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SUFFICIENT CONDITIONS AND INCLUSION PROPERTIES FOR ANALYTIC SUBCLASSES INVOLVING THE GENERALIZED IMAGINARY ERROR FUNCTION

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Abstract. In this paper, we investigate some characterizations of the generalized normalized imaginary error function to be in subclasses of analytic functions.

1. Introduction and preliminaries

Let $\digamma$ be the class of analytic functions of the form:

$$B(\Im) = \Im + \sum_{\wp=2}^{\infty} \gamma_{\wp} \Im^{\wp}, \quad \Im \in \Theta = \{\Im \in \mathbb{C} : |\Im| < 1\}. \tag{1.1}$$

A function B of the form (1.1) is in the subclass $S^*(\gimel_1, \gimel_2)$ if it satisfies the following condition:

$$\Re\left\{\frac{B'(\Im) + \gimel_1 \Im^2 B''(\Im)}{B(\Im)}\right\} > \gimel_2 \qquad (\Im \in \Theta;\ \gimel_1, \gimel_2 \in [0,1))$$

and a function B is in the subclass $C(\gimel_1, \gimel_2)$ if

$$\Re\left\{\frac{\left[\Im B'(\Im) + \gimel_1 \Im^2 B''(\Im)\right]'}{B'(\Im)}\right\} > \gimel_2 \qquad (\Im \in \Theta;\ \gimel_1, \gimel_2 \in [0,1)).$$

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Remark 1.1. For $B \in \digamma$, observe that $B \in C(\gimel_1, \gimel_2)$ if and only if $\Im B'(\Im) \in S^*(\gimel_1, \gimel_2)$.

Remark 1.2. For $\gimel_1 = 0$, we get $S^*(\gimel_1, \gimel_2) = S^*(\gimel_2)$ and $C(\gimel_1, \gimel_2) = C(\gimel_2)$, where $S^*(\gimel_2)$ and $C(\gimel_2)$ are the will-known classes of starlike and convex of order $\gimel_2 (0 \leq \gimel_2 < 1)$, respectively (see, Robertson [20]).

Definition 1.3. ([8]) A function $B \in \digamma$ be in the class $H^\tau(O_1, O_2)$, $\tau \in \mathbb{C} \backslash \{0\}$, $-1 \leq O_2 < O_1 \leq 1$, if it fulfills the condition

$$\left| \frac{B'(\Im) - 1}{(O_1 - O_2)\tau - O_2[B'(\Im) - 1]} \right| < 1, \quad \Im \in \Theta.$$

If we put $\tau = 1$, $O_1 = \xi$, and $O_2 = -\xi$ $(0 < \xi \leq 1)$, we get the class of functions $B \in \digamma$ satisfying the condition

$$\left| \frac{B'(\Im) - 1}{B'(\Im) + 1} \right| < \xi, \quad (\Im \in \Theta, 0 < \xi \leq 1).$$

Geometric function theory is known to heavily rely on special functions. It's also common knowledge that special functions are not just used in the theory of geometric functions. These functions have several uses in various issues as well as in other areas of applied sciences and mathematics, see [3, 5, 11, 16, 17, 18].

Abramowitz and Stegun [1] defined the error function erB as:

$$erB\,(\Im) = \frac{2}{\sqrt{\pi}} \int_0^{\Im} e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \sum_{\wp=0}^{\infty} \frac{(-1)^{\wp}\, \Im^{2\wp+1}}{(2\wp + 1)\, \wp!}, \qquad (\Im \in \mathbb{C}), \tag{1.2}$$

whereas the imaginary error function

$$erBi\,(\Im) = \frac{2}{\sqrt{\pi}} \int_0^{\Im} e^{t^2} dt = \frac{2}{\sqrt{\pi}} \sum_{\wp=0}^{\infty} \frac{\Im^{2\wp+1}}{(2\wp + 1)\, \wp!}, \qquad (\Im \in \mathbb{C}). \tag{1.3}$$

Statistics, applied mathematics, and the physics of partial differential equations all heavily rely on the error function. An essential instrument in quantum physics for calculating the probability of observing a particle in a specific location is the error function. Numerous features and inequalities of the error function demonstrated by Alzer [2] and Coman [6], while the characteristics of the complementary error function examined by Elbert et al. [9].

A generalization of the error function (1.2) is

$$erB_t(\Im) = \frac{t!}{\sqrt{\pi}} \int_0^{\Im} e^{-t^t} dt, \quad t \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$$
$$= \frac{t!}{\sqrt{\pi}} \sum_{\wp=0}^{\infty} \frac{(-1)^{\wp} \Im^{t\wp+1}}{(t\wp+1)\,\wp!}, \qquad (\Im \in \mathbb{C}). \tag{1.4}$$

While a generalization of the imaginary error function (1.3) is

$$erBi_t(\Im) = \frac{t!}{\sqrt{\pi}} \int_0^{\Im} e^{t^t} dt, \quad t \in \mathbb{N}_0$$
$$= \frac{t!}{\sqrt{\pi}} \sum_{\wp=0}^{\infty} \frac{\Im^{t\wp+1}}{(t\wp+1)\,\wp!}, \qquad (\Im \in \mathbb{C}). \tag{1.5}$$

From (1.4) and (1.5), we get

$$erB_0(\Im) = \frac{\Im}{e\sqrt{\pi}},\ erB_1(\Im) = \frac{1-e^{\Im}}{\sqrt{\pi}} = -erBi_1(\Im),\ erB_2(\Im) = erB(\Im),$$

$$\text{and} \quad erBi_2(\Im) = erBi(\Im).$$

The functions $erB_t(\Im)$ and $erBi_t(\Im)$ are not in $\digamma$. So, we examine the following functions (see, Al-Hawary et al. [14]).

$$\wp_t(\Im) = \frac{\sqrt{\pi}}{t!} \Im^{\left(1-\frac{1}{t}\right)} erB\left(\Im^{1/t}\right)$$
$$= \Im + \sum_{\wp=2}^{\infty} \frac{(-1)^{\wp-1}}{((\wp-1)t+1)(\wp-1)!} \Im^{\wp}, \qquad (t \in \mathbb{N}) \tag{1.6}$$

and

$$\wp i_t(\Im) = \frac{\sqrt{\pi}}{t!} \Im^{\left(1-\frac{1}{t}\right)} erBi_t\left(\Im^{1/t}\right)$$
$$= \Im + \sum_{\wp=2}^{\infty} \frac{1}{((\wp-1)t+1)(\wp-1)!} \Im^{\wp}, \quad (t \in \mathbb{N}). \tag{1.7}$$

From (1.6) and (1.7), we have

$$\wp_1 B(\Im) = \sqrt{\pi} erB_1(\Im) = 1 - e^{\Im},\ \wp i_1 B(\Im) = \sqrt{\pi} erBi_1(\Im) = e^{\Im} - 1$$

and

$$\wp_2 B(\Im) = \frac{\sqrt{\pi \Im}}{2} erB_2\left(\sqrt{\Im}\right) \quad \text{and} \quad \wp i_2 B(\Im) = \frac{\sqrt{\pi \Im}}{2} erBi_2\left(\sqrt{\Im}\right).$$

Let the function $\Pi i_t(\Im)$ be defined as:

$$\Pi i_t(\Im) = 2\Im - \wp i_t(\Im) = \Im - \sum_{\wp=2}^{\infty} \frac{1}{((\wp-1)t+1)(\wp-1)!}\Im^{\wp}, \qquad \Im \in \Theta \tag{1.8}$$

and the linear operator $\mathcal{I}i_t : F \to F$ defined as:

$$\mathcal{I}i_t(\Im) = \wp i_t(\Im) * B(\Im) = \Im + \sum_{\wp=2}^{\infty} \frac{1}{((\wp-1)t+1)(\wp-1)!}\gamma_{\wp}\Im^{\wp}. \tag{1.9}$$

Recently, several academics have employed a variety of special functions to determine certain requirements to be in subclasses of analytic functions (see [4, 7, 10, 12, 13, 15, 21, 22]). Inspired by these works, we will determine some sufficient properties of analytic subclasses functions $S^*(\daleth_1, \daleth_2)$ and $C(\daleth_1, \daleth_2)$.

Lemma 1.4. ([19]) (i) *A sufficient condition for a function B of the form* (1.1) *to be in the subclass $S^*(\daleth_1, \daleth_2)$ is that*

$$\sum_{\wp=2}^{\infty}(\wp + \daleth_1\wp(\wp-1) - \daleth_2)|\gamma_{\wp}| \leq 1 - \daleth_2. \tag{1.10}$$

Specifically, when $\daleth_1 = 0$, we get a sufficient condition for a function B to be in $S^*(\daleth_2)$ is that

$$\sum_{\wp=2}^{\infty}(\wp - \daleth_2)|\gamma_{\wp}| \leq 1 - \daleth_2. \tag{1.11}$$

(ii) *A sufficient condition for a function B of the form* (1.1) *to be in the subclass $C(\daleth_1, \daleth_2)$ is that*

$$\sum_{\wp=2}^{\infty}\wp(\wp + \daleth_1\wp(\wp-1) - \daleth_2)|\gamma_{\wp}| \leq 1 - \daleth_2. \tag{1.12}$$

Specifically, when $\daleth_1 = 0$, we get a sufficient condition for a function B to be in the subclass $C(\daleth_2)$ is that

$$\sum_{\wp=2}^{\infty}\wp(\wp - \daleth_2)|\gamma_{\wp}| \leq 1 - \daleth_2. \tag{1.13}$$

Lemma 1.5. ([8]) *If B of the form* (1.1) *and $B \in H^{\tau}(O_1, O_2)$, then*

$$|\gamma_{\wp}| \leq \frac{(O_1 - O_2)|\tau|}{\wp}, \qquad \wp \in \mathbb{N} - \{1\}. \tag{1.14}$$

The last result is sharp for the function $B(\Im)$ given by

$$B(\Im)=\int\limits_{0}^{\Im}\left(1+\frac{(O_1-O_2)\tau t^{\wp-1}}{1+O_2t^{\wp-1}}\right)dt \quad (\ \Im\in\Theta,\ \wp\geq 2). \tag{1.15}$$

The following series sums are used in the sequel.

$$\sum_{\wp=2}^{\infty}\frac{1}{(\wp-1)\,2^{\wp}}=\frac{1}{2}\ln 2, \tag{1.16}$$

$$\sum_{\wp=3}^{\infty}\frac{1}{2^{\wp}\,(\wp-1)}=\frac{1}{2}\ln 2-\frac{1}{4} \tag{1.17}$$

and

$$\sum_{\wp=4}^{\infty}\frac{1}{2^{\wp}\,(\wp-1)}=\frac{1}{2}\ln 2-\frac{1}{4}-\frac{1}{16}. \tag{1.18}$$

Note that

$$\sum_{\wp=d}^{\infty}\frac{1}{(\wp-1)\,2^{\wp}}=\frac{1}{2}\ln 2-\sum_{\wp=2}^{d-1}\frac{1}{(\wp-1)\,2^{\wp}},\ d=3,4,\cdots. \tag{1.19}$$

The following inequalities are also required

$$(\wp-1)t+1>(\wp-1)t \quad (\wp,t\in\mathbb{N}) \tag{1.20}$$

and

$$\wp!\geq 2^{\wp-1} \quad (\wp\in\mathbb{N}). \tag{1.21}$$

2. Sufficient conditions for the subclasses $S^*(\gimel_1,\gimel_2)$ and $C(\gimel_1,\gimel_2)$

In this section, we examine sufficient conditions for the function $\Pi i_t\,(\Im)$ to be in the subclasses $S^*(\gimel_1,\gimel_2)$ and $C(\gimel_1,\gimel_2)$.

Theorem 2.1. *If $t\in\mathbb{N}$, then $\Pi i_t\,(\Im)$ is in the subclass $S^*(\gimel_1,\gimel_2)$ if*

$$2\left[(8\gimel_1-\gimel_2+3)\ln 2-2\gimel_1\right]\leq t\,(1-\gimel_2) \tag{2.1}$$

and

$$(3\gimel_1-\gimel_2+2).e\leq(1-\gimel_2)\,(t+2). \tag{2.2}$$

Proof. Since

$$\Pi i_t\,(\Im)=\Im-\sum_{\wp=2}^{\infty}\frac{1}{((\wp-1)\,t+1)\,(\wp-1)!}\Im^{\wp}, \tag{2.3}$$

by virtue of (1.10) it suffices to show that

$$\begin{aligned} L_1(\gimel_1,\gimel_2) &= \sum_{\wp=2}^{\infty}(\wp+\gimel_1\wp(\wp-1)-\gimel_2)\ \frac{1}{((\wp-1)\,t+1)\,(\wp-1)!} \\ &= \sum_{\wp=2}^{\infty}(\gimel_1\wp^2+(1-\gimel_1)\wp-\gimel_2)\ \frac{1}{((\wp-1)\,t+1)\,(\wp-1)!} \\ &\leq 1-\gimel_2. \end{aligned}$$

Letting

$$\wp=(\wp-1)+1 \tag{2.4}$$

and

$$\wp^2=(\wp-1)(\wp-2)+3(\wp-1)+1, \tag{2.5}$$

we get

$$\begin{aligned} L_1(\gimel_1,\gimel_2) &= \sum_{\wp=2}^{\infty}\frac{\gimel_1(\wp-1)(\wp-2)}{((\wp-1)\,t+1)\,(\wp-1)!}+\sum_{\wp=2}^{\infty}\frac{(2\gimel_1+1)(\wp-1)}{((\wp-1)\,t+1)\,(\wp-1)!} \\ &\quad+\sum_{\wp=2}^{\infty}\frac{1-\gimel_2}{((\wp-1)\,t+1)\,(\wp-1)!} \\ &= \sum_{\wp=3}^{\infty}\frac{\gimel_1}{((\wp-1)\,t+1)\,(\wp-3)!}+\sum_{\wp=2}^{\infty}\frac{2\gimel_1+1}{((\wp-1)\,t+1)\,(\wp-2)!} \\ &\quad+\sum_{\wp=2}^{\infty}\frac{1-\gimel_2}{((\wp-1)\,t+1)\,(\wp-1)!}. \end{aligned}$$

By (1.20), we have

$$\begin{aligned} L_1(\gimel_1,\gimel_2) \leq \frac{1}{t}\Bigg(&\sum_{\wp=3}^{\infty}\frac{\gimel_1}{(\wp-1)\,(\wp-3)!} \\ &+\sum_{\wp=2}^{\infty}\frac{2\gimel_1+1}{(\wp-1)\,(\wp-2)!}+\sum_{\wp=2}^{\infty}\frac{1-\gimel_2}{(\wp-1)\,(\wp-1)!}\Bigg). \end{aligned}$$

By (1.21), we have

$$L_1(\gimel_1,\gimel_2) \leq \frac{1}{t}\left(\sum_{\wp=3}^{\infty}\frac{16\gimel_1}{(\wp-1)\,2^{\wp}}+\sum_{\wp=2}^{\infty}\frac{8\,(2\gimel_1+1)}{(\wp-1)\,2^{\wp}}+\sum_{\wp=2}^{\infty}\frac{4\,(1-\gimel_2)}{(\wp-1)\,2^{\wp}}\right).$$

Using (1.16) and (1.17), we have

$$L_1(\gimel_1,\gimel_2) \leq \frac{\gimel_1}{t}\,(8\ln 2-4)+\frac{4\,(2\gimel_1+1)}{t}(\ln 2)+\frac{2(1-\gimel_2)}{t}\ln 2. \tag{2.6}$$

But the inequality (2.6) is bounded above by $1-\daleth_2$ if (2.1) holds. □

Theorem 2.2. *If $t \in \mathbb{N}$, then $\Pi i_t(\Im)$ is in the subclass $C(\daleth_1, \daleth_2)$ if*

$$2\left[(36\daleth_1 - 3\daleth_2 + 11)\ln 2 - (15\daleth_1 + 2)\right] \le t(1-\daleth_2) \tag{2.7}$$

and

$$(10\daleth_1 - 2\daleth_2 + 5).e \le (1-\daleth_2)\,(t+2). \tag{2.8}$$

Proof. Since $\Pi i_t(\Im)$ is given by (2.3) and by virtue of (1.12) it suffices to show that

$$\begin{aligned} L_2(\daleth_1,\daleth_2) &= \sum_{\wp=2}^{\infty} \wp(\wp + \daleth_1\wp(\wp-1) - \daleth_2)\ \frac{1}{((\wp-1)\,t+1)\,(\wp-1)!} \\ &= \sum_{\wp=2}^{\infty} (\daleth_1\wp^3 + (1-\daleth_1)\wp^2 - \daleth_2\wp)\ \frac{1}{((\wp-1)\,t+1)\,(\wp-1)!} \\ &\le 1-\daleth_2. \end{aligned}$$

By (2.4), (2.5), and $\wp^3 = (\wp-1)(\wp-2)(\wp-3) + 6(\wp-1)(\wp-2) + 7(\wp-1) + 1$, we have

$$\begin{aligned} L_2(\daleth_1,\daleth_2) &= \sum_{\wp=2}^{\infty} \frac{\daleth_1(\wp-1)(\wp-2)(\wp-3)}{((\wp-1)\,t+1)\,(\wp-1)!} + \sum_{\wp=2}^{\infty} \frac{(5\daleth_1+1)(\wp-1)(\wp-2)}{((\wp-1)\,t+1)\,(\wp-1)!} \\ &\quad + \sum_{\wp=2}^{\infty} \frac{(4\daleth_1-\daleth_2+3)\,(\wp-1)}{((\wp-1)\,t+1)\,(\wp-1)!} + \sum_{\wp=2}^{\infty} \frac{1-\daleth_2}{((\wp-1)\,t+1)\,(\wp-1)!} \\ &= \sum_{\wp=4}^{\infty} \frac{\daleth_1}{((\wp-1)\,t+1)\,(\wp-4)!} + \sum_{\wp=3}^{\infty} \frac{5\daleth_1+1}{((\wp-1)\,t+1)\,(\wp-3)!} \\ &\quad + \sum_{\wp=2}^{\infty} \frac{4\daleth_1-\daleth_2+3}{((\wp-1)\,t+1)\,(\wp-2)!} + \sum_{\wp=2}^{\infty} \frac{1-\daleth_2}{((\wp-1)\,t+1)\,(\wp-1)!}. \end{aligned}$$

By (1.20), we have

$$\begin{aligned} L_2(\daleth_1,\daleth_2) &\le \frac{1}{t}\left(\sum_{\wp=4}^{\infty} \frac{\daleth_1}{(\wp-1)\,(\wp-4)!} + \sum_{\wp=3}^{\infty} \frac{5\daleth_1+1}{(\wp-1)\,(\wp-3)!}\right. \\ &\quad \left. + \sum_{\wp=2}^{\infty} \frac{4\daleth_1-\daleth_2+3}{(\wp-1)\,(\wp-2)!} + \sum_{\wp=2}^{\infty} \frac{1-\daleth_2}{(\wp-1)\,(\wp-1)!}\right). \end{aligned}$$

By (1.21), we have

$$\begin{aligned} L_2(\gimel_1,\gimel_2) &\leq \frac{1}{t}\left(\sum_{\wp=4}^{\infty}\frac{32\gimel_1}{(\wp-1)\,2^{\wp}}+\sum_{\wp=3}^{\infty}\frac{16(5\gimel_1+1)}{(\wp-1)\,2^{\wp}}\right. \\ &\qquad \left.+\sum_{\wp=2}^{\infty}\frac{8\,(4\gimel_1-\gimel_2+3)}{(\wp-1)\,2^{\wp}}+\sum_{\wp=2}^{\infty}\frac{4\,(1-\gimel_2)}{(\wp-1)\,2^{\wp}}\right). \end{aligned}$$

Using the series sums (1.16), (1.17) and (1.18), we have

$$\begin{aligned} L_2(\gimel_1,\gimel_2) &\leq \frac{\gimel_1}{t}\,(16\ln 2-10)+\frac{5\gimel_1+1}{t}\,(8\ln 2-4) \\ &\quad +\frac{4\gimel_1-\gimel_2+3}{t}(4\ln 2)+\frac{1-\gimel_2}{t}\,(2\ln 2)\,. \end{aligned} \tag{2.9}$$

But the inequality (2.9) is bounded above by $1-\gimel_2$ if (2.7) holds. □

3. Inclusion properties

In this section, we examine the action of the function $\mathcal{I}i_t(\Im)$ on the subclasses $S^*(\gimel_1,\gimel_2)$ and $C(\gimel_1,\gimel_2)$.

Theorem 3.1. *Let $t\in\mathbb{N}$. If $B\in H^{\tau}(O_1,O_2)$. Then $\mathcal{I}i_t(\Im)$ is in the subclass $S^*(\gimel_1,\gimel_2)$ if*

$$(O_1-O_2)|\tau|(4\gimel_1-\gimel_2+2)\ln 2\leq t\,(1-\gimel_2) \tag{3.1}$$

and

$$(\gimel_1+\gimel_2+1).e-1-2\gimel_2\leq\frac{(1-\gimel_2)\,(t+1)}{(O_1-O_2)|\tau|}. \tag{3.2}$$

Proof. In view of (1.10), it suffices to show that

$$M_1(\gimel_1,\gimel_2)=\sum_{\wp=2}^{\infty}(\wp+\gimel_1\wp(\wp-1)-\gimel_2)\frac{1}{((\wp-1)\,t+1)\,(\wp-1)!}\,|\gamma_{\wp}|\leq 1-\gimel_2.$$

Since $B\in H^{\tau}(O_1,O_2)$, then by virtue of (1.14), we have

$$\begin{aligned} M_1(\gimel_1,\gimel_2) &\leq (O_1-O_2)|\tau|\left(\sum_{\wp=2}^{\infty}\frac{\gimel_1\wp}{((\wp-1)\,t+1)\,(\wp-1)!}\right. \\ &\qquad \left.+\sum_{\wp=2}^{\infty}\frac{1-\gimel_1}{((\wp-1)\,t+1)\,(\wp-1)!}-\sum_{\wp=2}^{\infty}\frac{\gimel_2}{((\wp-1)\,t+1)\wp!}\right). \end{aligned}$$

By (2.4), we have

$$\begin{aligned} M_1(\beth_1,\beth_2) &\le (O_1-O_2)|\tau|\left(\sum_{\wp=2}^{\infty}\frac{\beth_1(\wp-1)}{((\wp-1)\,t+1)\,(\wp-1)!}\right. \\ &\quad \left.+\sum_{\wp=2}^{\infty}\frac{1}{((\wp-1)\,t+1)\,(\wp-1)!}-\sum_{\wp=2}^{\infty}\frac{\beth_2}{((\wp-1)\,t+1)\wp!}\right) \\ &= (O_1-O_2)|\tau|\left(\sum_{\wp=2}^{\infty}\frac{\beth_1}{((\wp-1)\,t+1)\,(\wp-2)!}\right. \\ &\quad \left.+\sum_{\wp=2}^{\infty}\frac{1}{((\wp-1)\,t+1)\,(\wp-1)!}-\sum_{\wp=2}^{\infty}\frac{\beth_2}{((\wp-1)\,t+1)\wp!}\right). \end{aligned}$$

By (1.20) and (1.21), we have

$$\begin{aligned} M_1(\beth_1,\beth_2) &\le \frac{(O_1-O_2)|\tau|}{t}\Big(8\beth_1\sum_{\wp=2}^{\infty}\frac{1}{(\wp-1)\,2^{\wp}} \\ &\quad +4\sum_{\wp=2}^{\infty}\frac{1}{(\wp-1)\,2^{\wp}}-2\beth_2\sum_{\wp=2}^{\infty}\frac{1}{(\wp-1)\,2^{\wp}}\Big). \end{aligned}$$

By (1.16), we get

$$M_1(\beth_1,\beth_2)\le\frac{(O_1-O_2)|\tau|}{t}(4\beth_1+2-\beth_2)\ln 2. \tag{3.3}$$

But the inequality (3.3) is bounded above by $1-\beth_2$ if (3.1) holds. □

Theorem 3.2. *Let $t\in\mathbb{N}$. If $B\in H^{\tau}(O_1,O_2)$, then $\mathcal{I}i_t(\Im)$ is in the subclass $C(\beth_1,\beth_2)$ if*

$$2(O_1-O_2)|\tau|\left[(8\beth_1-\beth_2+3)\ln 2-2\beth_1\right]\le t\,(1-\beth_2) \tag{3.4}$$

and

$$(3\beth_1-\beth_2+2).e+\beth_2-1\le\frac{(1-\beth_2)\,(t+1)}{(O_1-O_2)|\tau|}. \tag{3.5}$$

Proof. By view of (1.12), it suffices to show that

$$\begin{aligned} M_2(\beth_1,\beth_2) &= \sum_{\wp=2}^{\infty}\wp(\wp+\beth_1\wp(\wp-1)-\beth_2)\frac{1}{((\wp-1)\,t+1)\,(\wp-1)!}\,|\gamma_{\wp}| \\ &\le 1-\beth_2. \end{aligned}$$

Since $B \in H^{\tau}(O_1, O_2)$, then by virtue (1.14), we have

$$M_2(\daleth_1, \daleth_2) \le (O_1 - O_2)|\tau| \sum_{\wp=2}^{\infty} (\wp + \daleth_1 \wp(\wp - 1) - \daleth_2) \frac{1}{((\wp - 1)\, t + 1)\, (\wp - 1)!}$$

$$= (O_1 - O_2)|\tau| \sum_{\wp=2}^{\infty} (\daleth_1 \wp^2 + (1 - \daleth_1)\wp - \daleth_2) \frac{1}{((\wp - 1)\, t + 1)\, (\wp - 1)!}.$$

Using (2.4) and (2.5), then by a similar proof of Theorem 2.1, we have that $\mathcal{I}i_t(\Im) \in C(\daleth_1, \daleth_2)$ if (3.4) holds. □

4. Sufficient conditions of the integral operator $Ui_t(\Im)$

In this section, we examine the action of the operator

$$Ui_t(\Im) := \int_0^{\Im} \frac{\Pi i_t\, (\Im)}{t} dt, \ \Im \in \Theta \tag{4.1}$$

on the subclasses $S^*(\daleth_1, \daleth_2)$ and $C(\daleth_1, \daleth_2)$.

Theorem 4.1. *Let $t \in \mathbb{N}$. The integral operator $Ui_t(\Im)$ is in the subclass $S^*(\daleth_1, \daleth_2)$ if the following inequalities are holds*

$$|(4\daleth_1 - \daleth_2 + 2) \ln 2 \le t\, (1 - \daleth_2) \tag{4.2}$$

and

$$(\daleth_1 + \daleth_2 + 1).e - 2\daleth_2 - 1 \le (1 - \daleth_2)\, (t + 1). \tag{4.3}$$

Proof. According to (1.8) it follows that

$$Ui_t(\Im) = \Im - \sum_{\wp=2}^{\infty} \frac{1}{((\wp - 1)\, t + 1)\, (\wp - 1)!} \frac{\Im^{\wp}}{\wp}, \ \ \Im \in \Theta. \tag{4.4}$$

From (1.10), the function $Ui_t(\Im)$ belongs to $S^*(\daleth_1, \daleth_2)$ if

$$\sum_{\wp=2}^{\infty} (\wp + \daleth_1 \wp(\wp - 1) - \daleth_2)\, \frac{1}{\wp((\wp - 1)\, t + 1)\, (\wp - 1)!}$$

$$= \sum_{\wp=2}^{\infty} \frac{\daleth_1 \wp}{((\wp - 1)\, t + 1)\, (\wp - 1)!} + \sum_{\wp=2}^{\infty} \frac{1 - \daleth_1}{((\wp - 1)\, t + 1)\, (\wp - 1)!}$$

$$- \sum_{\wp=2}^{\infty} \frac{\daleth_2}{((\wp - 1)\, t + 1)\wp!} \le 1 - \daleth_2.$$

By a similar proof of Theorem 3.1, we get $Ui_t(\Im) \in S^*(\daleth_1, \daleth_2)$ if (4.2) holds. □

Theorem 4.2. *Let $t \in \mathbb{N}$. The integral operator $Ui_t(\Im)$ is in the subclass $C(\daleth_1, \daleth_2)$ if the inequalities* (2.1) *and* (2.2) *are holds.*

Proof. Since $Ui_t(\Im)$ is given by (4.4) and by (1.12), the function $Ui_t(\Im)$ belongs to $C(\gimel_1, \gimel_2)$ if

$$\begin{aligned}\sum_{\wp=2}^{\infty} \wp(\wp + \gimel_1 \wp(\wp-1) - \gimel_2) \ \frac{1}{\wp((\wp-1)\,t+1)\,(\wp-1)!} \\ = \sum_{\wp=2}^{\infty} (\gimel_1 \wp^2 + (1-\gimel_1)\wp - \gimel_2) \ \frac{1}{((\wp-1)\,t+1)\,(\wp-1)!} \\ \leq 1 - \gimel_2.\end{aligned}$$

By a similar proof of Theorem 3.2 we get $Ui_t(\Im) \in C(\gimel_1, \gimel_2)$ if (2.1) holds. □

For example, if $\gimel_1 = 0$, we get the following corollaries for the subclasses $S^*(\gimel_2)$ and $C(\gimel_2)$.

Corollary 4.3. *If* $t \in \mathbb{N}$, *then* $\Pi i_t\,(\Im) \in S^*(\gimel_2)$ *if*

$$2\,[(3-\gimel_2)\ln 2] \leq t\,(1-\gimel_2) \quad \textit{and} \quad (2-\gimel_2).e \leq (1-\gimel_2)\,(t+2). \tag{4.5}$$

Corollary 4.4. *If* $t \in \mathbb{N}$, *then* $\Pi i_t\,(\Im) \in C(\gimel_2)$ *if*

$$2\,[(11-3\gimel_2)\ln 2 - 2] \leq t(1-\gimel_2) \quad \textit{and} \quad (5-2\gimel_2).e \leq (1-\gimel_2)\,(t+2).$$

Corollary 4.5. *Let* $t \in \mathbb{N}$. *If* $B \in H^{\tau}(O_1, O_2)$, *then* $\mathcal{I}i_t(\Im) \in S^*(\gimel_1, \gimel_2)$ *if*

$$(O_1 - O_2)|\tau|(2-\gimel_2)\ln 2 \leq t\,(1-\gimel_2) \ \textit{ and } (\gimel_2+1).e - 2\gimel_2 - 1 \leq \frac{(1-\gimel_2)\,(t+1)}{(O_1-O_2)|\tau|}.$$

Corollary 4.6. *Let* $t \in \mathbb{N}$. *If* $B \in H^{\tau}(O_1, O_2)$, *then* $\mathcal{I}i_t(\Im) \in C(\gimel_1, \gimel_2)$ *if*

$$2(O_1 - O_2)|\tau|\,[(3-\gimel_2)\ln 2 - 2\gimel_1] \leq t\,(1-\gimel_2)$$

and

$$(2-\gimel_2).e + \gimel_2 - 1 \leq \frac{(1-\gimel_2)\,(t+1)}{(O_1-O_2)|\tau|}.$$

Corollary 4.7. *Let* $t \in \mathbb{N}$. *The integral operator* $Ui_t(\Im) \in S^*(\gimel_1, \gimel_2)$ *if the inequalities*

$$|(2-\gimel_2)\ln 2 \leq t\,(1-\gimel_2) \quad \textit{and} \quad (\gimel_2+1).e - 2\gimel_2 - 1 \leq (1-\gimel_2)\,(t+1)$$

holds.

Corollary 4.8. *Let* $t \in \mathbb{N}$. *The integral operator* $Ui_t(\Im) \in C(\gimel_1, \gimel_2)$ *if the inequality* (4.5) *holds.*

5. Conclusion

The conditions for the generalized normalized imaginary error function $\Pi i_t(\Im)$ to belong to the subclasses $S^*(\beth_1, \beth_2)$ and $C(\beth_1, \beth_2)$ of the analytic functions defined on the open unit disk Θ are established in this paper. We also investigate sufficient criteria for the integral operator $Ui_t(\Im)$ to belong to these subclasses, as well as the action of the function $\mathcal{I}i_t(\Im)$.

This study might inspire researchers to create additional conditions that allow the generalized normalized imaginary error function $\Pi i_t(\Im)$ to be a member of other analytic function subclasses defined in Θ.

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