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SOME FIXED POINT RESULTS FOR INTERPOLATIVE CONTRACTIVE MAPPINGS IN C^*-ALGEBRA VALUED METRIC SPACES

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Abstract. In this paper, we present a new concept of interpolative contraction mappings in C^*-algebra valued complete metric spaces and we prove the existence of fixed points and common fixed points for Kannan-Riech type contractions.

1. Introduction and preliminaries

In 1968 Kannan [9] introduced a new result that there is a discontinuous mapping that satisfies certain conditions and admits a fixed point in a complete metric spaces. This result is still one of the most important results that

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generalizes the famous Banach contraction principle [3]. For more information on contractions and applications, see [1, 4, 5, 18].

Definition 1.1. ([9]) Let $\mathcal{U}$ be a metric space. A self-mapping $T:\mathcal{U}\to\mathcal{U}$ is said to be a Kannan contraction if there exists $\tau\in\left[0,\frac{1}{2}\right)$ such that

$$d(Tx,Ty)\leq\tau(d(x,Tx)+d(y,Ty)),\quad\forall x,y\in\mathcal{U}.$$

In 1969, Kannan [10] proved that if $(\mathcal{U},d)$ is a complete metric space, then every Kannan contraction on $\mathcal{U}$ has a unique fixed point. Let $\mathcal{U}$ be a metric space. A self-mapping T on $\mathcal{U}$ is called a Reich contraction if there exists $r\in\left[0,\frac{1}{3}\right)$ such that

$$d(Tu,Tv)\leq rd(u,v)+rd(u,Tu)+rd(v,Tv),\ \ \forall u,v\in\mathcal{U}.$$

In 1972, Reich [19] proved that if $(\mathcal{U},d)$ is complete metric space, then every Reich contraction on $\mathcal{U}$ has a unique fixed point. It is interesting that the Riechs theorem is a generalization of the Banach contraction principle [3] and a Kannan contraction theorem (see [7, 12]).

Let $\mathcal{U}$ be a metric space. A self-mapping T on $\mathcal{U}$ is called a weak contraction if there exist a lower semicontinuous function $\varphi:\mathcal{U}\to\mathbb{R}^+$ and an altering distance function $\phi:\mathcal{U}\to\mathbb{R}^+$ such that

$$\phi(d(Tu,Tv))\leq\phi(d(u,v))-\varphi(d(u,v)).$$

In Hilbert spaces, weak contraction principle was first introduced by Alber et al. [2].

Recently, Errai et al. [6] announced the notion of interpolative Kannan contraction on b-metric space, and they also proved some fixed point results for interpolative Kannan contraction. Moreover, the notion of interpolative Kannan contraction has been studied by many authors in the field of fixed point theory (see [11, 16] and references therein).

Theorem 1.2. ([6, 10]) *If $(\mathcal{U},d)$ is a complete metric space, then every Kannan contraction on $\mathcal{U}$ has a unique fixed point.*

In 2018, Karapinar [11] published a new type of contraction obtained from the definition of the Kannan contraction by interpolation as follows.

Definition 1.3. ([11]) Let $(\mathcal{U},d)$ be a metric space. A self-mapping $T:\mathcal{U}\to\mathcal{U}$ is said to be an interpolative Kannan type contraction if there are two constants $\tau,\beta\in(0,1)$ such that

$$d(Tx,Ty)\leq\tau(d(x,Tx)^{\beta})(d(y,Ty))^{1-\beta}$$

for all $x,y\in\mathcal{U}$ with $x\neq Tx$ and $y\neq Ty$.

Karapinar [11] obtained the following result.

Theorem 1.4. ([11]) *Let* $(\mathcal{U}, d)$ *be a complete metric space and* $T : \mathcal{U} \to \mathcal{U}$ *be an interpolative Kannan type contraction mapping. Then* T *has a unique fixed point.*

Recently, Ma et al. [13] announced the notion of C^*-algebra-valued metric space and formulated some first fixed point theorems in the C^*-algebra-valued metric space. Many authors initiated and studied many existing fixed point theorems in such spaces, see [14, 15, 17].

Throughout this paper, we denote A by a unital C^*-algebra with unit element I and involution $*$ such that for all $x, y \in A$,

$$(xy)^* = y^* x^* \quad \text{and} \quad x^{**} = x.$$

We call an element $x \in A$ a positive element, denote it by $x \succeq \theta$ if $x \in A_h = \{x \in A : x = x^*\}$ and $\sigma(x) \subset \mathbb{R}_+$,where $\sigma(x)$ is the spectrum of x. Using positive element, we can define a partial ordering $\preceq$ on A_h as follows :

$$x \preceq y \quad \text{if and only if} \quad y - x \succeq \theta,$$

where θ means the zero element in A.

Definition 1.5. ([17]) Let $\mathcal{U}$ be a nonempty set. Suppose a mapping $d : \mathcal{U} \times \mathcal{U} \to A_+$ satisfies

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all distinct points $x, y \in \mathcal{U}$;
(iii) $d(x, y) \preceq d(x, u) + d(u, y)$ for all $x, y, u \in \mathcal{U}$.

Then $(\mathcal{U}, A, d)$ is called a C^*-algebra-valued metric space.

Lemma 1.6. *Let* $\{x_n\}$ *be a sequence in a* C^**-algebra-valued metric space* $(\mathcal{U}, A, d)$ *such that*

$$d(x_n, x_{n+1}) \preceq \delta d(x_{n-1}, x_n)$$

for some $\delta \in [0, 1[$ *and for all* $n \in \mathbb{N}$. *Then* $\{x_n\}$ *is a Cauchy sequence in* $(\mathcal{U}, A, d)$.

Proof. First we can easily show the following:

$$d(x_{n+1}, x_n) \preceq \delta^n d(x_1, x_0), \quad \forall n \in \mathbb{N}.$$

For $m \geq 1$ and $p \geq 1$,

$$\begin{aligned} d(x_m, x_{m+p}) &\preceq b\left(d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+p})\right) \\ &\preceq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2} + \cdots + d(x_{m+p-2}, x_{m+p-1}) \\ &\quad + d(x_{m+p-1}, x_{m+p}) \\ &\preceq \delta^m d(x_0, x_1) + \delta^{m+1} d(x_0, x_1) + \delta^{m+2} d(x_0, x_1) \\ &\quad + \delta^{m+3} d(x_0, x_1) + \cdots + \delta^{m+p-1} d(x_0, x_1) \\ &= (\delta^m + \delta^{m+1} + \cdots + \delta^{m+p-1}) d(x_0, x_1) \\ &= \sum_{k=0}^{p-1} \delta^{m+k} d(x_0, x_1). \end{aligned}$$

Since $\delta \in [0,1)$, $\|d(x_m, x_{m+p})\| \leq \sum_{k=0}^{p-1} (\delta)^{m+k} \|d(x_0, x_1)\| \to 0$ as $m \to \infty$. Thus

$$\lim_{n,m\to\infty} d(x_n, x_m) = \theta.$$

This implies that $\{x_n\}_n$ is a Cauchy sequence. □

2. Main results

Inspired by the notion of interpolative Kannan contraction in a complete metric space, we present a new concept of interpolative Kannan-Reich weak type contraction. We discuss the existence of fixed point results for these new contractive mappings in a C^*-algebra valued complete metric space.

Definition 2.1. Let $(\mathcal{U}, A, d)$ be a C^*-algebra-valued metric space. A self-mapping $T : \mathcal{U} \to \mathcal{U}$ is called an interpolative Kannan type contraction if there exist $\tau, \beta \in (0,1)$ such that

$$d(Tx, Ty) \preceq \tau d(x, Tx)^\beta d(y, Ty)^{1-\beta}, \quad \forall x, y \in \mathcal{U},$$

where $Tx \neq x$.

Theorem 2.2. *Let $(\mathcal{U}, A, d)$ be a complete C^*-algebra-valued metric space and $T : \mathcal{U} \to \mathcal{U}$ be an interpolative Kannan type contraction. Then T has a unique fixed point in $\mathcal{U}$.*

Proof. Let $x_0 \in \mathcal{U}$ and define a sequence $\{x_n\} \in \mathcal{U}$ by $Tx_0 = x_1$ and $x_{n+1} = Tx_n$, $\forall n \in \mathbb{N}$. Suppose that $x_n \neq Tx_n$, $\forall n \in \mathbb{N}$. Then we have

$$\begin{aligned} d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) &\preceq \tau d(x_n, Tx_n)^\beta d(x_{n-1}, Tx_{n-1})^{1-\beta} \\ &= \tau d(x_{n-1}, x_n)^{1-\beta} d(x_n, x_{n+1})^\beta. \end{aligned}$$

This implies that

$$d(x_n, x_{n+1})^{1-\beta} \preceq \tau d(x_{n-1}, x_n)^{1-\beta}. \tag{2.1}$$

Thus we have that the sequence $\{d(x_n, x_{n-1})\}$ is nonincreasing and nonnegative. From (2.1), we deduce that

$$d(x_n, x_{n+1}) \preceq \tau d(x_{n-1}, x_n) \preceq \tau^n d(x_0, x_1). \tag{2.2}$$

Letting $n \to \infty$ in (2.2), we obtain $\lim_{n\to\infty} d(x_n, x_{n+1}) = \theta$. For $m \geq 1$ and $r \geq 1$, it follows that

$$\begin{aligned} d(x_m, x_{m+r}) &\preceq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{m+r-1}, x_{m+r}) \\ &\preceq \tau^m d(x_0, x_1) + \cdots + \tau^{m+r-1} d(x_0, x_1) \\ &\preceq \frac{\tau^m}{1-\tau} d(x_0, x_1) \to \theta \ \ as \ \ m \to \infty. \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in $(\mathcal{U}, A, d)$. Hence there exists $z \in \mathcal{U}$ such that

$$\lim_{n\to\infty} d(x_n, z) = \theta.$$

Now, we shall show that z is a fixed point of T. We get

$$d(Tx_n, Tz) \preceq \tau d(x_n, Tx_n)^{\beta} d(z, Tz)^{1-\beta}.$$

Letting $n \to \infty$ and using the concept of continuity of T, we have $d(z, Tz) = \theta$.

For uniqueness, let y be another fixed point of T. Then

$$\begin{aligned} d(z, y) &= d(Tz, Ty) \\ &\preceq \tau d(z, Tz)^{\beta} d(y, Ty)^{1-\beta} \\ &= 0. \end{aligned}$$

So $z = y$. □

Example 2.3. Let $\mathcal{U} = \{\frac{1}{2}, \frac{1}{4}\}$, $A = \mathbb{M}_2(\mathbb{R})$ and d be the metric defined on $\mathcal{U}$ by $d : \mathcal{U} \times \mathcal{U} \to A$ such that

$$d(x, y) = \begin{pmatrix} |x-y|^{\frac{1}{3}} & 0 \\ 0 & |x-y|^{\frac{1}{3}} \end{pmatrix}; x, y \in U.$$

Define self-mapping T by $T(\frac{1}{2}) = \frac{1}{4}$ and $T(\frac{1}{2}) = \frac{1}{2}$. Then T is an interpolative Kannan contraction for $\tau = \frac{2}{5}$ and $\beta = \frac{1}{2}$.

Definition 2.4. Let $(\mathcal{U}, A, d)$ be a C^*-algebra-valued metric space. A self-mapping $T : \mathcal{U} \to \mathcal{U}$ is called a (τ, β, η)-interpolative Kannan contraction if there are $\tau, \beta, \eta \in (0,1)$ such that $\beta + \eta < 1$ and

$$d(Tx, Ty) \preceq \tau d(x, Tx)^{\beta} d(y, Ty)^{\eta}, \quad \forall x, y \in \mathcal{U},$$

where $Tx \neq x$ and $Ty \neq y$.

Theorem 2.5. *Let $(\mathcal{U}, A, d)$ be a complete C^*-algebra-valued metric space and $T : \mathcal{U} \to \mathcal{U}$ be a (τ, β, η)-interpolative Kannan contraction. Then T has a unique fixed point in $\mathcal{U}$.*

Proof. Let $x_0 \in \mathcal{U}$ and define a sequence $\{x_n\} \in \mathcal{U}$ by $Tx_0 = x_1$ and $x_{n+1} = Tx_n$, for all $n \in \mathbb{N}$. Suppose that $x_n \neq Tx_n$, $\forall n \in \mathbb{N}$. Then we have

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \preceq \tau d(x_{n-1}, x_n)^{\beta} d(x_n, x_{n+1})^{\eta},$$

that is,

$$d(x_n, x_{n+1})^{1-\eta} \preceq \tau d(x_{n-1}, x_n)^{\beta} \preceq \tau d(x_{n-1}, x_n)^{1-\eta},$$

since $\beta \leq 1 - \eta$. The rest of the proof is similarly to the proof of Theorem 2.2. □

Definition 2.6. Let $(\mathcal{U}, A, d)$ be a C^*-algebra-valued metric space and $T, S : \mathcal{U} \to \mathcal{U}$ be two self-mappings. We call (T, S) a (τ, β, η)-interpolative Kannan contraction pair if there exist $\tau \in [0, 1)$, $0 < \beta, \eta < 1$ with $\beta + \eta < 1$ such that

$$d(Tx, Sy) \preceq \tau d(x, Tx)^{\beta} d(y, Sy)^{\eta}$$

for all $x, y \in \mathcal{U}$ with $x \neq Tx$, $y \neq Sy$.

Theorem 2.7. *Let $(\mathcal{U}, A, d)$ be a complete C^*-algebra-valued metric space and (T, S) be a (τ, β, η)-interpolative Kannan contraction pair. Then T and S have a unique common fixed point in $\mathcal{U}$, that is, there exists $z \in \mathcal{U}$ such that $Tz = z = Sz$.*

Proof. Let $x_0 \in \mathcal{U}$ and we define a sequence $\{x_n\} \in \mathcal{U}$ by $Sx_0 = x_1$ and $Tx_{2n+1} = x_{2n+2}$ and $Sx_{2n} = x_{2n+1}$ for all $n \in \mathbb{N}$. Then we have

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &\preceq \tau d(x_{2n}, x_{2n+1})^{\beta} d(x_{2n}, x_{2n+1})^{\eta} \\ &\preceq \tau d(x_{2n}, x_{2n+1})^{\beta} d(x_{2n+1}, x_{2n+2})^{1-\beta}. \end{aligned}$$

Thus

$$d(x_{2n+1}, x_{2n+2})^{\beta} \preceq \tau d(x_{2n}, x_{2n+1})^{\beta}.$$

So

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &\preceq \tau^{\frac{1}{1-\beta}} d(x_{2n}, x_{2n+1}) \\ &\preceq \tau d(x_{2n}, x_{2n+1}). \end{aligned}$$

Hence

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &\preceq \tau d(x_{2n}, x_{2n+1}) \\ &\preceq \tau^2 d(x_{2n-1}, x_{2n}) \\ &\ \vdots \\ &\preceq \tau^{2n+1} d(x_0, x_1). \end{aligned}$$

Similarly,

$$\begin{aligned} d(x_{2n+1}, x_{2n}) &= d(Tx_{2n}, Sx_{2n-1}) \\ &\preceq \tau d(x_{2n}, x_{2n+1})^{\beta} d(x_{2n-1}, x_{2n})^{1-\beta}. \end{aligned}$$

So we have

$$d(x_{2n+1}, x_{2n})^{1-\beta} \preceq \tau d(x_{2n-1}, x_{2n})^{1-\beta}.$$

Thus

$$\begin{aligned} d(x_{2n+1}, x_{2n}) &\preceq \tau^{\frac{1}{1-\beta}} d(x_{2n-1}, x_{2n}) \\ &\preceq \tau d(x_{2n-1}, x_{2n}). \end{aligned}$$

Hence

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &\preceq \tau^{\frac{1}{1-\beta}} d(x_{2n}, x_{2n+1}) \\ &\preceq \tau d(x_{2n}, x_{2n+1}) \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} d(x_{2n+1}, x_{2n}) &\preceq \tau d(x_{2n-1}, x_{2n}) \\ &\preceq \tau^2 d(x_{2n-2}, x_{2n-1}) \\ &\ \vdots \\ &\preceq \tau^{2n} d(x_0, x_1). \end{aligned}$$

So

$$d(x_{2n+1}, x_{2n}) \preceq \tau^{2n} d(x_0, x_1). \tag{2.4}$$

From (2.3) and (2.4), it follows that

$$d(x_n, x_{n+1}) \preceq \tau^n d(x_0, x_1). \tag{2.5}$$

Using (2.5), we prove that $\{x_n\}$ is a Cauchy sequence.

For $m \geq 1$ and $r \geq 1$, it follows that

$$\begin{aligned} d(x_m, x_{m+r}) &\preceq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + ... + d(x_{m+r-1}, x_{m+r}) \\ &\preceq \tau^m d(x_0, x_1) + \cdots + \tau^{m+r-1} d(x_0, x_1) \\ &\preceq \frac{\tau^m}{1-\tau} d(x_0, x_1) \to \theta \ \ as \ \ m \to \infty. \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in $(\mathcal{U}, A, d)$. Hence there exists $z \in \mathcal{U}$ such that

$$\lim_{n\to\infty} d(x_n, z) = \theta.$$

Now, we shall show that z is a fixed point of T. We get

$$\begin{aligned} d(Tz, x_{2n+2}) &= d(Tz, Sx_{2n+1}) \\ &\preceq \tau d(z, Tz)^{\beta} d(x_{2n+1}, x_{2n+2})^{1-\beta}. \end{aligned}$$

Letting $n \to \infty$ and using the continuity of T, we have $d(z, Tz) = \theta$.

Similarly,

$$\begin{aligned} d(x_{2n+2}, Sz) &= d(Tx_{2n}, Sz) \\ &\preceq \tau d(x_{2n}, x_{2n+1})^{\beta} d(z, Sz)^{1-\beta}. \end{aligned}$$

Letting $n \to \infty$, we obtain $Sz = z$.

For uniqueness, let y be another common fixed point of T and S. Then

$$\begin{aligned} d(z, y) &= d(Tz, Sy) \\ &\preceq \tau d(z, Tz)^{\beta} d(y, Sy)^{1-\beta} \\ &= 0. \end{aligned}$$

Thus $z = y$. □

Definition 2.8. Let $(\mathcal{U}, A, d)$ be a C^*-algebra-valued metric space and $T, R : \mathcal{U} \to \mathcal{U}$ be two self-mappings. We call T an R-interpolative Kannan type contraction if there exist $\tau \in [0, 1)$ and $0 < \beta < 1$ such that

$$d(Tx, Ty) \preceq \tau d(Rx, Tx)^{\beta} d(Ry, Ty)^{1-\beta}$$

for all $x, y \in \mathcal{U}$ with $x \neq Tx$ and $y \neq Ty$.

Theorem 2.9. *Let $(\mathcal{U}, A, d)$ be a complete C^*-algebra-valued metric space and $T : \mathcal{U} \to \mathcal{U}$ be an R-interpolative Kannan contraction. Assume that $T\mathcal{U} \subset R\mathcal{U}$ and $R\mathcal{U}$ is closed. If there exist $\tau \in [0, 1)$ and $0 < \beta < 1$ such that*

$$d(Tx, Ty) \preceq \tau d(Rx, Tx)^{\beta} d(Ry, Ty)^{1-\beta}$$

for all $x, y \in \mathcal{U}$ with $x \neq Tx$, $y \neq Ty$, then T and R have a unique common fixed point in $\mathcal{U}$.

Proof. Let $x_0 \in \mathcal{U}$ and we define a sequence $\{x_n\} \in \mathcal{U}$ by $Rx_1 = Tx_0$ and $Rx_{n+1} = Tx_n$, $\forall n \in \mathbb{N}$. Then

$$\begin{aligned} d(Tx_{n+1}, Tx_n) &\preceq \tau d(Rx_{n+1}, Tx_{n+1})^{\beta} d(Rx_n, Tx_n)^{1-\beta} \\ &= \tau d(Tx_n, Tx_{n+1})^{\beta} d(Tx_{n-1}, Tx_n)^{1-\beta}. \end{aligned}$$

Then

$$d(Tx_{n+1}, Tx_n) \preceq \tau^{\frac{1}{1-\beta}} d(Tx_n, Tx_{n-1}) \preceq \tau d(Tx_n, Tx_{n-1}).$$

Lemma 1.6 implies that $\{Tx_n\}$ is a Cauchy sequence and consequently $\{Rx_n\}$ is also a Cauchy sequence.

Let $z \in \mathcal{U}$ such that $\lim_{n\to\infty} d(Tx_n, z) = \lim_{n\to\infty} d(Rx_{n+1}, z) = \theta$. Since $z \in R\mathcal{U}$, there exists $v \in \mathcal{U}$ such that $z = Rv$. Then, we obtain

$$d(Tx_n, Tz) \preceq \tau d(Rx_n, Tx_n)^{\beta} d(Rv, Tv)^{1-\beta}.$$

Letting $n \to \infty$, we obtain $z = Rv = Tv$. □

Example 2.10. Let $\mathcal{U} = (0, \infty)$, $A = \mathbb{R}^2$ and d be the metric defined on $\mathcal{U}$ by $d : \mathcal{U} \times \mathcal{U} \to A$ such that $d(x, y) = ((x + y)^2, 0)$. Define two self-mappings T and R by $Tx = \frac{1}{x}$ and $R(x) = x^2$. Then T is an R-interpolative Kannan contraction for $\tau = \frac{3}{4}$ and $\beta = \frac{2}{5}$.

Theorem 2.11. *Let $(\mathcal{U}, A, d)$ be a complete C^*-algebra-valued metric space and $T : \mathcal{U} \to \mathcal{U}$ satisfy the following:*

$$d(Tx, Ty) \preceq \tau d(x, y)^{\alpha} d(x, Ty)^{\beta} d(y, Ty)^{\eta}$$

for all $x, y \in \mathcal{U}$ with $x \neq Tx$, $y \neq Ty$, where $\tau \in (0, 1)$ and $\alpha, \beta, \eta \in (0, 1)$ such that $\alpha + \beta + \eta > 1$. If there exists $x_0 \in \mathcal{U}$ such that $d(x_0, Tx_0) \preceq I$, then T has a unique fixed point in $\mathcal{U}$.

Proof. Let $x_0 \in \mathcal{U}$ and define a sequence $\{x_n\} \in \mathcal{U}$ by $Tx_0 = x_1$ and $x_{n+1} = Tx_n$, for all $n \in \mathbb{N}$. Suppose that $x_n \neq Tx_n$, for all $n \in \mathbb{N}$. Then we have

$$\begin{aligned} d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) &\preceq \tau d(x_{n-1}, x_n)^{\alpha} d(x_{n-1}, Tx_{n-1})^{\beta} d(x_n, Tx_n)^{\eta} \\ &= \tau d(x_{n-1}, x_n)^{\alpha} d(x_{n-1}, x_n)^{\beta} d(x_n, x_{n+1})^{\eta} \\ &= \tau d(x_{n-1}, x_n)^{\alpha+\beta} d(x_n, x_{n+1})^{\eta}. \end{aligned}$$

Therefore,

$$\begin{aligned} d(x_n, x_{n+1}) &\preceq (\tau d(x_{n-1}, x_n)^{\alpha+\beta})^{\frac{1}{1-\eta}} \\ &= \tau^{\frac{1}{1-\eta}} d(x_{n-1}, x_n)^{\frac{\alpha+\beta}{1-\eta}} \\ &\preceq \tau^{\frac{2}{1-\eta}} d(x_{n-2}, x_{n-1})^{\frac{2(\alpha+\beta)}{1-\eta}} \\ &\vdots \\ &\preceq \tau^{\frac{n}{1-\eta}} d(x_0, x_1)^{\frac{n(\alpha+\beta)}{1-\eta}}. \end{aligned}$$

Thus

$$d(x_n, x_{n+1}) \preceq \tau^{\frac{n}{1-\eta}}.$$

Since $\tau < 1$ and $\frac{n}{1-\eta} > 1$, we have

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \theta.$$

For $m \geq 1$ and $r \geq 1$, it follows that

$$\begin{aligned} d(x_m, x_{m+r}) &\preceq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{m+r-1}, x_{m+r}) \\ &\preceq \tau^{\frac{m}{1-\eta}} + \cdots + \tau^{\frac{m+r-1}{1-\eta}} \\ &\preceq \tau^{\frac{m}{1-\eta}} \frac{1+\tau}{1-\tau^2} \to \theta \ \ as \ \ m \to \infty. \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in $(\mathcal{U}, A, d)$. So there exists $z \in \mathcal{U}$ such that

$$\lim_{n\to\infty} d(x_n, z) = \theta.$$

Now, we shall show that z is a fixed point of T. We get

$$d(Tx_n, Tz) \preceq \tau d(x_n, z)^{\alpha} d(x_n, Tz)^{\beta} d(z, Tz)^{\eta}.$$

Letting $n \to \infty$ and using the continuity of T, we have $d(z, Tz) = \theta$. □

Definition 2.12. A mapping $\psi : A^+ \to A^+$ is called an alterning distance function if it satisfies the following conditions:

(i) ψ is continuous,
(ii) ψ is nondecreasing,
(iii) $\psi(x) = \theta$ if and only if $x = \theta$.

Definition 2.13. Let $(\mathcal{U}, A, d)$ be a complete C^*-algebra-valued metric space. A self-mapping $T : \mathcal{U} \to \mathcal{U}$ is called an interpolative weakly contractive mapping of the Reich type if it satisfies the following:

$$\phi(d(Tx, Ty)) \preceq \phi(d(x, y)^{\alpha} d(x, Ty)^{\beta} d(y, Ty)^{\eta}) - \psi(d(x, y)^{\alpha} d(x, Ty)^{\beta} d(y, Ty)^{\eta})$$

for all $x, y \in \mathcal{U}$, where ϕ is an alterning distance function and ψ is a mapping from A_+ to A_+, which is characterized by its lower semicontinuity satisfying the condition that $\psi(x) = \theta \Leftrightarrow x = \theta$.

Theorem 2.14. *Let $(\mathcal{U}, A, d)$ be a complete C^*-algebra-valued metric space and $T : \mathcal{U} \to \mathcal{U}$ be an interpolative weakly contractive mapping of Reich type for all $x, y \in \mathcal{U}$ with $x \neq Tx$ and $y \neq Ty$. Let $\alpha, \beta, \eta \in (0, 1)$ such that $\alpha + \beta + \eta = 1$. If there exists a point $x_0 \in \mathcal{U}$ satisfying $d(x_0, Tx_0) \preceq I$, then T has a fixed point in $\mathcal{U}$.*

Proof. Let $x_0 \in \mathcal{U}$ and define a sequence $\{x_n\} \in \mathcal{U}$ by $Tx_0 = x_1$ and $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. Suppose that $x_n \neq Tx_n$ for all $n \in \mathbb{N}$. Then we have

$$\begin{aligned}\phi(d(x_n, x_{n+1})) &\preceq \phi(d(x_{n-1}, x_n)^{\alpha} d(x_{n-1}, Tx_{n-1})^{\beta} d(x_n, Tx_n)^{\eta}) \\ &- \psi(d(x_{n-1}, x_n)^{\alpha+\beta} d(x_n, Tx_n)^{\eta}) \\ &\preceq \phi(d(x_{n-1}, x_n)^{\alpha+\beta} d(x_n, x_{n+1})^{\eta}).\end{aligned}$$

This implies that

$$\begin{aligned}d(x_n, x_{n+1}) &\preceq d(x_{n-1}, x_n)^{\alpha+\beta} d(x_n, x_{n+1})^{\eta} \\ &\preceq d(x_{n-1}, x_n)^{\frac{\alpha+\beta}{1-\eta}} \\ &\preceq d(x_{n-1}, x_n) \\ &\preceq d(x_0, x_1) \\ &\preceq I.\end{aligned}$$

Then there exists $\gamma \succeq \theta$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \gamma.$$

Letting $n \to \infty$ in

$$\phi(d(x_n, x_{n+1})) \preceq \phi(d(x_{n-1}, x_n)^{\alpha+\beta} d(x_n, Tx_n)^{\eta}) - \psi(d(x_{n-1}, x_n)^{\alpha+\beta} d(x_n, Tx_n)^{\eta}),$$

we obtain

$$\phi(\gamma) \preceq \phi(\gamma^{\alpha+\beta}\gamma^{\eta}) - \psi(\gamma^{\alpha+\beta}\gamma^{\eta}) \preceq \phi(\gamma) - \psi(\gamma).$$

This implies that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \theta.$$

For $m \geq 1$ and $p \geq 1$, it follows that

$$\begin{aligned}\phi(d(x_{m+1}, x_{p+1})) &\preceq \phi(d(x_m, x_p)^{\alpha} d(x_m, x_{m+1})^{\beta} d(x_p, x_{p+1})^{\eta}) \\ &\quad - \psi(d(x_m, x_p)^{\alpha} d(x_m, x_{m+1})^{\beta} d(x_p, x_{p+1})^{\eta}) \\ &\to \theta \ \ as \ \ m, p \to \infty.\end{aligned}$$

Thus $\{x_n\}$ is a Cauchy sequence, and there exists $z \in \mathcal{U}$ such that $\lim_{n\to\infty} d(x_n, z) = \theta$. So we have

$$d(x_n, z)^{\alpha} d(x_n, x_{n+1})^{\beta} d(z, Tz)^{\eta} \to \theta.$$

Hence, we get

$$\begin{aligned}\phi(d(z, Tz)) &\preceq \phi(d(x_n, z)^{\alpha} d(x_n, x_{n+1})^{\beta} d(z, Tz)^{\eta}) \\ &\quad - \psi(d(x_n, z)^{\alpha} d(x_n, x_{n+1})^{\beta} d(z, Tz)^{\eta}).\end{aligned}$$

So, we have $d(z, Tz) = \theta$. □

Corollary 2.15. *Let $(\mathcal{U}, A, d)$ be a complete C^*-algebra-valued metric space and $T : \mathcal{U} \to \mathcal{U}$ be a mapping such that there exists $k \in (0, 1)$ such that*

$$d(Tx, Ty) \preceq k[d(x, y)^{\alpha} d(x, Tx)^{\beta} d(y, Ty)^{\eta}],$$

where $\alpha, \beta, \eta \in]0, 1[$ and $\alpha + \beta + \eta = 1$. If there exists a point $x_0 \in \mathcal{U}$ satisfying $d(x_0, Tx_0) \preceq I$, then T has a fixed point in $\mathcal{U}$.

Proof. It is sufficient to apply $\phi(x) = x$ and $\psi(x) = (I - k)x$ in Theorem 2.14. □

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