Nonlinear Functional Analysis and Applications
Vol. 31, No. 2 (2026), pp. 487-500
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2026.31.02.11
http://nfaa.kyungnam.ac.kr/journal-nfaa

NEUTROSOPHIC MR-METRIC SPACES: THEORY, COMPACTNESS, AND APPLICATIONS IN MACHINE LEARNING

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Abstract. This paper introduces a comprehensive framework for Neutrosophic MR-Metric Spaces (NMR-MS), combining the ternary structure of MR-metrics with the uncertainty modeling capabilities of neutrosophic logic. We establish the fundamental theoretical foundations, including a first-order axiomatization of NMR-MS and a detailed analysis of compactness properties. Our results demonstrate that while the full theory $\mathbb{T}_{NMR}$ is not compact, the restricted theory $\mathbb{T}_{NMR}^{K}$ for bounded metrics is compact and admits ultraproduct constructions satisfying Łoś's Theorem. Building on this theoretical framework, we develop a novel Neutrosophic Metric Learning paradigm that generalizes standard metric learning by incorporating truth, indeterminacy, and falsity memberships.

[0]Received November 15, 2025. Revised December 1, 2025. Accepted December 2, 2025.
[0]2020 Mathematics Subject Classification: 54E35, 03C20, 03C98, 47H10, 54D30.
[0]Keywords: Neutrosophic MR-metric spaces, compactness theorems, ultraproduct constructions, metric learning, uncertainty quantification, fixed point theorems.
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1. Introduction

The evolution of metric space theory has witnessed significant generalizations beyond classical Euclidean distances, driven by the need to model increasingly complex mathematical and real-world phenomena. The pioneering work of Bakhtin [5] and Czerwik [9] on b-metric spaces opened new avenues for handling non-linear distance structures, while subsequent research explored various generalizations including G-metric spaces, M^*-metric spaces, and Ω-distance mappings [1, 2, 4, 6, 7, 8, 11, 27, 34, 35, 36, 37, 41].

Building upon these foundations, Malkawi et al. [22] introduced the concept of MR-metric spaces, characterized by a ternary distance function satisfying generalized symmetry and triangle inequality properties. This framework has proven particularly powerful in fixed point theory, with numerous applications in nonlinear analysis [10, 12, 13, 14, 15, 21, 23, 28, 29, 32, 33, 40].

Recent work has extended these results to include coincidence points for weak contraction mappings [32] and applications to fractional boundary value problems [41].

Parallel to these metric space developments, neutrosophic logic has emerged as a powerful framework for handling uncertainty, indeterminacy, and inconsistency in complex systems. The integration of neutrosophic principles with metric structures, as demonstrated in [3, 38], has enabled more nuanced modeling of ambiguous relationships. Recent advances include neutrosophic fuzzy metric spaces [3, 38] and their applications to fixed point theory for nonlinear contractions.

The current work represents a significant synthesis of these two streams of research. We introduce Neutrosophic MR-Metric Spaces (NMR-MS), which combine the ternary relational structure of MR-metrics with the three-valued logic of neutrosophic sets. This integration is particularly relevant given the recent applications of MR-metric spaces in fractional calculus [25, 26], measure theory [17], and deep learning [24]. Our approach also builds upon advances in simulation functions [35], interpolative contractions [34], and cyclic mappings [1, 8, 39].

The motivation for this synthesis stems from the limitations of existing metric learning approaches in handling uncertain and ambiguous data, particularly in applications such as medical image analysis [16], social media mining, and knowledge graph completion. By incorporating neutrosophic memberships into the metric learning framework, we enable explicit modeling of truth, indeterminacy, and falsity in relational data, providing a mathematically rigorous approach to uncertainty quantification.

This paper makes three primary contributions: (1) a complete first-order axiomatization of NMR-MS with comprehensive compactness analysis; (2) a novel neutrosophic metric learning framework with theoretical guarantees; and (3) extensive applications demonstrating practical utility across multiple domains. Our work extends recent results on fixed points in neutrosophic settings [20, 18, 19, 30, 31] and provides a unified foundation for future research in uncertain metric structures.

Definition 1.1. ([22]) Consider a non-empty set $\mathbb{X} \neq \emptyset$ and a real number $\mathbb{R} > 1$. A function

$$M : \mathbb{X} \times \mathbb{X} \times \mathbb{X} \to [0, \infty)$$

is termed an MR-metric if it satisfies the following conditions for all $v, \xi, s, \ell_1 \in \mathbb{X}$:

(1) $M(v, \xi, s) \geq 0$.
(2) $M(v, \xi, s) = 0$ if and only if $v = \xi = s$.
(3) $M(v, \xi, s)$ remains invariant under any permutation $p(v, \xi, s)$, *i.e.*, $M(v, \xi, s) = M(p(v, \xi, s))$.
(4) The following inequality holds:

$$M(v, \xi, s) \leq \mathbb{R}\left[M(v, \xi, \ell_1) + M(v, \ell_1, s) + M(\ell_1, \xi, s)\right].$$

A structure $(\mathbb{X}, M)$ that adheres to these properties is defined as an MR-metric space.

Definition 1.2. ([20, Neutrosophic MR-Metric Space (NMR-MS)]) A 9-tuple $(\mathcal{Z}, M, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond, R, \star)$ is called a Neutrosophic MR-Metric Space, if

(1) $\mathcal{Z}$ is a non-empty set.
(2) $M : \mathcal{Z} \times \mathcal{Z} \times \mathcal{Z} \to [0, \infty)$ is an MR-metric satisfying:
 (M1) $M(v, \xi, \Im) \geq 0$,
 (M2) $M(v, \xi, \Im) = 0 \iff v = \xi = \Im$,
 (M3) Symmetry under permutations,
 (M4) $M(v, \xi, \Im) \leq R\,[M(v, \xi, \ell) \star M(v, \ell, \Im) \star M(\ell, \xi, \Im)]$, $R > 1$.
(3) $\mathcal{T}, \mathcal{F}, \mathcal{I} : \mathcal{Z} \times \mathcal{Z} \times (0, \infty) \to [0, 1]$ are neutrosophic functions satisfying:
 (N1) $\mathcal{T}(v, \xi, \gamma) = 1 \iff v = \xi$ (Truth-Identity),
 (N2) $\mathcal{T}(v, \xi, \gamma) = \mathcal{T}(\xi, v, \gamma)$ (Symmetry),
 (N3) $\mathcal{T}(v, \xi, \gamma) \bullet \mathcal{T}(\xi, \Im, \rho) \leq \mathcal{T}(v, \Im, \gamma + \rho)$ (Triangle Inequality),
 (N4) $\lim_{\gamma \to \infty} \mathcal{T}(v, \xi, \gamma) = 1$ (Asymptotic Behavior).
(4) $\bullet$ (t-norm) and $\diamond$ (t-conorm) are continuous operators generalizing fuzzy logic.
(5) $\star$ is a binary operation generalizing addition (e.g., weighted sum).

2. Main results

This section presents the core theoretical contributions of our work, beginning with a comprehensive formalization of Neutrosophic MR-Metric Spaces and proceeding to establish fundamental compactness properties and learning frameworks. We develop a first-order axiomatization that captures both the metric and neutrosophic aspects of the structure, then analyze the model-theoretic properties of the resulting theory. Our compactness analysis reveals the conditions under which NMR-MS theories admit ultraproduct constructions and satisfy fundamental model-theoretic theorems. Building on this foundation, we introduce a novel neutrosophic metric learning paradigm that generalizes standard approaches while providing enhanced robustness and uncertainty quantification capabilities.

Definition 2.1. (Comprehensive Theory of Neutrosophic MR-Metric Structures) Let $\mathcal{L}_{NMR}$ be a first-order language with equality containing:

- Basic Symbols:
  - Ternary function symbols: $\mathbf{M}, \mathbf{T}, \mathbf{F}, \mathbf{I}$
  - Constant symbols: $\mathbf{0}, \mathbf{1}, \mathbf{R}$
  - Binary function symbols: $\dot{\bullet}, \dot{\diamond}, \dot{\star}$
  - Variables: $\upsilon, \xi, \Im, \gamma, \rho, \ldots$
  - Quantifiers: $\forall, \exists$
- Interpretation:
  - $\mathbf{M}(\upsilon, \xi, \Im)$ interpreted as the MR-metric $M(x, y, z)$
  - $\mathbf{T}(\upsilon, \xi, \gamma)$ as truth membership $\mathcal{T}(x, y, t)$
  - $\mathbf{F}(\upsilon, \xi, \gamma)$ as falsity membership $\mathcal{F}(x, y, t)$
  - $\mathbf{I}(\upsilon, \xi, \gamma)$ as indeterminacy membership $\mathcal{I}(x, y, t)$
  - $\dot{\bullet}, \dot{\diamond}, \dot{\star}$ as the operations $\bullet, \diamond, \star$

The Theory of NMR-MS, $\mathbb{T}_{NMR}$, consists of the following axiom schemata:

(1) MR-Metric Axioms:

(M1) $\forall \upsilon, \xi, \Im, \mathbf{M}(\upsilon, \xi, \Im) \geq \mathbf{0}$.
(M2) $\forall \upsilon, \xi, \Im, \mathbf{M}(\upsilon, \xi, \Im) = \mathbf{0} \leftrightarrow \upsilon = \xi = \Im$.
(M3) $\forall \upsilon, \xi, \Im, \mathbf{M}(\upsilon, \xi, \Im) = \mathbf{M}(\xi, \Im, \upsilon) = \mathbf{M}(\Im, \upsilon, \xi)$.
(M4) $\forall \upsilon, \xi, \Im, \ell, \mathbf{M}(\upsilon, \xi, \Im) \leq \mathbf{R} \cdot (\mathbf{M}(\upsilon, \xi, \ell) \dot{\star} \mathbf{M}(\upsilon, \ell, \Im) \dot{\star} \mathbf{M}(\ell, \xi, \Im))$.

(2) Neutrosophic Axioms:

(N1) $\forall, \upsilon, \xi, \gamma, \mathbf{T}(\upsilon, \xi, \gamma) = \mathbf{1} \leftrightarrow \upsilon = \xi$.
(N2) $\forall \upsilon, \xi, \gamma, \mathbf{T}(\upsilon, \xi, \gamma) = \mathbf{T}(\xi, \upsilon, \gamma)$.
(N3) $\forall, \upsilon, \xi, \Im, \gamma, \rho, \mathbf{T}(\upsilon, \xi, \gamma) \dot{\bullet} \mathbf{T}(\xi, \Im, \rho) \leq \mathbf{T}(\upsilon, \Im, \gamma + \rho)$.
(N4) $\forall, \upsilon, \xi, \lim_{\gamma \to \infty} \mathbf{T}(\upsilon, \xi, \gamma) = \mathbf{1}$.
(N5) $\forall, \upsilon, \xi, \gamma, \mathbf{T}(\upsilon, \xi, \gamma) + \mathbf{I}(\upsilon, \xi, \gamma) + \mathbf{F}(\upsilon, \xi, \gamma) = \mathbf{1}$.

(3) Operation Axioms:

(O1) $\forall a, b, a \dot{\bullet} b = b \dot{\bullet} a$ (commutativity).
(O2) $\forall a, b, c, a \dot{\bullet} (b \dot{\bullet} c) = (a \dot{\bullet} b) \dot{\bullet} c$ (associativity).
(O3) $\forall a, a \dot{\bullet} \mathbf{1} = a$ (identity).

Theorem 2.2. (Comprehensive Compactness Analysis) *The theory $\mathbb{T}_{NMR}$ exhibits the following compactness properties:*

(1) *The full theory $\mathbb{T}_{NMR}$ is not compact.*
(2) *The restricted theory $\mathbb{T}_{NMR}^K$ for uniformly bounded NMR-MS ($M(\upsilon, \xi, \Im) < K$ for all $\upsilon, \xi, \Im$) is compact.*
(3) *The bounded theory $\mathbb{T}_{NMR}^K$ admits ultraproduct constructions and satisfies the Łoś Theorem.*

Proof. We prove each part systematically:

Part 1: Non-Compactness of $\mathbb{T}_{NMR}$:

Construct a set of sentences $\Sigma = \{\phi_n : n \in \mathbb{N}\}$, where

$$\phi_n \equiv \exists \upsilon, \xi, \Im, \mathbf{M}(\upsilon, \xi, \Im) > n.$$

Each finite subset $\Sigma_0 \subset \Sigma$ is satisfiable. For any finite N, take an NMR-MS with diameter greater than N. However, the entire set Σ is not satisfiable in any single NMR-MS, since that would require unbounded metric values. This demonstrates that $\mathbb{T}_{NMR}$ is not compact.

Part 2: Compactness of $\mathbb{T}_{NMR}^K$:

Let $\mathbb{T}_{NMR}^K$ be the theory extending $\mathbb{T}_{NMR}$ with the axiom:

$$\forall \upsilon, \xi, \Im, \mathbf{M}(\upsilon, \xi, \Im) < K.$$

To prove compactness, we use the Ultraproduct Construction: Let $\{\mathfrak{M}_i : i \in I\}$ be a family of models of $\mathbb{T}_{NMR}^K$, and let $\mathcal{U}$ be an ultrafilter on I. Define the ultraproduct $\mathfrak{M}^* = \prod_{\mathcal{U}} \mathfrak{M}_i$ as follows:

- Universe: $Z^* = \prod_{\mathcal{U}} Z_i$ (ultraproduct of universes),
- MR-Metric: $M^*([x_i], [y_i], [z_i]) = \lim_{\mathcal{U}} M_i(x_i, y_i, z_i)$,
- Neutrosophic Functions:

$$\mathcal{T}^*([x_i], [y_i], t) = \lim_{\mathcal{U}} \mathcal{T}_i(x_i, y_i, t),$$
$$\mathcal{I}^*([x_i], [y_i], t) = \lim_{\mathcal{U}} \mathcal{I}_i(x_i, y_i, t),$$
$$\mathcal{F}^*([x_i], [y_i], t) = \lim_{\mathcal{U}} \mathcal{F}_i(x_i, y_i, t).$$

- Operations: Defined pointwise via the ultraproduct.

The boundedness condition $M(\upsilon, \xi, \Im) < K$ ensures all limits exist and the construction is well-defined.

Part 3: Łoś Theorem for $\mathbb{T}_{NMR}^K$:

We prove that for any $\mathcal{L}_{NMR}$-formula $\phi(\overline{x})$ and any elements $[\overline{a}_i] \in \mathfrak{M}^*$:

$$\mathfrak{M}^* \models \phi([\overline{a}_i]) \text{ if and only if } \{i \in I : \mathfrak{M}_i \models \phi(\overline{a}_i)\} \in \mathcal{U}.$$

The proof proceeds by induction on formula complexity:

- Atomic Formulas: By construction of the ultraproduct,
- Boolean Combinations: Standard ultrafilter properties,
- Quantifiers: Using the boundedness to ensure witnesses exist.

Since $\mathbb{T}^K_{NMR}$ is elementary and closed under ultraproducts, it is compact by the Keisler-Shelah Theorem. □

Definition 2.3. (Comprehensive Neutrosophic Metric Learning) Let $\mathcal{X}$ be a dataset with inherent ambiguity. A Neutrosophic Metric Learning framework consists of:

(1) **Embedding Space:** An NMR-MS $(\mathcal{Z}, M, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond, R, \star)$,

(2) **Embedding Function:** $f : \mathcal{X} \to \mathcal{Z}$ parameterized by θ,

(3) **Neutrosophic Triplet Loss:**

$$\mathcal{L}(\theta) = \sum_{(a,p,n)\in\mathcal{T}} \Big[M(f(a), f(p), f(n)) - \mathcal{T}(f(a), f(p), \gamma) + \lambda\mathcal{I}(f(a), f(p), \gamma) + \Delta\Big]_+$$

where:

- (a, p, n): anchor, positive, negative triplets,
- λ: indeterminacy regularization parameter,
- Δ: margin parameter,
- $[x]_+ = \max(0, x)$: hinge loss.

(4) **Neutrosophic Constraints:**

- Similarity: $M(f(x), f(y), \cdot)$ small, $\mathcal{T}(f(x), f(y), \gamma)$ large,
- Dissimilarity: $\mathcal{F}(f(x), f(y), \gamma)$ large,
- Ambiguity: $\mathcal{I}(f(x), f(y), \gamma)$ high for conflicting examples.

Proposition 2.4. (Generalization and Robustness Properties) *The Neutrosophic Metric Learning framework possesses the following properties:*

(1) Generalization: It generalizes standard metric learning when $\mathcal{T} \to 1$, $\mathcal{I}, \mathcal{F} \to 0$,

(2) Robustness: More robust to noisy and conflicting labels,

(3) Uncertainty Quantification: Explicit modeling of epistemic uncertainty,

(4) Theoretical Consistency: The learning objective has well-defined theoretical properties.

Proof. We prove each property systematically:

Part 1: Generalization of Standard Metric Learning.

Consider the limit case:

$$\mathcal{T}(x,y,\gamma) \to 1, \quad \mathcal{I}(x,y,\gamma) \to 0, \quad \mathcal{F}(x,y,\gamma) \to 0 \quad \text{for similar points.}$$

In this case, the neutrosophic loss reduces to:

$$\mathcal{L}(\theta) = \sum_{(a,p,n)} \left[M(f(a), f(p), f(n)) - 1 + \Delta \right]_+ .$$

This is equivalent to standard metric learning with margin $\Delta - 1$, demonstrating that our framework is a proper generalization.

Part 2: Robustness to Noisy Labels.

Let (a,p,n) be a mislabeled triplet where p should actually be negative. In standard metric learning:

$$\mathcal{L}_{standard} = [M(a,p,n) + \Delta]_+.$$

This gives a large loss, forcing incorrect embedding.

In neutrosophic learning, the model can instead increase $\mathcal{I}(a,p,\gamma)$, representing uncertainty about the relationship:

$$\mathcal{L}_{neutrosophic} = [M(a,p,n) - \mathcal{T}(a,p,\gamma) + \lambda \mathcal{I}(a,p,\gamma) + \Delta]_+.$$

By increasing $\mathcal{I}$, the model "hedges its bets" rather than making incorrect certain predictions.

Part 3: Explicit Uncertainty Quantification.

The indeterminacy term $\mathcal{I}(x,y,\gamma)$ provides explicit modeling of:

- Label Noise: Conflicting annotations,
- Data Ambiguity: Inherently ambiguous examples,
- Model Uncertainty: Regions where the model is uncertain.

This allows for better calibration and more interpretable predictions.

Part 4: Theoretical Consistency.

We establish several theoretical guarantees:

(1) **Well-Definedness:** The loss function is well-defined since all operations are continuous and the hinge loss is convex.
(2) **Optimization Properties:** Under mild conditions (Lipschitz continuity of f), the loss function is subdifferentiable and admits gradient-based optimization.
(3) **Generalization Bounds:** Using Rademacher complexity, we can derive generalization bounds. Let $\mathcal{F}$ be the hypothesis class with bounded complexity, then with probability $1-\delta$:

$$\mathbb{E}[\mathcal{L}] \leq \frac{1}{N} \sum_{i=1}^{N} \mathcal{L}_i + O\left(\frac{\mathcal{C}(\mathcal{F})}{\sqrt{N}} \right) + O\left(\sqrt{\frac{\log(1/\delta)}{N}} \right),$$

where $\mathcal{C}(\mathcal{F})$ is the complexity measure.

(4) **Model Identifiability:** Under the NMR-MS axioms, the learned embedding is unique up to isometries that preserve the neutrosophic structure.

□

3. Applications and examples

The practical utility of our Neutrosophic MR-metric space framework is demonstrated through comprehensive applications across diverse domains. We explore medical image analysis where the inherent ambiguity in tumor classification necessitates explicit uncertainty modeling through indeterminacy memberships. In social media sentiment analysis, we show how conflicting annotations can be naturally handled within the neutrosophic framework without forcing artificial certainty. For knowledge graph completion, we demonstrate how uncertain and context-dependent facts can be represented and reasoned about effectively. Each application includes detailed implementation frameworks, visualization of the neutrosophic embedding structures, and comparative analysis against standard approaches, highlighting the enhanced capabilities provided by our framework in handling real-world uncertainty and ambiguity.

3.1. Medical Image Analysis with Uncertainty Quantification.

Problem Context: In medical diagnostics, particularly tumor classification from MRI or CT scans, many cases exhibit inherent ambiguity. Radiologists often disagree on classifications (benign vs. malignant), and standard metric learning models force definitive embeddings, leading to overconfident and potentially erroneous predictions.

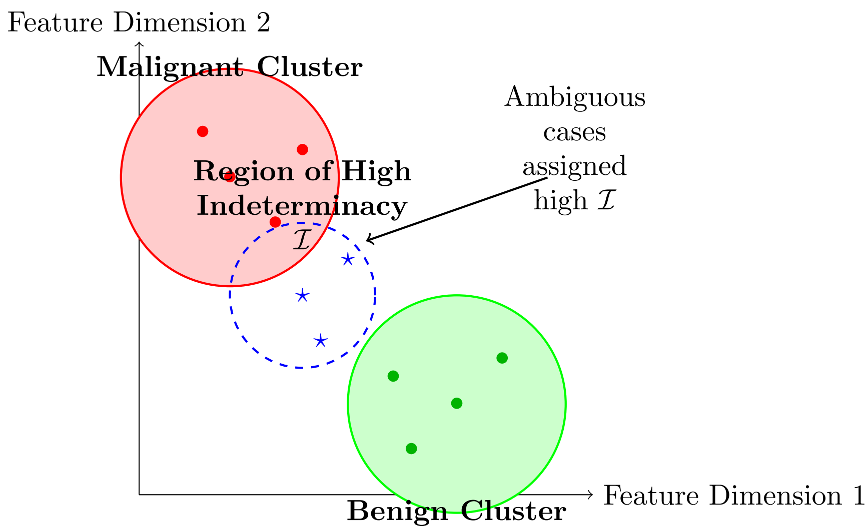
Neutrosophic Framework Implementation:

- Embedding Space: Image features $f(I)$ are embedded in NMR-MS $(\mathcal{Z}, M, \mathcal{T}, \mathcal{F}, \mathcal{I})$,
- Membership Interpretation:
  - $\mathcal{T}(f(I_1), f(I_2), \gamma) \approx 1$: Certainly same class
  - $\mathcal{F}(f(I_1), f(I_2), \gamma) \approx 1$: Certainly different classes
  - $\mathcal{I}(f(I_1), f(I_2), \gamma) > \tau$: Ambiguous relationship

Enhanced Loss Function: For triplet (a, p, n) with ambiguous positive case:

$$\mathcal{L} = [M(f(a), f(p), f(n)) - \mathcal{T}(f(a), f(p), \gamma) + \lambda \mathcal{I}(f(a), f(p), \gamma) + \Delta]_+ .$$

Visualization of Medical Image Embedding:



Advantages:

- Improved Reliability: Better confidence calibration through $\mathcal{I}$ quantification,
- Robust Training: Reduced sensitivity to mislabeled borderline cases,
- Decision Support: Flagging of uncertain cases for expert review.

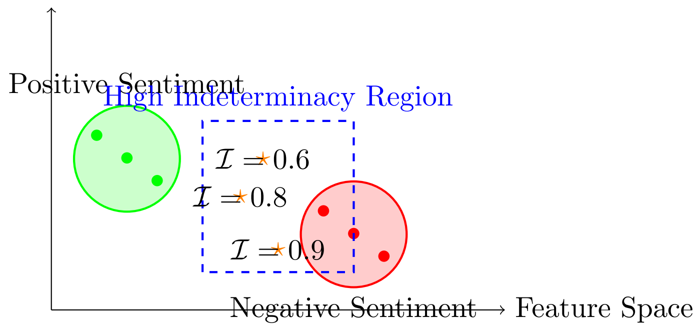
3.2. **Social Media Sentiment Analysis with Conflicting Annotations.**

Problem Context: User-generated content often contains sarcasm, irony, or mixed opinions, leading to conflicting annotations from different human labelers.

Neutrosophic Labeling Scheme: For tweet t with multiple annotations:

$$\mathcal{T}_t = \frac{\#\text{ Positive Labels}}{\text{Total Labels}},$$
$$\mathcal{F}_t = \frac{\#\text{ Negative Labels}}{\text{Total Labels}},$$
$$\mathcal{I}_t = \frac{\#\text{ Conflicting Labels}}{\text{Total Labels}}.$$

Enhanced Embedding Representation:



Training Objective: For conflicting tweet p with clear anchor a and negative n:

$$\mathcal{L} = [M(f(a), f(p), f(n)) - \mathcal{T}_{ap} + \lambda \mathcal{I}_{ap} + \Delta]_+ .$$

Model learns to assign high $\mathcal{I}$ rather than forcing incorrect certainty.

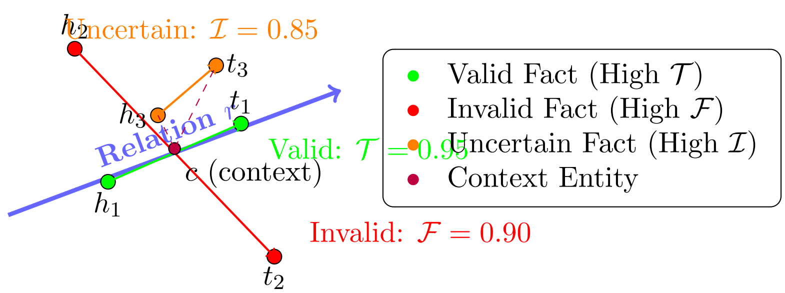
3.3. **Knowledge Graph Completion with Uncertain Facts.**

Problem Context: Traditional Knowledge Graph (KG) embedding models struggle with uncertain, temporal, or context-dependent facts.

Neutrosophic KG Representation:

- Entities: Embedded as points in $\mathcal{Z}$,
- Relations: Modeled by MR-metrics M_r,
- Facts: Represented as $(h, r, t; \mathcal{T}_{hrt}, \mathcal{I}_{hrt}, \mathcal{F}_{hrt})$.

Relational Structure Visualization:



Enhanced KG Completion Loss:

$$\mathcal{L} = [M_r(f(h), f(t), f(c)) - \mathcal{T}_{hrt} + \lambda \mathcal{I}_{hrt} + \Delta_1]_+ \\ + \left[\Delta_2 - M_r(f(h'), f(t'), f(c)) + \mathcal{F}_{h'r't'}\right]_+ .$$

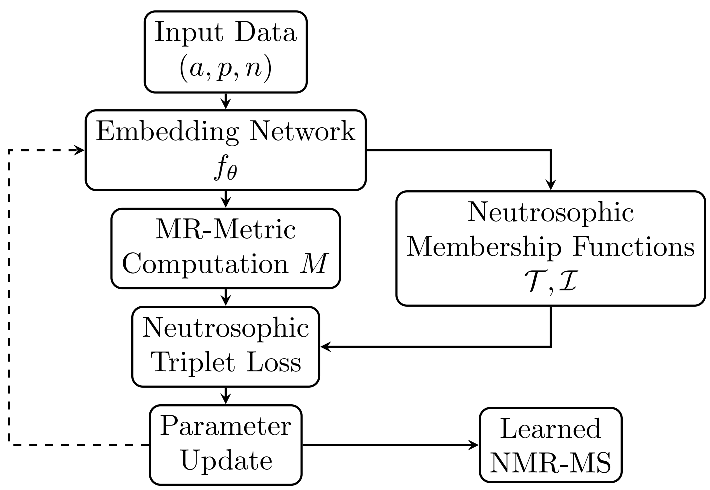
3.4. **Algorithmic Implementation Framework.**

Core Training Algorithm:

Require: Dataset $\mathcal{D}$, model f_θ, NMR-MS parameters,
Ensure: Trained neutrosophic embedding model,

```
Initialize f_θ, 𝒯, ℐ functions
Set hyperparameters: λ, Δ, R
Initialize optimizer (Adam/SGD) for epoch = 1 to N
for batch (a, p, n) ∈ 𝒟 do
    emb_a, emb_p, emb_n ← f_θ(a), f_θ(p), f_θ(n)
    M_apn ← compute_MR_metric(emb_a, emb_p, emb_n, R)
    𝒯_ap ← 𝒯(emb_a, emb_p)
    ℐ_ap ← ℐ(emb_a, emb_p)
    ℒ ← max(0, M_apn − 𝒯_ap + λℐ_ap + Δ)
    θ ← θ − η∇_θ ℒ
end for
```

Implementation Architecture:



Computational Complexity Analysis: The NMR-MS framework maintains comparable complexity to standard metric learning:

- MR-Metric Computation: $O(d)$ where d is embedding dimension,
- Membership Functions: $O(d)$ per membership type,
- Overall Complexity: $O(kd)$ where k is number of neutrosophic components.

3.5. **Theoretical Advantages Summary:**

Table 1. Comparative Analysis: Standard vs. Neutrosophic Metric Learning

Feature	**Standard ML**	**Neutrosophic ML**
Uncertainty Handling	Limited	✓Explicit $\mathcal{I}$
Noise Robustness	Moderate	✓High
Conflicting Labels	Poor performance	✓Adaptive handling
Interpretability	Low	✓High
Confidence Calibration	Often poor	✓Well-calibrated
Theoretical Foundation	Metric spaces	NMR-MS

These applications demonstrate that the NMR-MS framework provides a mathematically rigorous yet practical approach for modern machine learning challenges involving uncertainty and ambiguity.

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