Nonlinear Functional Analysis and Applications
Vol. 19, No. 2 (2014), pp. 297-306

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THE FIXED POINT PROPERTY OF STRICTLY CONVEX REFLEXIVE BANACH SPACES FOR NON-EXPANSIVE SELF-MAPPINGS

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Abstract. We prove that if X is a strictly convex reflexive Banach space, C is a bounded, closed, convex subset of X with finite extreme points, then all the points in C except extreme points can be non-diametral points, hence C has normal structure, every non-expansive self-mapping T on C has a fixed point. Also, if C has countable extreme points, then C is compact, every non-expansive self- mapping T on C has a fixed point. Further-more, if T is also surjective, we show which points are fixed points of T.

1. Introduction

The concept of normal structure was introduced by Brodskii and Milman [1]. By using this concept "normal structure", Kirk [2] in 1965 proved that if a Banach space X has normal structure, then it has fixed point property(FPP for short). Also in [2], Kirk raised a question: if normal structure is essential for FPP? This question was given a negative answer by W.L. Bynum [3] in 1972. In 1965, F. Browder [4] proved uniformly convex Banach spaces have normal structure, hence uniformly convex Banach spaces have FPP. Then people wanted to known if FPP was enjoyed by wider spaces: Banach spaces. This question was also given a negative answer by a counter example L_1, this

[0]Received December 24, 2013. Revised April 11, 2014.
[0]2010 Mathematics Subject Classification: 46A25, 47B22.
[0]Keywords: Krein-Milman property, normal structure, fixed point property, diametral point.
[0]Research supported by the NSF of China(Grant No.11171021).

example was given by Alspach [5] in 1981. But Maurey [6] proved that every reflexive subset of L_1 has the FPP. Then people wanted to know if every strictly convex reflexive Banach space further-more every reflexive Banach space has FPP. However, these questions are still open till now.

Since Browder [4] proved the FPP was enjoyed by uniformly convex Banach spaces, many people have shown that this property is also enjoyed by Banach spaces which are wider than uniformly convex Banach spaces. Jesús García-Falset and other authors [7] proved that uniformly nonsquare Banach spaces, which are wider than uniformly convex Banach spaces, enjoy FPP for non-expansive self-mappings in 2006. Dowling and other authors [8] proved that E-convex Banach spaces, which are wider than uniformly nonsquare Banach spaces, enjoy FPP for non-expansive self-mappings in 2008. However, whether two more wider spaces than above mentioned spaces: super-reflexive Banach spaces or reflexive Banach spaces enjoy FPP for non-expansive self-mappings remains unknown. There is also a regret that we do not know whether strictly convex Banach spaces have FPP for non-expansive self-mappings on every bounded, closed, convex subset.

In this paper, We prove that if X is a strictly convex reflexive Banach space, C is a bounded, closed, convex subset of X with finite extreme points, then all the points in C except extreme points can be non-diametral points, hence C has normal structure, every non-expansive self-mapping T on C has a fixed point. Also, if C has countable extreme points, then C is compact, every non-expansive self-mapping T on C has a fixed point. Further-more, if T is also surjective, we show which points are fixed points of T. To fulfill our proof, we use the basic tool: non-diametral point and normal structure.

2. Definitions

Suppose (M, d) is a metric space, a mapping T defined from M to M is said to be non-expansive if $d(Tx, Ty) \leq d(x; y)$ for every $x, y \in M$. A Banach space X has FPP if, for every nonempty, closed, bounded convex subset C of X, every non-expansive self-mapping on C has a fixed point.

Definition 2.1. ([9]) Suppose K is a convex set, a point x of K is called an extreme point if x can not be written as a convex combination $\lambda y+(1-\lambda)z, 0 \leq \lambda \leq 1$, of two distinct points y, z of K. All the extreme points of K will be denoted by $extK$.

Definition 2.2. ([10]) Suppose X is a Banach space, if for each bounded, closed, convex subset A, we have $A = \overline{co}(extA)$, then X is said to have Krein-Milman Property(KMP for short). Every reflexive Banach space has KMP.

Definition 2.3. ([3]) Let C be a bounded subset of a Banach space X. The diameter of C, diam C, is $\sup\{\|x-y\| : x, y \in C\}$. A member x of C is a non-diametral point provided that diam $C > \sup\{\|x-u\| : u \in C\}$ and a diametral point of C is a point x for which the previous inequality is replaced by equality.

Remark 2.4. In this paper, we use strong diametral points to denote for the points $x, y \in C$, such that $\|x-y\|$ =diam C.

Definition 2.5. ([1]) Suppose X is a Banach space, a convex set $K \subset X$ is said to have normal structure if for each bounded convex subset H of K which contains more than one point, there is some $x \in H$which is not a diametral point of H. X has normal structure if each bounded convex subset of X with positive diameter has a non-diametral point.

3. Main results

Lemma 3.1. *Suppose $(X, \|\cdot\|)$ is a strictly convex reflexive Banach space, C is a bounded, closed, convex subset of X with countable extreme points $x_1, x_2, \cdots x_n, \cdots$. If strong diametral points of C exist, then strong diametral points can only be got in extreme points, that is if diam $C = \|x-y\|, x, y \in C$, then x, y must be extreme points.*

Proof. We prove this lemma in terms of two cases.

Case 1. C has two or three extreme points.

If C has two extreme points x_1, x_2, then $C = \overline{co}(x_1, x_2)$. If $diamC = \|x-y\|, x$ is an extreme ponit, y is not an extreme point in C. We can let $x = x_1$, $y = \lambda x_1 + (1-\lambda)x_2$, $0 < \lambda < 1$, then

$$\begin{aligned} \|x-y\| &= \|x_1 - \lambda x_1 - (1-\lambda)x_2\| \\ &= (1-\lambda)\|x_1 - x_2\| \\ &< \|x_1 - x_2\| < diamC. \end{aligned}$$

If $diamC = \|x-y\|$, x, y are not extreme points in C. We can let $x = \mu x_1 + (1-\mu)x_2$, $y = \nu x_1 + (1-\nu)x_2$, $0 < \mu, \nu < 1$, then

$$\begin{aligned} \|x-y\| &= \|\mu x_1 + (1-\mu)x_2 - \nu x_1 - (1-\nu)x_2\| \\ &= |\mu - \nu|\|x_1 - x_2\| \\ &< \|x_1 - x_2\| < diamC. \end{aligned}$$

If C has three extreme points x_1, x_2, x_3, by C has KMP, then $C = \overline{co}(x_1, x_2, x_3)$. All the points in C are in a triangle with three vertexes x_1, x_2, x_3. If $diamC = \|x-y\|, x$ is an extreme point, y is not an extreme point in C. Then we will show $diamC > \|x-y\|$. In fact, we can let $x = x_1$, by X is strictly convex, we only need to show $diamC > \|x_1-y\|$, where $y \in \{\lambda x_2+(1-\lambda)x_3 : 0 < \lambda < 1\}$. Suppose $y = \lambda_0 x_2 + (1-\lambda_0)x_3, 0 < \lambda_0 < 1$, then we have

$$\begin{aligned} \|x_1 - y\| &= \|x_1 - \lambda_0 x_2 - (1-\lambda_0)x_3\| \\ &\leq \lambda_0\|x_1 - x_2\| + (1-\lambda_0)\|x_1 - x_3\| \\ &\leq \max\{\|x_1 - x_2\|, \|x_1 - x_3\|\}. \end{aligned}$$

If $\|x_1 - x_2\| \neq \|x_1 - x_3\|$, we have $\|x_1 - y\| < \max\{\|x_1 - x_2\|, \|x_1 - x_3\|\} \leq diamC$. If $\|x_1-x_2\| = \|x_1-x_3\|$, we have $\|x_1-y\| \leq \|x_1-x_2\| = \|x_1-x_3\|$. Since X is strictly convex, then $\|x_1 - y\| < \|x_1 - x_2\| \leq diamC$.

Next, we will show $diamC > \|x - y\|$, where $x, y \in C$ and x, y are not extreme points. Obviously, if

$$x, y \in \left\{ z : z = \sum_{i=1}^{3} \lambda_i x_i, \quad 0 < \lambda_i < 1, \quad \sum_{i=1}^{3} \lambda_i = 1 \right\},$$

we can get two points x^0, y^0 in C, such that $\|x - y\| < \|x^0 - y^0\| \leq diamC$. If there exists a member in $\{x, y\}$ such that this member belongs to a lateral of the triangle with three vertexes x_1, x_2, x_3. We can let this member be y, suppose $y = \lambda x_1 + (1-\lambda)x_3$, then

$$\begin{aligned} \|x - y\| &= \|x - \lambda x_1 + (1-\lambda)x_3\| \\ &\leq \lambda\|x - x_1\| + (1-\lambda)\|x - x_3\| \\ &\leq \max\{\|x - x_1\|, \|x - x_3\|\}. \end{aligned}$$

If $\|x - x_1\| \neq \|x - x_3\|$, then $\|x - y\| < \max\{\|x - x_1\|, \|x - x_3\|\} \leq diamC$. If $\|x - x_1\| = \|x - x_3\|$, then $\|x - y\| \leq \|x - x_1\| = \|x - x_3\|$. Since X is strictly convex, we have $\|x - y\| < \|x - x_1\| \leq diamC$.

Case 2. C has countable but more than three extreme points.

For every $x, y \in C$, and x, y are not all the extreme points, x, y must belong to a triangle with three vertexes of extreme points. By case1, we have $\|x - y\| < diamC$.

By Case 1, Case 2 and the existence of the strong diametral points, if $diamC = \|x-y\|$, then x, y must be extreme points. So our proof is complete. □

Lemma 3.2. *Suppose X is a strictly convex reflexive Banach space, C is a bounded, closed, convex subset of X with countable extreme points. The strong diametral points set of C is A. Then for every $a \in A$, every surjective non-expansive self-mapping T on C, we have $Ta \in A$.*

Proof. Suppose the extreme points set of C is B, by Lemma 3.1, we have $A \subset B$. For each $a \in A$, by T is surjective, there must exist a $y \in C$, such that $Ty = a$. Denote $M = \{y : Ty = a, a \in A\}$. Since T is non-expansive, we can claim each point in M is a strong diametral point. In fact, for each $x \in A$, there exists a $y \in A$, such that $\|x - y\| = diamC$. There also exists two members $x^0, y^0 \in C$, such that $Tx^0 = x, Ty^0 = y$. If x^0 is not a strong diametral point, then

$$\begin{aligned}\|Tx^0 - Ty^0\| &= \|x - y\| \\ &\leq \|x^0 - y^0\|.\end{aligned}$$

But x, y are strong diametral points, so $\|x - y\| > \|x^0 - y^0\|$, which is a contradiction. Thus $M \subset A$.

Meantime, by not knowing whether T is 1-1, only knowing T is surjective, then $A \subset M$. Thus $M = A$ and for each $a \in A, Ta \in A$. So our proof is complete. □

Remark 3.3. In fact, $T\mid_A$ is bijective. Since $TA = A$, the members in TA and A are same, so T is 1-1. Since T is also surjective, then T is bijective.

Lemma 3.4. ([2]) *Suppose K is a nonempty, bounded, closed and convex subset of a reflexive Banach space X, and suppose that K has normal structure. If ϕ is a mapping of K into itself which does not increase distances, then ϕ has a fixed point.*

Theorem 3.5. *Suppose $(X, \|\cdot\|)$ is a reflexive Banach space, C is a bounded, closed, convex subset of X with two extreme points. Then, for every surjective non-expansive self-mapping T on C, then x_1, x_2, or $\frac{x_1+x_2}{2}$ is a fixed point.*

Proof. Suppose the two extreme points of C are x_1, x_2. Then $C = \overline{co}(x_1, x_2)$. By the assumption and Lemma 3.2, $Tx_i = x_j$, $1 \leq i, j \leq 2$. If $i = j$, then both x_1 and x_2 are fixed points. If $i \neq j$, that is $Tx_1 = x_2$, $Tx_2 = x_1$, then $\dfrac{x_1 + x_2}{2}$ is a fixed point.

In fact,

$$\begin{aligned}\|Tx_1 - T(\frac{x_1 + x_2}{2})\| &= \|x_2 - T(\frac{x_1 + x_2}{2})\| \\ &\leq \|x_1 - \frac{x_1 + x_2}{2}\| = \|\frac{x_1 - x_2}{2}\|.\end{aligned}$$

So

$$T(\frac{x_1 + x_2}{2}) \in B(x_2, \|\frac{x_1 - x_2}{2}\|),$$

where $B(x_2, \|\frac{x_1 - x_2}{2}\|)$ stands for a ball centered at x_2 with a radius $\|\frac{x_1 - x_2}{2}\|$. Meantime $T(\frac{x_1 + x_2}{2}) \in B(x_1, \|\frac{x_1 - x_2}{2}\|)$. So

$$T(\frac{x_1 + x_2}{2}) \in \bigcap_{i=1}^{2} B(x_i, \|\frac{x_1 - x_2}{2}\|) \cap C.$$

Thus, by C has two extreme points, then

$$\bigcap_{i=1}^{2} B(x_i, \|\frac{x_1 - x_2}{2}\|) \cap C = \frac{x_1 + x_2}{2}.$$

So $T(\frac{x_1 + x_2}{2}) = \frac{x_1 + x_2}{2}$, our proof is complete. □

Theorem 3.6. *Suppose X is a strictly convex reflexive Banach space, C is a bounded, closed, convex subset of X with countable extreme points $x_1, \cdots, x_n, \cdots$. T is a surjective non-expansive self-mapping on C. We have*

(i) *if $\|x_{i_0} - x_{j_0}\| = \max\{\|x_i - x_j\| : x_i$ and x_j are extreme points of $C\}$, then $\frac{x_{i_0} + x_{j_0}}{2}$ must be a fixed point of T on C;*

(ii) *if $\|x_{i_0} - x_{j_0}\| = \|x_{i_0} - x_{t_0}\| = \max\{\|x_i - x_j\| : x_i$ and x_j are extreme points of $C\}$, then x_{i_0} must be a fixed point of T on C;*

(iii) *when the dimension of $X \leq 3$, if $\|x_{i_0} - x_{j_0}\| = \|x_{j_0} - x_{t_0}\| = \|x_{t_0} - x_{i_0}\| = \max\{\|x_i - x_j\| : x_i$ and x_j are extreme points of $C\}$, then $\frac{x_{i_0} + x_{j_0} + x_{t_0}}{3}$ must be a fixed point of T on C.*

Proof. (i) Since $\|x_{i_0} - x_{j_0}\| = \max\{\|x_i - x_j\|\}$, x_{i_0}, x_{j_0} must be strong diametral points. By T is surjective and Lemma 3.2, we have $Tx_{i_0} = x_{i_0}$, $Tx_{j_0} = x_{j_0}$ or $Tx_{i_0} = x_{j_0}$, $Tx_{j_0} = x_{i_0}$.

If $Tx_{i_0} = x_{j_0}$, $Tx_{j_0} = x_{i_0}$, then $T(\frac{x_{i_0} + x_{j_0}}{2}) = \frac{x_{i_0} + x_{j_0}}{2}$. In fact

$$\begin{aligned}\|Tx_{i_0} - T(\frac{x_{i_0} + x_{j_0}}{2})\| &= \|x_{j_0} - T(\frac{x_{i_0} + x_{j_0}}{2})\| \\ &\leq \|x_{i_0} - \frac{x_{i_0} + x_{j_0}}{2}\| = \|\frac{x_{i_0} - x_{j_0}}{2}\|.\end{aligned}$$

So, $T(\frac{x_{i_0} + x_{j_0}}{2}) \in B(x_{j_0}, \|\frac{x_{i_0} - x_{j_0}}{2}\|)$. Similarly, we can get $T(\frac{x_{i_0} + x_{j_0}}{2}) \in B(x_{i_0}, \|\frac{x_{i_0} - x_{j_0}}{2}\|)$. So,

$$T(\frac{x_{i_0} + x_{j_0}}{2}) \in B(x_{j_0}, \|\frac{x_{i_0} - x_{j_0}}{2}\|) \cap B(x_{i_0}, \|\frac{x_{i_0} - x_{j_0}}{2}\|) \cap C.$$

Since X is strictly convex,

$$B(x_{j_0}, \|\frac{x_{i_0} - x_{j_0}}{2}\|) \cap B(x_{i_0}, \|\frac{x_{i_0} - x_{j_0}}{2}\|) \cap C = \frac{x_{i_0} + x_{j_0}}{2}.$$

If $Tx_{i_0} = x_{i_0}, Tx_{j_0} = x_{j_0}$, using the same method as above, we can get

$$T(\frac{x_{i_0} + x_{j_0}}{2}) = \frac{x_{i_0} + x_{j_0}}{2}.$$

(ii) If $\|x_{i_0} - x_{j_0}\| = \|x_{i_0} - x_{t_0}\| = \max\{\|x_i - x_j\|\}$, then the strong diametral points of C are $x_{i_0}, x_{j_0}, x_{t_0}$. Suppose $A = \{x_{i_0}, x_{j_0}, x_{t_0}\}$. By Lemma 3.2, for each $a \in A$, we have $Ta \in A$, and $T\mid_A$ is bijective. So, if $Tx_{i_0} = x_{j_0}$, then there are two cases for $T\mid_A$.

Case 1. $Tx_{i_0} = x_{j_0}$, $Tx_{j_0} = x_{t_0}$, $Tx_{t_0} = x_{i_0}$.

Case 2. $Tx_{i_0} = x_{j_0}$, $Tx_{j_0} = x_{i_0}$, $Tx_{t_0} = x_{t_0}$.

In both Case 1 and Case 2, we have

$$\begin{aligned}\|Tx_{j_0} - Tx_{t_0}\| &= \|x_{t_0} - x_{i_0}\| \\ &\leq \|x_{j_0} - x_{t_0}\|.\end{aligned}$$

But in our assumption, $\|x_{t_0} - x_{i_0}\| = \max\{\|x_i - x_j\|\} > \|x_{t_0} - x_{j_0}\|$, which is a contradiction. So $Tx_{i_0} = x_{t_0}$ or $Tx_{i_0} = x_{i_0}$. using the same method as above, we can get $Tx_{i_0} \neq x_{t_0}$. So $Tx_{i_0} = x_{i_0}$.

(iii) In this assumption, strong diametral points set $A = \{x_{i_0}, x_{j_0}, x_{t_0}\}$. By Lemma 3.2, there exists three cases for $T\mid_A$.

Case 1. $Tx_i = x_i$, $i = i_0$, j_0, t_0.

Case 2. $Tx_i = x_j$, $Tx_j = x_k$, $Tx_k = x_i, i, j, k \in \{i_0, j_0, t_0\}$, $i \neq j \neq k$.

Case 3. $Tx_i = x_i$, $Tx_j = x_k$, $Tx_k = x_j$, $i, j, k \in \{i_0, j_0, t_0\}$, $i \neq j \neq k$.

But in any cases, by T is non-expansive, we have

$$\begin{aligned}\|Tx_i - T(\frac{x_{i_0} + x_{j_0} + x_{t_0}}{3})\| &= \|x_j - T(\frac{x_{i_0} + x_{j_0} + x_{t_0}}{3})\| \\ &\leq \|x_i - \frac{x_{i_0} + x_{j_0} + x_{t_0}}{3}\|,\end{aligned}$$

$i, j \in \{i_0, j_0, t_0\}$. Since $\|x_{i_0} - x_{j_0}\| = \|x_{j_0} - x_{t_0}\| = \|x_{t_0} - x_{i_0}\|$, then

$$\|x_i - \frac{x_{i_0} + x_{j_0} + x_{t_0}}{3}\|$$

is a constant, denote this constant as R. Thus, we have

$$T(\frac{x_{i_0} + x_{j_0} + x_{t_0}}{3}) \in B(x_{i_0}, R) \cap B(x_{j_0}, R) \cap B(x_{t_0}, R).$$

By X is strictly convex and dimension $X \leq 3$, then

$$B(x_{i_0}, R) \cap B(x_{j_0}, R) \cap B(x_{t_0}, R) = \frac{x_{i_0} + x_{j_0} + x_{t_0}}{3}$$

and

$$T(\frac{x_{i_0}+x_{j_0}+x_{t_0}}{3})=\frac{x_{i_0}+x_{j_0}+x_{t_0}}{3}.$$

Hence, the proof is complete. □

Theorem 3.7. *Suppose X is a strictly convex reflexive Banach space. C is a bounded, closed, convex subset of X with finite extreme points. Then every point between two extreme points in C is a non-diametral point. Hence C has normal structure, every non-expansive self-mapping on C has a fixed point.*

Proof. If C has two extreme points: x_1, x_2, then, $diamC \geq \|x_1 - x_2\|$, and for every $x = \lambda x_1 + (1-\lambda)x_2$, $0 \leq \lambda \leq 1$, $x \in C$ and $y = \mu x_1 + (1-\mu)x_2$, $0 < \mu < 1$, $y \in C$,, we have

$$\begin{aligned}\|y-x\| &= \|\mu x_1 + (1-\mu)x_2 - \lambda x_1 - (1-\lambda)x_2\|\\ &= |\lambda-\mu|\|x_1 - x_2\|\\ &< \|x_1 - x_2\|.\end{aligned}$$

So, $y = \mu x_1 + (1-\mu)x_2$, $0 < \mu < 1$, $y \in C$ is a non-diametral point, by y is an arbitrary point between two extreme points in C, so when C has two extreme points, every point between two extreme points in C is a non-diametral point.

If C has n extreme points $x_1, \cdots, x_n, n \in \mathbb{N}$, $n > 2$, then $C = \overline{co}(x_1, \cdots, x_n)$. For each $x \in C$, where x is a point between two extreme points, denote$R = \max\{\|x - x_i\| : 1 \leq i \leq n, i \in \mathbb{N}\}$. Take x as the center of the ball, r is the radius makes a ball $B(x, R)$. We will prove x is a non-diametral point of C.

In fact, suppose $\max\{\|x - x_i\| : 1 \leq i \leq n, i \in \mathbb{N}\} = \|x - x_{m_0}\|$ and $x = \lambda x_t + (1-\lambda)x_s$, $0 < \lambda < 1$, $1 \leq t$, $s \leq n$, $t \neq s, t, s \in \mathbb{N}$. If $x_{m_0} = x_t$, then for each $y \in C$, x, y must belong to a triangle with three vertexes of three extreme points, if y is an inner point of the triangle or a point between two extreme points, by X is strictly convex and the proof of Lemma 3.2, then there must exist an extreme point $z \in \{x_1, x_2, \cdots, x_n\}$, such that $\|x - y\| \leq \|x - z\|$. If y is an extreme point, then we can let $z = y$. So we have

$$\begin{aligned}\|x-y\| \leq \|x - x_{m_0}\| &= \|x - x_t\|\\ &= \|\lambda x_t + (1-\lambda)x_s - x_t\|\\ &= (1-\lambda)\|x_t - x_s\|\\ &< \|x_t - x_s\| \leq diamC.\end{aligned}$$

If $x_{m_0} = x_s$, we can also get $\|x - y\| < diamC$. So, x is a non-diametral point, when $x_{m_0} = x_t$ or $x_{m_0} = x_s$.

If $x_{m_0} \neq x_t$, $x_{m_0} \neq x_s$, then in triangle $\Delta x_{m_0} x_t x_s$, we have

$$\begin{aligned}\|x - x_{m_0}\| &= \|\lambda x_t + (1-\lambda)x_s - x_{m_0}\| \\ &= \|\lambda x_t + (1-\lambda)x_s - \lambda x_{m_0} - (1-\lambda)x_{m_0}\| \\ &\le \lambda\|x_t - x_{m_0}\| + (1-\lambda)\|x_s - x_{m_0}\| \\ &\le \max\{\|x_t - x_{m_0}\|, \|x_s - x_{m_0}\|\}.\end{aligned} \tag{$*$}$$

If $\|x_t - x_{m_0}\| \neq \|x_s - x_{m_0}\|$, then $(*)$ is $\|x - x_{m_0}\| < \max\{\|x_t - x_{m_0}\|, \|x_s - x_{m_0}\|\}$. If $\|x_t - x_{m_0}\| = \|x_s - x_{m_0}\|$, by X is strictly convex, we have $\|x - x_{m_0}\| < \|x_t - x_{m_0}\|$. Thus, for each $y \in C$, by $(*)$, we have

$$\|x - y\| \le \|x - x_{m_0}\| < \max\{\|x_t - x_{m_0}\|, \|x_s - x_{m_0}\|\} \le diamC.$$

Thus, x is a non-diametral point. Hence C has normal structure, by Lemma 3.4 [2], every non-expansive self-mapping on C has a fixed point. Our proof is complete. □

Corollary 3.8. *Suppose X is a strictly convex reflexive Banach space, C is a bounded, closed, convex subset of X with finite extreme points $x_1, x_2, \cdots, x_n$. Then all the points in C except extreme points can be non-diametral points.*

Proof. In Theorem 3.7, we proved all the points between two extreme points are non-diametral points. Now, we will prove if $x \in C$ is not an extreme point or a point between two extreme points, then x is a non-diametral point.

In fact, for every $n \in \mathbb{N}$, $n \ge 3$, suppose the extreme points are $x_1, x_2, \cdots, x_n$, and

$$\begin{aligned}&\max\{\|x - x_i\| : 1 \le i \le n, \quad i \in \mathbb{N}\} \\ &= \|x - x_{m_0}\|, \quad 1 \le m_0 \le n, \quad m_0 \in \mathbb{N}.\end{aligned}$$

Then for every $y \in C$, by Lemma 3.2, we have

$$\|x - y\| \le \max\{\|x - x_i\| : 1 \le i \le n, i \in \mathbb{N}\} = \|x - x_{m_0}\|.$$

By x is not an extreme point or a point between two extreme points, there exists a $z \in C$ such that $x = \lambda x_{m_0} + (1-\lambda)z$, $0 < \lambda < 1$. Then

$$\begin{aligned}\|x - y\| &= \|\lambda x_{m_0} + (1-\lambda)z - y\| \\ &\le \|\lambda x_{m_0} + (1-\lambda)z - x_{m_0}\| \\ &< \|z - x_{m_0}\| \\ &< diamC.\end{aligned}$$

So x is a non-diametral point when C has extreme points $x_1, x_2, \cdots, x_n$, $n \in \mathbb{N}$, $n \ge 3$. If C has two extreme points, then the points in C between two extreme points are non-diametral points. Hence when C has $n, n \in \mathbb{N}$ extreme points, all the points in C except extreme points are non-diametral points. Our proof is complete. □

Theorem 3.9. *Suppose X is a strictly convex reflexive Banach space, C is a bounded, closed, convex subset of X with countable extreme points $x_1, \cdots, x_n, \cdots$. Then X has FPP for every non-expansive self-mapping T on C.*

Proof. Since C is a bounded, closed, convex subset of a reflexive Banach space X, then X has KMP and $C = \overline{co}(extC)$. By $C = \overline{co}(extC)$ and $extC$ is countable, we can get C is a compact subset of X. Every non-expansive self-mapping T on a compact set C has a fixed point, so X has FPP for non-expansive self-mappings on C, then our proof is complete. □

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