, y(0)) \Big] \\ &+ \frac{2\upsilon}{(2-\upsilon)\mathcal{B}(\upsilon)} \int_0^{\varkappa} f(s, y(s)) ds, \quad 0 \leq \varkappa \leq \mathfrak{T} \end{aligned}$$

and $y(\varkappa) = v(\varkappa)$ for $\varkappa \in [-\zeta, 0]$. By the inequality (2.4), for each $0 \le \varkappa \le \mathfrak{T}$, we have

$$\begin{aligned}
|v(\varkappa) - y(\varkappa)| &= \Big| v(\varkappa) - v(0) - \frac{2(1-\upsilon)}{(2-\upsilon)\mathcal{B}(\upsilon)} \Big[f(\varkappa, y(\varkappa)) - f(0, y(0)) \Big] \\
&\quad - \frac{2\upsilon}{(2-\upsilon)\mathcal{B}(\upsilon)} \int_0^{\varkappa} f(s, y(s)) ds \Big| \\
&\le \Big| v(\varkappa) - v(0) - \frac{2(1-\upsilon)}{(2-\upsilon)\mathcal{B}(\upsilon)} \Big[f(\varkappa, v(\varkappa)) - f(0, v(0)) \Big] \\
&\quad - \frac{2\upsilon}{(2-\upsilon)\mathcal{B}(\upsilon)} \int_0^{\varkappa} f(s, v(s)) ds \Big| \\
&\quad + \frac{2(1-\upsilon)}{(2-\upsilon)\mathcal{B}(\upsilon)} \Big| f(\varkappa, v(\varkappa)) - f(\varkappa, y(\varkappa)) \Big| \\
&\quad + \frac{2\upsilon}{(2-\upsilon)\mathcal{B}(\upsilon)} \int_0^{\varkappa} \Big| f(s, v(s)) - f(s, y(s)) \Big| ds. \qquad (4.1)
\end{aligned}$$

For each $0 \le \varkappa \le \mathfrak{T}$, we have

$$\begin{aligned}
\Big| f(\varkappa, v(\varkappa)) - f(\varkappa, y(\varkappa)) \Big| &= |P v(\varkappa - \zeta) \exp(-\gamma v(\varkappa - \zeta)) - \delta v(\varkappa) \\
&\quad - P y(\varkappa - \zeta) \exp(-\gamma y(\varkappa - \zeta)) + \delta y(\varkappa) \\
&\quad - y(\varkappa - \zeta) \exp\left(-\gamma y(\varkappa - \zeta)\right)| + \delta \left| v(\varkappa) - y(\varkappa) \right| \\
&\le P \left| v(\varkappa - \zeta) - y(\varkappa - \zeta) \right| + \delta \left| v(\varkappa) - y(\varkappa) \right|. \quad (4.2)
\end{aligned}$$

Using (4.2) in (4.1) and applying (2.4), we obtain

$$\begin{aligned}
|v(\varkappa) - y(\varkappa)| &\le \frac{2\varepsilon}{(2-\upsilon)\mathcal{B}(\upsilon)} \left[2(1-\upsilon) + \upsilon \mathfrak{T} \right] \\
&\quad + \frac{2(1-\upsilon)}{(2-\upsilon)\mathcal{B}(\upsilon)} \Big[P \left| v(\varkappa - \zeta) - y(\varkappa - \zeta) \right| + \delta \left| v(\varkappa) - y(\varkappa) \right| \Big] \\
&\quad + \frac{2\upsilon}{(2-\upsilon)\mathcal{B}(\upsilon)} \int_0^{\varkappa} \Big[P \left| v(s - \zeta) - y(s - \zeta) \right| + \delta \left| v(s) - y(s) \right| \Big] ds \\
&\le \frac{2\varepsilon}{(2-\upsilon)\mathcal{B}(\upsilon)} \left[2(1-\upsilon) + \upsilon \mathfrak{T} \right] + \frac{4r(1-\upsilon)(P+\delta)}{(2-\upsilon)\mathcal{B}(\upsilon)} \\
&\quad + \frac{4rP\mathfrak{T}\upsilon}{(2-\upsilon)\mathcal{B}(\upsilon)} + \frac{2\upsilon\delta}{(2-\upsilon)\mathcal{B}(\upsilon)} \int_0^{\varkappa} \left| v(s) - y(s) \right| ds. \qquad (4.3)
\end{aligned}$$

Denote by

$$x(s) = |v(s) - y(s)|,$$

$$c = \frac{2\varepsilon}{(2-\upsilon)\mathcal{B}(\upsilon)} \left[2(1-\upsilon) + \upsilon \mathfrak{T} \right] + \frac{4r(1-\upsilon)(P+\delta)}{(2-\upsilon)\mathcal{B}(\upsilon)} + \frac{4rP\mathfrak{T}\upsilon}{(2-\upsilon)\mathcal{B}(\upsilon)}$$

and

$$d = \frac{2\upsilon\delta}{(2-\upsilon)\mathcal{B}(\upsilon)}.$$

Clearly, x is a nonnegative function defined on $[0, \mathfrak{T}]$, and c, d are nonnegative constants. From (4.3), we have

$$x(\varkappa) \leq c + d\int_0^{\varkappa} x(s)ds, \quad 0 \leq \varkappa \leq \mathfrak{T}.$$

It follows from Gronwall's inequality that

$$x(\varkappa) \leq ce^{d\varkappa}, \quad 0 \leq \varkappa \leq \mathfrak{T}.$$

That is,

$$|v(\varkappa) - y(\varkappa)| \leq ce^{d\varkappa}, \quad 0 \leq \varkappa \leq \mathfrak{T}.$$

Since $e^{d\varkappa}$ is a nondecreasing function of $\varkappa$ on $[0, \mathfrak{T}]$, we obtain

$$|v(\varkappa) - y(\varkappa)| \leq ce^{d\mathfrak{T}}, \quad 0 \leq \varkappa \leq \mathfrak{T}.$$

Choose

$$c_f = ce^{d\mathfrak{T}}.$$

Thus, the model (1.1) is Ulam–Hyers stable in $\mathcal{M}$. □

5. Extended discussion

Our analysis also opens the door to several natural extensions which highlight the adaptability of the CF framework in modeling complex real-world biological phenomena with memory effects, while preserving the analytical advantages offered by its non-singular kernel.

Extension to Harvesting Models. A biologically relevant generalization involves incorporating harvesting effects into the model [30]. For instance, the following extension of system (1.1) includes a continuous harvesting term H:

$$\begin{cases} \left({}^{CF}D_0^{\upsilon} y\right)(\varkappa) = Py(\varkappa-\zeta)e^{-\gamma y(\varkappa-\zeta)} - \delta y(\varkappa) - H(y(\varkappa)), & \varkappa \in [0, \mathfrak{T}], \\ y(\varkappa) = \phi(\varkappa), & \varkappa \in [-\zeta, 0], \end{cases} \tag{5.1}$$

where $H(y)$ represents the harvesting rate (e.g., $H(y) = hu$ for constant harvesting). Under suitable Lipschitz and boundedness conditions on H, the existence, uniqueness, and stability results established earlier remain valid. This demonstrates the flexibility of the Caputo–Fabrizio framework in accommodating external influences such as controlled population removal or resource exploitation.

Extension to Impulsive Models. Another important extension arises when considering impulsive effects, which describe sudden changes in population dynamics due to external shocks like seasonal harvesting, pesticide application, or migration bursts [3]. A possible impulsive version of the model is given by:

$$\begin{cases} \left({}^{CF}D_0^{\upsilon} y\right)(\varkappa) = Py(\varkappa-\zeta)e^{-\gamma y(\varkappa-\zeta)} - \delta y(\varkappa), & \varkappa \in [0,\mathfrak{T}],\ \varkappa \neq \varkappa_k, \\ \Delta y(\varkappa_k) = I_k(y(\varkappa_k^-)), & k = 1,2,\ldots,m, \\ y(\varkappa) = \phi(\varkappa), & \varkappa \in [-\zeta, 0], \end{cases} \tag{5.2}$$

where $\Delta y(\varkappa_k) = y(\varkappa_k^+) - y(\varkappa_k^-)$ denotes the jump at impulse point $\varkappa_k$, and $I_k : \mathbb{R} \to \mathbb{R}$ is a continuous function representing the magnitude of the impulse. Using similar fixed point techniques and adjusting the integral representation accordingly, one can derive analogous existence, uniqueness, and stability results for this impulsive system.

Extension to Stochastic Models. Another natural extension involves incorporating stochastic effects to account for environmental fluctuations and random disturbances, which are especially relevant in ecological systems [25]. The stochastic counterpart of the CF fractional Nicholsons model can be formulated as follows:

$$\begin{cases} \left({}^{CF}D_0^{\upsilon} u\right)(\varkappa) = Py(\varkappa-\zeta)e^{-\gamma y(\varkappa-\zeta)} - \delta y(\varkappa) + \sigma y(\varkappa)\,\dfrac{dB(\varkappa)}{d\varkappa}, & \varkappa \in [0,\mathfrak{T}], \\ y(\varkappa) = \varphi(\varkappa), & \varkappa \in [-\zeta, 0], \end{cases} \tag{5.3}$$

where $\mathcal{B}(\varkappa)$ denotes a standard Brownian motion, and $\sigma > 0$ is the intensity of the stochastic perturbation. This model captures the influence of random environmental variability on population dynamics. The inclusion of the multiplicative noise term $\sigma y(\varkappa)\,\frac{dB(\varkappa)}{d\varkappa}$ reflects the fact that larger populations are more susceptible to stochastic influences, a common feature in ecological modeling. Analytical investigation of such systems typically focuses on mean-square stability and the existence of moment-bounded solutions. While rigorous analysis in the CF setting requires further development of stochastic fractional calculus, preliminary approaches using Itô-type stochastic integrals and fixed point theory can be adapted to establish well-posedness. Moreover, by extending the techniques used in this paperparticularly the fixed point framework and fractional integral inequalitiesone can derive analogous UH-type stability results for the stochastic model under suitable assumptions on the noise intensity and the Lipschitz continuity of the nonlinear terms.

6. Applications and concluding remarks

In this section, we present numerical examples to validate the theoretical results and demonstrate the effectiveness of the proposed Caputo–Fabrizio fractional model of Nicholsons blowflies equation.

Example 6.1. Consider the model:

$$\begin{cases} \left({}^{CF}D_0^{0.5}y\right)(\varkappa) = (0.5)y(\varkappa - 0.5)\exp\left(-y(\varkappa - 0.5)\right) - (0.5)y(\varkappa), & \varkappa \in [0,1], \\ y(\varkappa) = \varkappa^2, \quad \varkappa \in [-0.5, 0]. \end{cases} \tag{6.1}$$

Here $\upsilon = 0.5$, $\mathcal{B}(\upsilon) = 1$, $P = 0.5$, $\delta = 0.5$, $\gamma = 1$, $\mathfrak{T} = 1$, $\zeta = 0.5$ and $\phi(\varkappa) = \varkappa^2$. If we choose $r \geq 0.7367$, then the conditions (3.1) and (3.2) hold. Then, by Theorem 3.1, the model (6.1) has a solution in the set $\mathcal{M}$ with $r \geq 0.7367$.

To support the theoretical results, we perform a numerical simulation for Example 6.1 using MATLAB. Figure 1 below shows the numerical solution $y(\varkappa)$ on the interval $[-\zeta, \mathfrak{T}]$. The smooth growth in the early stages and stabilization reflect the interplay between reproduction and mortality dynamics modeled by the Caputo–Fabrizio derivative.

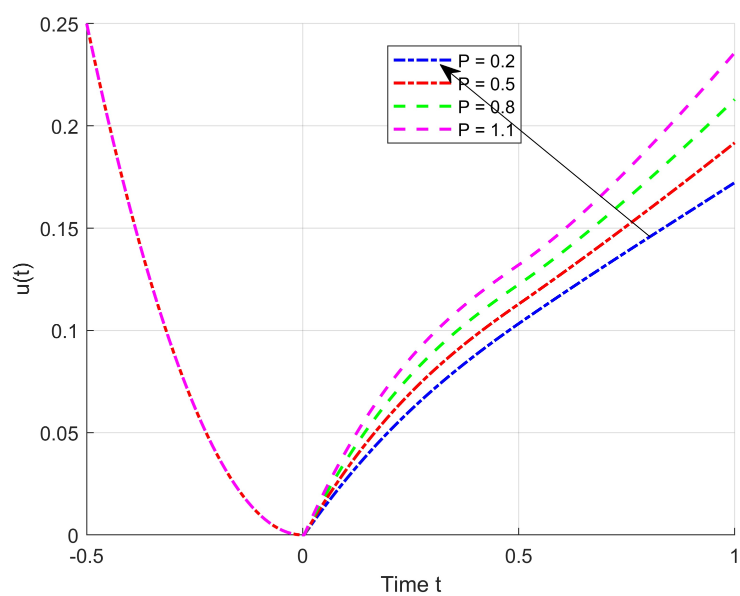

Figure 1. Numerical solution $y(\varkappa)$ of the fractional Nicholsons blowflies model using MATLAB.

Example 6.2. Consider the model

$$\begin{cases} \left({}^{CF}D_0^{0.5}y\right)(\varkappa) = (0.25)y(\varkappa - 0.5)\exp\left(-y(\varkappa - 0.5)\right) - (0.25)y(\varkappa), & \varkappa \in [0,1], \\ y(\varkappa) = \varkappa^2, \quad \varkappa \in [-0.5, 0]. \end{cases} \tag{6.2}$$

Here $\upsilon = 0.5$, $\mathcal{B}(\upsilon) = 1$, $P = 0.25$, $\delta = 0.25$, $\gamma = 1$, $\mathfrak{T} = 1$, $\zeta = 0.5$ and $\phi(\varkappa) = \varkappa^2$. If we choose $r \geq 0.0597$, then the conditions (3.1) and (3.3) hold. Then, by Theorem 3.3, the model (6.2) has a unique solution in the set $\mathcal{M}$ with $r \geq 0.0597$. Furthermore, the model (6.2) is UH stable in $\mathcal{M}$ with $r \geq 0.0597$.

To demonstrate the UH stability property, we simulate the exact solution $y(\varkappa)$ and a perturbed solution $v(\varkappa)$ satisfying

$$\left|\left({}^{CF}D_0^{0.5}v\right)(\varkappa) - f(\varkappa, v(\varkappa))\right| \leq \varepsilon,$$

where $\varepsilon > 0$ is a small error.

We use MATLAB to numerically compute both the exact and perturbed solutions, and compare them to confirm stability behavior.

Figure 2 shows the close match between the exact solution $y(\varkappa)$ and the perturbed solution $v(\varkappa)$, while Figure 3 demonstrates that the error $|y(\varkappa) - v(\varkappa)|$ remains uniformly bounded over time. This numerical observation is consistent with the UH stability result proven theoretically in Theorem 4.1.

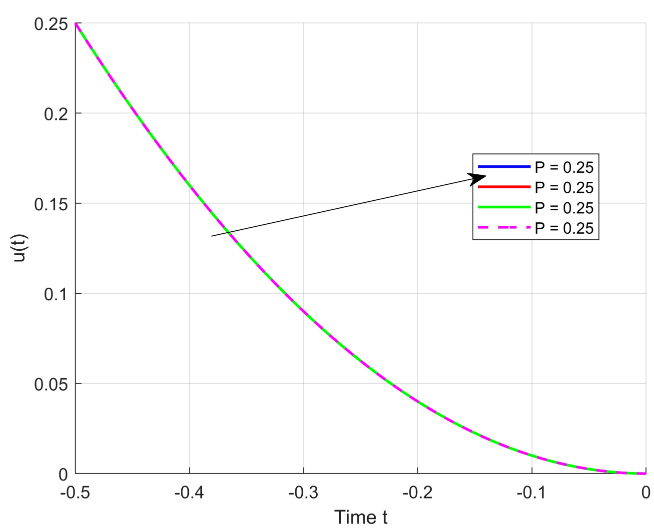

Figure 2. Comparison of the true and perturbed solutions in Example 3.

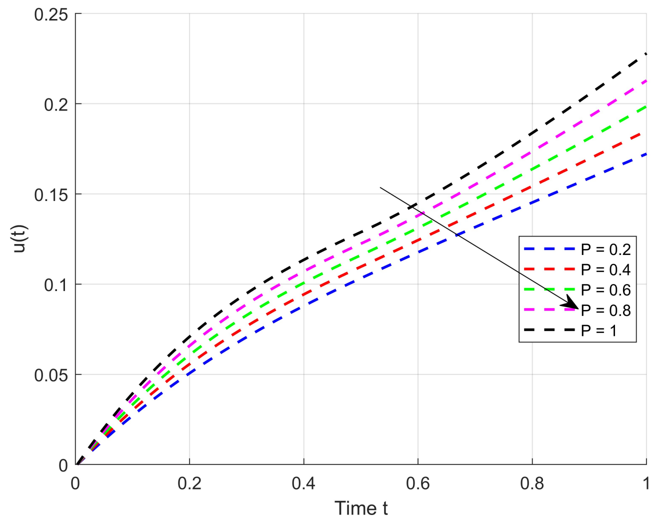

FIGURE 3. Error between solutions $|y(\varkappa) - v(\varkappa)|$ confirming Ulam–Hyers stability.

Remark 6.3. The illustrative examples and simulations in this study substantiate the analytical results, providing insight into the behavior of the Nicholson's blowflies model under the CF derivative framework. In Example 1 and Example 2, the numerical solutions exhibit biologically realistic dynamics: a gradual rise in population followed by stabilization, aligning with empirical observations of blowfly populations. Furthermore, the numerical investigation of UlamHyers stability in Example 2 demonstrates that small perturbations in initial data or model structure do not lead to large deviations in population trajectories. This stability property is crucial in biological modeling, where uncertainty is inherent.

7. Conclusion

In this work, we presented and analyzed a new fractional-order formulation of the Nicholsons blowflies model using the CF derivative, which incorporates a non-singular exponential kernel. This derivative provides significant advantages over traditional fractional operators by allowing for better handling of memory effects without singularities and ensuring compatibility with classical initial conditions. We established rigorous analytical results regarding the existence, uniqueness, and UlamHyers stability of solutions using fixed point theory. Furthermore, we extended the model to include biologically relevant factors such as harvesting, impulsive perturbations, and stochastic noise, demonstrating the flexibility and broad applicability of the proposed framework.

Numerical simulations were conducted to validate the theoretical findings and illustrate the model's dynamic behavior under various conditions. These simulations confirmed the stability of the system and showed that small perturbations do not lead to significant deviations, reinforcing the robustness of the model. Our analysis also included corollaries that relax certain assumptions on initial conditions, thereby expanding the scope of admissible data in practical applications.

This study represents the first integration of the CF fractional derivative into the Nicholsons blowflies model, contributing a novel perspective to the field of fractional biological modeling. Future research directions include investigating additional qualitative properties such as persistence, bifurcation, and global attractivity, as well as extending the current framework to incorporate spatial heterogeneity or multi-species interactions.

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