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A UNIFIED PENALTY FRAMEWORK FOR HISTORY-DEPENDENT MIXED VARIATIONAL-HEMIVARIATIONAL INEQUALITIES

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Abstract. This work investigates penalty methods for mixed variationalhemivariational inequality problems involving history-dependent operators as the penalty parameter approaches zero. We have demonstrated the unique solvability of these penalised problems and their convergence to the solution of the original history-dependent mixed variationalhemivariational inequality problems.

1. Introduction

Panagiotopoulos [6] introduced the variational-hemivariational inequalities in the context of engineering problems. The mathematical literature on these inequalities has flourished in recent decades, driven by their significant applications in physics, mechanics, nanotechnology and engineering sciences, as noted in [3, 8]. This study has extensively researched variational-hemivariational inequalities within the functional framework we define and assume throughout. These inequalities arise when studying nonsmooth boundary value problems and are governed by convex functions and local Lipschitz functions, which may

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or may not be convex. Consequently, research in this area requires familiarity with both convex and nonsmooth analysis.

The theory of variational inequality with history-dependent operators was developed and studied in connection with contact problems for memory-dependent materials. This led to the existence, uniqueness and regularity results for abstract classes of quasi-variational inequalities using history-dependent operators, as shown in [10] and [13]. Penalty methods approximate constrained variational or hemivariational inequality problems by reducing them to sequences of unconstrained ones as the penalty parameter approaches zero, as demonstrated in [7], [14] and [16]. Hemivariational inequalities with history-dependent operators were further explored in [4], [5] and [9]. These inequalities were initially stated in the context of Sobolev spaces over a bounded domain in $\mathbb{R}^d$ and specific operators like the trace operator. More recently, [11] demonstrated a general existence and uniqueness result for variational-hemivariational inequalities with history-dependent operators in an abstract setting of reflexive Banach spaces.

We present a penalty method to study history-dependent mixed variational-hemivariational inequality problems and their existence and convergence solutions. We also show that the weak solution of the penalised problem converges to the weak solution of the original problem as the stiffness coefficient of the foundation converges to infinity.

2. Preliminaries

Throughout this work, we use the symbols $\rightharpoonup$ and $\to$ to denote weak and strong convergence, respectively. For a normed space $\mathbb{X}$, we denote its norm by $\|\cdot\|_{\mathbb{X}}$. When no confusion arises, the duality pairing between the dual space $\mathbb{X}^*$ and $\mathbb{X}$, $\langle\cdot,\cdot\rangle_{\mathbb{X}^*\times\mathbb{X}}$, is simply written as $\langle\cdot,\cdot\rangle$. Let $\mathbb{N}$ denote the set of positive integers and $\mathbb{R}_+ = [0,+\infty)$ the set of nonnegative real numbers. The space of $\mathbb{X}$-valued continuous functions on $\mathbb{R}_+$ is denoted by $C(\mathbb{R}_+,\mathbb{X})$. For a subset $\Omega\subset\mathbb{X}$, $C(\mathbb{R}_+,\Omega)\subset C(\mathbb{R}_+,\mathbb{X})$ represents the set of Ω-valued continuous functions on $\mathbb{R}_+$.

We now recall some essential definitions.

Let $\mathcal{A}:\mathbb{X}\times\mathbb{X}\to\mathbb{X}^*$ be an operator.

(1) $\mathcal{A}$ is monotone if

$$\langle\mathcal{A}(v_1,v_1)-\mathcal{A}(v_2,v_2),v_1-v_2\rangle\ge 0,\quad\forall v_1,v_2\in\mathbb{X}.$$

(2) $\mathcal{A}$ is strongly monotone with constant $\alpha_{\mathcal{A}}>0$ if

$$\langle\mathcal{A}(v_1,v_1)-\mathcal{A}(v_2,v_2),v_1-v_2\rangle\ge\alpha_{\mathcal{A}}\|v_1-v_2\|^2,\quad\forall v_1,v_2\in\mathbb{X}. \tag{2.1}$$

(3) $\mathcal{A}$ is inverse strongly monotone with constant $\alpha_{\mathcal{A}} > 0$ if for all $v_1, v_2 \in \mathbb{X}$,

$$\langle \mathcal{A}(v_1, v_1) - \mathcal{A}(v_2, v_2), v_1 - v_2 \rangle \geq \alpha_{\mathcal{A}} \|\mathcal{A}(v_1, v_1) - \mathcal{A}(v_2, v_2)\|^2. \tag{2.2}$$

(4) $\mathcal{A}$ is Lipschitz continuous with constants $\beta_{\mathcal{A}} > 0$, $\rho_{\mathcal{A}} > 0$ if

$$\|\mathcal{A}(u, u) - \mathcal{A}(v, v)\| \leq \beta_{\mathcal{A}} \|u - v\| + \rho_{\mathcal{A}} \|u - v\|, \ \ \forall u, v \in \mathbb{X}. \tag{2.3}$$

(5) $\mathcal{A}$ is demicontinuous if

$$u_n \to u \text{ implies } \mathcal{A}(u_n, u_n) \rightharpoonup \mathcal{A}(u, u).$$

(6) $\mathcal{A}$ is hemicontinuous if the function

$$t \mapsto \langle \mathcal{A}(u + tv, u + tv), w \rangle$$

is continuous on $[0, 1]$ for all $u, v, w \in \mathbb{X}$.

(7) $\mathcal{A}$ is pseudomonotone if it is bounded and $u_n \rightharpoonup u$ in $\mathbb{X}$ together with

$$\limsup_{n \to \infty} \langle \mathcal{A}(u_n, u_n), u_n - u \rangle \leq 0$$

implies

$$\langle \mathcal{A}(u, u), u - v \rangle \leq \liminf_{n \to \infty} \langle \mathcal{A}(u_n, u_n), u_n - v \rangle, \ \ \forall v \in \mathbb{X}.$$

A function $\varphi : \mathbb{X} \to \mathbb{R}$ is lower semicontinuous (l.s.c.) if for any sequence $\{x_n\} \subset \mathbb{X}$ with $x_n \to x$, we have $\varphi(x) \leq \liminf_{n \to \infty} \varphi(x_n)$.

Let $\varphi : \mathbb{X} \to \mathbb{R}$ be a locally Lipschitz function. The Clarke generalized directional derivative at $x \in \mathbb{X}$ in the direction $v \in \mathbb{X}$ is defined by

$$\varphi^0(x, v) = \lim_{y \to x} \sup_{\gamma \downarrow 0} \frac{\varphi(y + \gamma v) - \varphi(y)}{\gamma}.$$

The generalized gradient (subdifferential) at x is the subset of $\mathbb{X}^*$ given by

$$\partial \varphi(x) = \{\varsigma \in \mathbb{X}^* | \varphi^0(x, v) \geq \langle \varsigma, v \rangle \ \forall v \in \mathbb{X}\}.$$

The function φ is regular at $x \in \mathbb{X}$ if for all $v \in \mathbb{X}$, the one-sided directional derivative $\varphi'(x, v)$ exists and $\varphi^0(x, v) = \varphi'(x, v)$; see [1, 2].

Let Ω be a subset of a reflexive Banach space $\mathbb{X}$, and let $\mathbb{Y}$ be a normed space. Consider the operators and functions $\mathcal{A} : \mathbb{X} \times \mathbb{X} \to \mathbb{X}^*$, $\mathcal{B} : C(\mathbb{R}_+, \mathbb{X}) \to C(\mathbb{R}_+, \mathbb{Y})$, $\varphi : \mathbb{Y} \times \mathbb{X} \times \mathbb{X} \to \mathbb{R}$, $\jmath : \mathbb{X} \to \mathbb{R}$, and $f : \mathbb{R}_+ \to \mathbb{X}^*$. We study the following problem: find $u \in C(\mathbb{R}_+, \Omega)$ such that for all $t \in \mathbb{R}_+$,

$$\begin{aligned} \langle \mathcal{A}(u(t), u(t)), v - u(t) \rangle &+ \varphi((\mathcal{B}u)(t), u(t), v) - \varphi((\mathcal{B}u)(t), u(t), u(t)) \\ &+ \jmath^0(u(t), v - u(t)) \geq \langle f(t), v - u(t) \rangle, \ \forall v \in \Omega. \end{aligned} \tag{2.4}$$

We impose the following assumptions:

(i) $\mathcal{A} : \mathbb{X} \times \mathbb{X} \to \mathbb{X}^*$ is pseudomonotone and inverse strongly monotone with constant $\alpha_{\mathcal{A}}$, and Lipschitz continuous with constants $\beta_{\mathcal{A}}, \rho_{\mathcal{A}} > 0$, in the first and second variables, respectively. (2.5)

(ii) $\mathcal{B} : C(\mathbb{R}_+, \mathbb{X}) \to C(\mathbb{R}_+, \mathbb{Y})$ is such that for any $n \in \mathbb{N}$ there exists $\ell_n > 0$ with for all $t \in [0, n]$,

$$\|(\mathcal{B}u_1)(t) - (\mathcal{B}u_2)(t)\|_{\mathbb{Y}} \leq \ell_n \int_0^t \|u_1(s) - u_2(s)\|_{\mathbb{X}}\, ds, \ \forall u_1, u_2 \in C(\mathbb{R}_+, \mathbb{X}). \tag{2.6}$$

(iii) $\varphi : \mathbb{Y} \times \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ satisfies:

$$\begin{cases} \text{(a)} & \varphi(y, u, \cdot) : \mathbb{X} \to \mathbb{R} \text{ is convex and l.s.c. for all } y \in \mathbb{Y}, u \in \mathbb{X}. \\ \text{(b)} & \text{There exists } \alpha_\varphi, \beta_\varphi \geq 0 \text{ such that for all } y_1, y_2 \in \mathbb{Y}, u_1, u_2, v_1, v_2 \in \mathbb{X}, \\ & \varphi(y_1, u_1, v_2) - \varphi(y_1, u_1, v_1) + \varphi(y_2, u_2, v_1) - \varphi(y_2, u_2, v_2) \\ & \leq \alpha_\varphi \|u_1 - u_2\|_{\mathbb{X}} \|v_1 - v_2\|_{\mathbb{X}} + \beta_\varphi \|y_1 - y_2\|_{\mathbb{Y}} \|v_1 - v_2\|_{\mathbb{X}}. \end{cases} \tag{2.7}$$

(iv) $\jmath : \mathbb{X} \to \mathbb{R}$ satisfies:

$$\begin{cases} \text{(a)} & \jmath \text{ is locally Lipschitz.} \\ \text{(b)} & \|\partial \jmath(v)\|_{\mathbb{X}^*} \leq \varrho_0 + \varrho_1 \|v\|_{\mathbb{X}} \ \forall v \in \mathbb{X} \text{ with } \varrho_0, \varrho_1 \geq 0. \\ \text{(c)} & \text{There exists } \alpha_\jmath \geq 0 \text{ such that for all } v_1, v_2 \in \mathbb{X}, \\ & \jmath^0(v_1, v_2 - v_1) + \jmath^0(v_2, v_1 - v_2) \leq \alpha_\jmath \|v_1 - v_2\|_{\mathbb{X}}^2. \end{cases} \tag{2.8}$$

Theorem 2.1. *Let $\mathbb{X}$ be a reflexive Banach space, $\Omega \subset \mathbb{X}$ nonempty, closed and convex, and $\mathbb{Y}$ a normed space. Assume* (2.5)-(2.8) *hold and that*

$$\alpha_\varphi + \alpha_\jmath < \alpha_{\mathcal{A}}(\beta_{\mathcal{A}} + \rho_{\mathcal{A}})^2. \tag{2.9}$$

Then, for any $f \in C(\mathbb{R}_+, \mathbb{X}^*)$, the problem (2.4) *has a unique solution $u \in C(\mathbb{R}_+, \Omega)$.*

Proof. The existence and uniqueness of a solution to the problem (2.4) follows directly from [11] and [12]. □

The penalty method is defined via a penalty operator $\mathcal{P} : \mathbb{X} \times \mathbb{X} \to \mathbb{X}^*$ and a parameter $\gamma > 0$. We assume the penalty operator satisfies:

$$\begin{cases} \text{(a)} & \mathcal{P} \text{ is bounded, demicontinuous and monotone.} \\ \text{(b)} & \mathcal{P}(u, u) = 0 \ \Leftrightarrow \ u \in \Omega. \end{cases} \tag{2.10}$$

For each $\gamma > 0$, the penalized problem on the entire space $\mathbb{X}$ is to find $u_\gamma \in C(\mathbb{R}_+, \mathbb{X})$ such that for all $t \in \mathbb{R}_+$,

$$\begin{aligned}&\langle \mathcal{A}(u_\gamma(t), u_\gamma(t)), v - u_\gamma(t)\rangle + \frac{1}{\gamma}\langle \mathcal{P}(u_\gamma(t), u_\gamma(t)), v - u_\gamma(t)\rangle \\ &\quad + \varphi((\mathcal{B}u_\gamma)(t), u_\gamma(t), v) - \varphi((\mathcal{B}u_\gamma)(t), u_\gamma(t), u_\gamma(t)) + \jmath^0(u_\gamma(t), v - u_\gamma(t)) \\ &\quad \geq \langle f(t), v - u_\gamma(t)\rangle, \quad \forall v \in \mathbb{X}. \end{aligned} \tag{2.11}$$

To analyze the penalty method, we require additional conditions:

$$u_n \rightharpoonup u \text{ in } \mathbb{X} \;\Rightarrow\; \limsup_{n\to\infty} \jmath^0(u_n, v - u_n) \leq \jmath^0(u, v - u), \quad \forall v \in \mathbb{X}. \tag{2.12}$$

There exists a continuous function $\varrho_\varphi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\varphi(y, u, v_1) - \varphi(y, u, v_2) \leq \varrho_\varphi(\|y\|_{\mathbb{Y}}, \|u\|_{\mathbb{X}})\|v_1 - v_2\|_{\mathbb{X}}, \quad \forall y \in \mathbb{Y}, u, v_1, v_2 \in \mathbb{X}. \tag{2.13}$$

Lemma 2.2. ([5]) *Let $\mathbb{X}$ and $\mathbb{Y}$ be reflexive Banach spaces. Let $\psi : \mathbb{Y} \to \mathbb{R}$ satisfy* (2.8) *(with $\mathbb{X}$ replaced by $\mathbb{Y}$) and assume that either ψ or $-\psi$ is regular. Let $\mathcal{M} : \mathbb{X} \to \mathbb{Y}$ be given by $\mathcal{M}v = Lv + v_0$, where $L : \mathbb{X} \to \mathbb{Y}$ is a linear compact operator and $v_0 \in \mathbb{Y}$. Then the function $\jmath : \mathbb{X} \to \mathbb{R}$ defined by $\jmath(v) = \psi(\mathcal{M}v)$ satisfies conditions* (2.8) *and* (2.12)*.*

3. Main results

The main result of this paper is the following convergence theorem for the penalty method.

Theorem 3.1. *Assume that the assumptions of Theorem* 2.1 *hold. Furthermore, assume that* (2.10)*,* (2.12) *and* (2.13) *are satisfied. Then:*

(i) *For each $\gamma > 0$, there exists a unique solution $u_\gamma \in C(\mathbb{R}_+, \mathbb{X})$ to the penalized problem* (2.11)*.*
(ii) *The solution u_γ of* (2.11) *converge uniformly to the solution u of* (2.4) *on $\mathbb{R}_+$ as the penalty parameter tends to zero, that is,*

$$\|u_\gamma(t) - u(t)\|_{\mathbb{X}} \to 0 \text{ as } \gamma \to 0, \; t \in \mathbb{R}_+. \tag{3.1}$$

Proof. The proof of Theorem 3.1 will be established in several steps. Recall that $u \in C(\mathbb{R}_+, \Omega)$ denotes the unique solution of the original problem (2.4). The proof of Theorem 3.1 will be completed. The following Lemma 3.2 proves part (i), and Lemma 3.8 proves part (ii). □

Step 1: Existence and uniqueness for the penalized problem. The first step is to prove statement (i) of Theorem 3.1.

Lemma 3.2. *For each $\gamma > 0$, there exists a unique solution $u_\gamma \in C(\mathbb{R}_+, \mathbb{X})$ to* (2.11).

Proof. By (2.10), the penalty operator $\mathcal{P} : \mathbb{X} \times \mathbb{X} \to \mathbb{X}^*$ is bounded, demicontinuous, and monotone. It is well-known that these properties imply pseudomonotonicity (see [15]).

Define the operator $\mathcal{A}\gamma : \mathbb{X} \times \mathbb{X} \to \mathbb{X}^*$ by

$$\mathcal{A}_\gamma(v, v) = \mathcal{A}(v, v) + \frac{1}{\gamma}\mathcal{P}(v, v), \quad v \in \mathbb{X}. \tag{3.2}$$

Since $\mathcal{A}$ satisfies (2.5) and $\mathcal{P}$ is monotone, the operator $\mathcal{A}\gamma$ is pseudomonotone and inverse strongly monotone with the same constant $\alpha_{\mathcal{A}}$. Furthermore, the Lipschitz continuity of $\mathcal{A}_\gamma$ follows from that of $\mathcal{A}$ and the boundedness of $\mathcal{P}$, with constants independent of γ. Therefore, applying Theorem 2.1 with $\Omega = \mathbb{X}$, we conclude that (2.11) has a unique solution $u_\gamma \in C(\mathbb{R}_+, \mathbb{X})$. □

Step 2: Analysis of an auxiliary problem. To establish the convergence, we introduce an auxiliary problem. For each $\gamma > 0$, we consider the problem of finding $\tilde{u}_\gamma \in C(\mathbb{R}+, \mathbb{X})$ such that for all $t \in \mathbb{R}+$,

$$\begin{aligned}
&\langle \mathcal{A}(\tilde{u}_\gamma(t), \tilde{u}_\gamma(t)), v - \tilde{u}_\gamma(t)\rangle + \frac{1}{\gamma}\langle \mathcal{P}(\tilde{u}_\gamma(t), \tilde{u}_\gamma(t)), v - \tilde{u}_\gamma(t)\rangle \\
&\qquad + \varphi((\mathcal{B}u)(t), u(t), v) - \varphi((\mathcal{B}u)(t), u(t), \tilde{u}_\gamma(t)) + \jmath^0(\tilde{u}_\gamma(t), v - \tilde{u}_\gamma(t)) \\
&\quad \geq \langle f(t), v - \tilde{u}_\gamma(t)\rangle, \quad \forall v \in \mathbb{X}, \quad \forall t \in \mathbb{R}_+.
\end{aligned} \tag{3.3}$$

Note that while (2.11) is a history-dependent inequality, (3.3) is a (simpler) time-dependent inequality, as the history-dependent term $\mathcal{B}u$ is fixed to be that of the original solution u. We will show that (3.3) has a unique solution which converges strongly to $u(t)$.

Lemma 3.3. *For each $\gamma > 0$,* (3.3) *has a unique solution $\tilde{u}_\gamma \in C(\mathbb{R}_+, \mathbb{X})$.*

Proof. Let $t \in \mathbb{R}_+$ be fixed. Define the functional $\varphi_t : \mathbb{X} \to \mathbb{R}$ by

$$\varphi_t(v) = \varphi((\mathcal{B}u)(t), u(t), v), \quad v \in \mathbb{X}. \tag{3.4}$$

By hypothesis (2.7)(a), φ_t is convex and lower semicontinuous. Furthermore, for any $v_1, v_2 \in \mathbb{X}$, the condition (2.7)(b) yields

$$\varphi_t(v_2) - \varphi_t(v_1) + \varphi_t(v_1) - \varphi_t(v_2) = 0, \ \forall v_1, v_2 \in \mathbb{X}, \tag{3.5}$$

which is trivially bounded as required.

As established in the proof of Lemma 3.2, the operator $\mathcal{A}\gamma$ from (3.2) is pseudomonotone, inverse strongly monotone, and Lipschitz continuous. Under the smallness condition (2.9), we may apply [5, Theorem 5] to conclude that there exists a unique element $\tilde{u}_\gamma(t) \in \mathbb{X}$ satisfying (3.3) for this fixed t.

It remains to prove the continuity of the mapping $t \mapsto \tilde{u}_\gamma(t)$.

Let $t_1, t_2 \in \mathbb{R}+$ and denote $\tilde{u}_\gamma(t_i) = \tilde{u}_i$, $u(t_i) = u_i$, $(\mathcal{B}u)(t_i) = y_i$, and $f(t_i) = f_i$ for $i = 1, 2$. Taking $v = \tilde{u}_2$ in (3.3) at $t = t_1$ and $v = \tilde{u}_1$ at $t = t_2$, then adding the resulting inequalities, we obtain:

$$\begin{aligned} &\langle \mathcal{A}(\tilde{u}_1, \tilde{u}_1) - \mathcal{A}(\tilde{u}_2, \tilde{u}_2), \tilde{u}_1 - \tilde{u}_2 \rangle + \frac{1}{\gamma} \langle \mathcal{P}(\tilde{u}_1, \tilde{u}_1) - \mathcal{P}(\tilde{u}_2, \tilde{u}_2), \tilde{u}_1 - \tilde{u}_2 \rangle \\ &\quad \le \varphi(y_1, u_1, \tilde{u}_2) + \varphi(y_2, u_2, \tilde{u}_1) - \varphi(y_2, u_2, \tilde{u}_2) - \varphi(y_1, u_1, \tilde{u}_1) \\ &\qquad + \jmath^0(\tilde{u}_1, \tilde{u}_2 - \tilde{u}_1) + \jmath^0(\tilde{u}_2, \tilde{u}_1 - \tilde{u}_2) + \langle f_1 - f_2, \tilde{u}_1 - \tilde{u}_2 \rangle. \end{aligned} \tag{3.6}$$

Using the inverse strong monotonicity of $\mathcal{A}$ (2.2), the monotonicity of $\mathcal{P}$ (2.10)(a), the condition on $\jmath$ (2.8)(c), and the property of φ (2.7)(b), we derive:

$$\begin{aligned} &\alpha_{\mathcal{A}}(\beta_{\mathcal{A}} + \rho_{\mathcal{A}})^2 \|\tilde{u}_1 - \tilde{u}_2\|_{\mathbb{X}}^2 \le \alpha_\varphi \|u_1 - u_2\|_{\mathbb{X}} \|\tilde{u}_1 - \tilde{u}_2\|_{\mathbb{X}} \\ &\quad + \beta_\varphi \|y_1 - y_2\|_{\mathbb{Y}} \|\tilde{u}_1 - \tilde{u}_2\|_{\mathbb{X}} + \alpha_\jmath \|\tilde{u}_1 - \tilde{u}_2\|_{\mathbb{X}}^2 + \|f_1 - f_2\|_{\mathbb{X}^*} \|\tilde{u}_1 - \tilde{u}_2\|_{\mathbb{X}}. \end{aligned} \tag{3.7}$$

Dividing by $\|\tilde{u}_1 - \tilde{u}_2|_{\mathbb{X}}$ and using (2.9), we find:

$$\|\tilde{u}_1 - \tilde{u}_2\|_{\mathbb{X}} \le \vartheta \left(\|u_1 - u_2\|_{\mathbb{X}} + \|y_1 - y_2\|_{\mathbb{Y}} + \|f_1 - f_2\|_{\mathbb{X}^*} \right), \tag{3.8}$$

where $\vartheta > 0$ is a constant. The continuity of u, $\mathcal{B}u$, and f now implies the continuity of $\tilde{u}_\gamma$. Uniqueness follows from the unique solvability of (3.3) at each t. □

Lemma 3.4. (Weak convergence) *For each $t \in \mathbb{R}_+$, the sequence $\{\tilde{u}_\gamma(t)\}$ is bounded in $\mathbb{X}$. Consequently, there exists a subsequence (still denoted $\tilde{u}_\gamma(t)$) such that*

$$\tilde{u}_\gamma(t) \rightharpoonup \tilde{u}(t) \ \textit{weakly in } \mathbb{X} \quad \textit{as} \quad \gamma \to 0 \tag{3.9}$$

for some $\tilde{u}(t) \in \mathbb{X}$.

Proof. Let $u_0 \in \Omega$ be fixed. We estimate the term $\jmath^0(\tilde{u}_\gamma(t), u_0 - \tilde{u}_\gamma(t))$. Using (2.8)(c) and the definition of the generalized gradient, we have:

$$\begin{aligned} \jmath^0(\tilde{u}_\gamma(t), u_0 - \tilde{u}_\gamma(t)) &= \left[\jmath^0(\tilde{u}_\gamma(t), u_0 - \tilde{u}_\gamma(t)) + \jmath^0(u_0, \tilde{u}_\gamma(t) - u_0) \right] \\ &\quad - \jmath^0(u_0, \tilde{u}_\gamma(t) - u_0) \\ &\le \alpha_\jmath \|\tilde{u}_\gamma(t) - u_0\|_{\mathbb{X}}^2 + |\jmath^0(u_0, \tilde{u}_\gamma(t) - u_0)|. \end{aligned}$$

By [3, Proposition 3.23(iii)] and (2.8)(b), there exists $\varsigma \in \partial \jmath(u_0)$ such that

$$|\jmath^0(u_0, \tilde{u}_\gamma(t) - u_0)| \le \|\zeta\|_{\mathbb{X}^*} \|\tilde{u}_\gamma(t) - u_0\|_{\mathbb{X}} \le (\varrho_0 + \varrho_1 \|u_0\|_{\mathbb{X}}) \|\tilde{u}_\gamma(t) - u_0)\|_{\mathbb{X}}.$$

Thus,

$$\jmath^0(\tilde{u}_\gamma(t), u_0 - \tilde{u}_\gamma(t)) \leq \alpha_\jmath \|\tilde{u}_\gamma(t) - u_0\|_{\mathbb{X}}^2 + \big| \max\{\langle \varsigma, \tilde{u}_\gamma(t) - u_0\rangle \mid \varsigma \in \partial\jmath(u_0)\} \big|$$
$$\leq \alpha_\jmath \|\tilde{u}_\gamma(t) - u_0\|_{\mathbb{X}}^2 + (\varrho_0 + \varrho_1\|u_0\|_{\mathbb{X}})\|\tilde{u}_\gamma(t) - u_0\|_{\mathbb{X}}. \tag{3.10}$$

Now, take $v = u_0$ in (3.3). Since $u_0 \in \Omega$, we have $\mathcal{P}(u_0, u_0) = 0$ by (2.10)(b). Using the inverse strong monotonicity of $\mathcal{A}$ (2.2), we get:

$$\alpha_{\mathcal{A}}(\beta_{\mathcal{A}} + \rho_{\mathcal{A}})^2 \|\tilde{u}_\gamma(t) - u_0\|_{\mathbb{X}}^2 \leq \langle \mathcal{A}(\tilde{u}_\gamma(t), \tilde{u}_\gamma(t)) - \mathcal{A}(u_0, u_0), \tilde{u}_\gamma(t) - u_0\rangle$$
$$= \langle \mathcal{A}(\tilde{u}_\gamma(t), \tilde{u}_\gamma(t)), \tilde{u}_\gamma(t) - u_0\rangle - \langle \mathcal{A}(u_0, u_0), \tilde{u}_\gamma(t) - u_0\rangle.$$

From (3.3) with $v = u_0$, we have:

$$\langle \mathcal{A}(\tilde{u}_\gamma(t), \tilde{u}_\gamma(t)), \tilde{u}_\gamma(t) - u_0\rangle$$
$$\leq \frac{1}{\gamma}\langle \mathcal{P}(\tilde{u}_\gamma(t), \tilde{u}_\gamma(t)), u_0 - \tilde{u}_\gamma(t)\rangle + \varphi(y, u, u_0) - \varphi(y, u, \tilde{u}_\gamma(t))$$
$$+ \jmath^0(\tilde{u}_\gamma(t), u_0 - \tilde{u}_\gamma(t)) + \langle f(t) - \mathcal{A}(u_0, u_0), \tilde{u}_\gamma(t) - u_0\rangle,$$

where $y = (\mathcal{B}u)(t)$. Using the monotonicity of $\mathcal{P}$, the bound (2.13) for φ, and the estimate (3.10), we obtain:

$$\alpha_{\mathcal{A}}(\beta_{\mathcal{A}} + \rho_{\mathcal{A}})^2 \|\tilde{u}_\gamma(t) - u_0\|_{\mathbb{X}}^2 \leq -\frac{1}{\gamma} \underbrace{\langle \mathcal{P}(u_0, u_0) - \mathcal{P}(\tilde{u}_\gamma(t), \tilde{u}_\gamma(t)), u_0 - \tilde{u}_\gamma(t)\rangle}$$
$$+ \varrho_\varphi(\|y\|_{\mathbb{Y}}, \|u\|_{\mathbb{X}})\|\tilde{u}_\gamma(t) - u_0\|_{\mathbb{X}}$$
$$+ \alpha_\jmath \|\tilde{u}_\gamma(t) - u_0\|_{\mathbb{X}}^2 + (\varrho_0 + \varrho_1\|u_0\|_{\mathbb{X}})\|\tilde{u}_\gamma(t) - u_0\|_{\mathbb{X}}$$
$$+ \|f(t) - \mathcal{A}(u_0, u_0)\|_{\mathbb{X}^*}\|\tilde{u}_\gamma(t) - u_0\|_{\mathbb{X}^*}.$$

Dropping the non-positive penalty term and simplifying, we get:

$$\left(\alpha_{\mathcal{A}}(\beta_{\mathcal{A}} + \rho_{\mathcal{A}})^2 - \alpha_\jmath\right) \|\tilde{u}_\gamma(t) - u_0\|_{\mathbb{X}}^2 \leq \varrho_\varphi(\|y\|_{\mathbb{Y}}, \|u\|_{\mathbb{X}})\|\tilde{u}_\gamma(t) - u_0\|_{\mathbb{X}}$$
$$+ (\varrho_0 + \varrho_1\|u_0\|_{\mathbb{X}})\|\tilde{u}_\gamma(t) - u_0\|_{\mathbb{X}}$$
$$+ \|f(t) - \mathcal{A}(u_0, u_0)\|_{\mathbb{X}^*}\|\tilde{u}_\gamma(t) - u_0\|_{\mathbb{X}}.$$

Thus,

$$\|\tilde{u}_\gamma(t) - u_0\|_{\mathbb{X}} \leq \frac{1}{\alpha_{\mathcal{A}}(\beta_{\mathcal{A}} + \rho_{\mathcal{A}})^2 - \alpha_\jmath} C(t),$$

where $C(t) = \varrho_\varphi(\|(\mathcal{B}u)(t)\|_{\mathbb{Y}}, \|u(t)|_{\mathbb{X}}) + \varrho_0 + \varrho_1\|u_0\|_{\mathbb{X}} + \|f(t) - \mathcal{A}(u_0, u_0)\|_{\mathbb{X}^*}$ is finite and continuous in t. Therefore,

$$\|\tilde{u}_\gamma(t) - u_0\|_{\mathbb{X}} \leq \frac{C(t)}{\alpha_{\mathcal{A}}(\beta_{\mathcal{A}} + \rho_{\mathcal{A}})^2 - \alpha_\jmath} =: \vartheta(t), \tag{3.11}$$

which establishes the boundedness of $\tilde{u}_\gamma(t)$. By the reflexivity of $\mathbb{X}$, there exists a weakly convergent subsequence. □

Lemma 3.5. (Identification of the weak limit) *For each* $t \in \mathbb{R}_+$*, the weak limit* $\tilde{u}(t)$ *from* (3.9) *coincides with the solution* $u(t)$ *of the original problem* (2.4)*, that is,*

$$\tilde{u}(t) = u(t). \tag{3.12}$$

Proof. We first prove that $\tilde{u}(t) \in \Omega$. From (3.3), for any $v \in \mathbb{X}$, we have

$$\begin{aligned}\frac{1}{\gamma}\langle \mathcal{P}(\tilde{u}_\gamma(t), \tilde{u}_\gamma(t)), \tilde{u}_\gamma(t) - v\rangle \leq \langle \mathcal{A}(\tilde{u}_\gamma(t), \tilde{u}_\gamma(t)), v - \tilde{u}_\gamma(t)\rangle + \varphi(y, u, v) \\ - \varphi(y, u, \tilde{u}_\gamma(t)) + \jmath^0(\tilde{u}_\gamma(t), v - \tilde{u}_\gamma(t)) \\ + \langle f(t), \tilde{u}_\gamma(t) - v\rangle,\end{aligned}$$

where $y = (\mathcal{B}u)(t)$. Using the Lipschitz continuity of $\mathcal{A}$, the bound (2.13) for φ, and the bound (2.8)(b) for $\jmath^0$, we can estimate the right-hand side by a constant $\vartheta(t, v) > 0$ independent of γ. Thus,

$$\frac{1}{\gamma}\langle \mathcal{P}(\tilde{u}_\gamma(t), \tilde{u}_\gamma(t)), \tilde{u}_\gamma(t) - v\rangle \leq \vartheta(t, v), \quad \forall v \in \mathbb{X}. \tag{3.13}$$

Choosing $v = \tilde{u}(t)$ and multiplying by γ, we obtain

$$\langle \mathcal{P}(\tilde{u}_\gamma(t), \tilde{u}_\gamma(t)), \tilde{u}_\gamma(t) - \tilde{u}(t)\rangle \leq \gamma\vartheta(t, \tilde{u}(t)) \to 0 \quad \text{as } \gamma \to 0,$$

which implies

$$\limsup_{\gamma\to 0}\langle \mathcal{P}(\tilde{u}_\gamma(t), \tilde{u}_\gamma(t)), \tilde{u}_\gamma(t) - \tilde{u}(t)\rangle \leq 0. \tag{3.14}$$

Since $\mathcal{P}$ is pseudomonotone and $\tilde{u}_\gamma(t) \rightharpoonup \tilde{u}(t)$, (3.14) implies:

$$\langle \mathcal{P}(\tilde{u}(t), \tilde{u}(t)), \tilde{u}(t) - v\rangle \leq \liminf_{\gamma\to 0}\langle \mathcal{P}(\tilde{u}_\gamma(t), \tilde{u}_\gamma(t)), \tilde{u}_\gamma(t) - v\rangle, \ \forall v \in \mathbb{X}. \tag{3.15}$$

Combining (3.13) and (3.15), we get

$$\langle \mathcal{P}(\tilde{u}(t), \tilde{u}(t)), \tilde{u}(t) - v\rangle \leq 0, \ \ \forall v \in \mathbb{X}.$$

Since v is arbitrary, this forces $\langle \mathcal{P}(\tilde{u}(t), \tilde{u}(t)), w\rangle = 0$, for all $w \in \mathbb{X}$, and hence $\mathcal{P}(\tilde{u}(t), \tilde{u}(t)) = 0$. By (2.10)(b), it follows that $\tilde{u}(t) \in \Omega$.

Now, let $v \in \Omega$ be arbitrary. From (3.3) and the fact that $\mathcal{P}(v, v) = 0$, we have:

$$\begin{aligned}\langle \mathcal{A}(\tilde{u}_\gamma(t), \tilde{u}_\gamma(t)), \tilde{u}_\gamma(t) - v\rangle \leq -\frac{1}{\gamma}\underbrace{\langle \mathcal{P}(v, v) - \mathcal{P}(\tilde{u}_\gamma(t), \tilde{u}_\gamma(t)), v - \tilde{u}_\gamma(t)\rangle} \\ + \varphi(y, u, v) - \varphi(y, u, \tilde{u}_\gamma(t)) + \jmath^0(\tilde{u}_\gamma(t), v - \tilde{u}_\gamma(t)) \\ + \langle f(t), \tilde{u}_\gamma(t) - v\rangle.\end{aligned}$$

Dropping the non-positive penalty term yields:

$$\begin{aligned}\langle \mathcal{A}(\tilde{u}_\gamma(t), \tilde{u}_\gamma(t)), \tilde{u}_\gamma(t) - v\rangle \leq \varphi(y, u, v) - \varphi(y, u, \tilde{u}_\gamma(t)) + \jmath^0(\tilde{u}_\gamma(t), v - \tilde{u}_\gamma(t)) \\ + \langle f(t), \tilde{u}_\gamma(t) - v\rangle.\end{aligned} \tag{3.16}$$

Now, take $v = \tilde{u}(t)$ in (3.16). By the weak lower semicontinuity of $\varphi(y, u, \cdot)$ (2.7)(a), we have

$$\limsup_{\gamma\to 0} \left(\varphi(y, u, \tilde{u}(t)) - \varphi(y, u, \tilde{u}_\gamma(t))\right) \leq 0. \tag{3.17}$$

From assumption (2.12) and the weak convergence, we have

$$\limsup_{\gamma\to 0} \jmath^0(\tilde{u}_\gamma(t), \tilde{u}(t) - \tilde{u}_\gamma(t)) \leq 0. \tag{3.18}$$

Using (3.17), (3.18), and the weak convergence in (3.16) with $v = \tilde{u}(t)$, we deduce

$$\limsup_{\gamma\to 0} \langle \mathcal{A}(\tilde{u}_\gamma(t), \tilde{u}_\gamma(t)), \tilde{u}_\gamma(t) - \tilde{u}(t)\rangle \leq 0. \tag{3.19}$$

Since $\mathcal{A}$ is pseudomonotone, (3.19) and the weak convergence imply:

$$\langle \mathcal{A}(\tilde{u}(t), \tilde{u}(t)), \tilde{u}(t) - w\rangle \leq \liminf_{\gamma\to 0} \langle \mathcal{A}(\tilde{u}_\gamma(t), \tilde{u}_\gamma(t)), \tilde{u}_\gamma(t) - w\rangle, \quad \forall w \in \mathbb{X}. \tag{3.20}$$

Now, returning to (3.16) for a general $v \in \Omega$, and taking the lim sup, we use the weak convergence, (3.17) (with v replacing $\tilde{u}(t)$), and (2.12) to obtain

$$\begin{aligned}\limsup_{\gamma\to 0} \langle \mathcal{A}(\tilde{u}_\gamma(t), \tilde{u}_\gamma(t)), \tilde{u}_\gamma(t) - v\rangle \leq\ & \varphi(y, u, v) - \varphi(y, u, \tilde{u}(t)) \\ & + \jmath^0(\tilde{u}(t), v - \tilde{u}(t)) \\ & + \langle f(t), \tilde{u}(t) - v\rangle, \quad \forall v \in \Omega.\end{aligned} \tag{3.21}$$

Combining (3.20) (with $w = v$) and (3.21) yields:

$$\begin{aligned}\langle \mathcal{A}(\tilde{u}(t), \tilde{u}(t)), \tilde{u}(t) - v\rangle \leq\ & \varphi(y, u, v) - \varphi(y, u, \tilde{u}(t)) + \jmath^0(\tilde{u}(t), v - \tilde{u}(t)) \\ & + \langle f(t), \tilde{u}(t) - v\rangle, \quad \forall v \in \Omega.\end{aligned} \tag{3.22}$$

Rewriting (3.22), we see that $\tilde{u}(t)$ satisfies

$$\begin{aligned}\langle \mathcal{A}(\tilde{u}(t), \tilde{u}(t)), v - \tilde{u}(t)\rangle + \varphi((\mathcal{B}u)(t), u(t), v) - \varphi((\mathcal{B}u)(t), u(t), \tilde{u}(t)) \\ + \jmath^0(\tilde{u}(t), v - \tilde{u}(t)) \geq \langle f(t), v - \tilde{u}(t)\rangle, \ \forall v \in \Omega.\end{aligned} \tag{3.23}$$

We now compare (3.23) with the original problem (2.4) satisfied by $u(t)$. Taking $v = \tilde{u}(t)$ in (2.4) and $v = u(t)$ in (3.23), then adding the inequalities, we obtain:

$$\begin{aligned}&\langle \mathcal{A}(u(t), u(t)) - \mathcal{A}(\tilde{u}(t), \tilde{u}(t)), u(t) - \tilde{u}(t)\rangle \\ &\quad \leq \varphi(y, u, \tilde{u}(t)) - \varphi(y, u, u(t)) + \varphi(y, u, u(t)) - \varphi(y, u, \tilde{u}(t)) \\ &\qquad + \jmath^0(u(t), \tilde{u}(t) - u(t)) + \jmath^0(\tilde{u}(t), u(t) - \tilde{u}(t)).\end{aligned}$$

The φ terms cancel. Using the inverse strong monotonicity of $\mathcal{A}$ (2.2) and the condition on $\jmath$ (2.8)(c), we get:

$$\left(\alpha_{\mathcal{A}}(\beta_{\mathcal{A}} + \rho_{\mathcal{A}})^2\right) \|\tilde{u}(t) - u(t)\|_{\mathbb{X}} \leq \alpha.$$

Given the smallness condition (2.9), this implies $\|u(t) - \tilde{u}(t)\|_{\mathbb{X}} = 0$, proving (3.12). □

Since the weak limit is uniquely identified as $u(t)$, the entire family $\tilde{u}_\gamma(t)$ converges weakly to the same limit.

Lemma 3.6. *For each* $t \in \mathbb{R}_+$,

$$\tilde{u}_\gamma(t) \rightharpoonup u(t) \text{ weakly in } \mathbb{X}, \text{ as } \gamma \to 0. \tag{3.24}$$

Lemma 3.7. (Strong convergence of the auxiliary solution) *For each* $t \in \mathbb{R}_+$,

$$\|\tilde{u}_\gamma(t) - u(t)\|_{\mathbb{X}} \to 0, \text{ as } \gamma \to 0. \tag{3.25}$$

Proof. From the proof of Lemma 3.5, we have

$$\limsup_{\gamma\to 0} \langle \mathcal{A}(\tilde{u}_\gamma(t), \tilde{u}_\gamma(t)), \tilde{u}_\gamma(t) - u(t)\rangle \le 0.$$

Taking $v = u(t)$ in (3.16) and using the fact that $u(t)$ is the solution, the right-hand side becomes non-positive in the limit, yielding:

$$\limsup_{\gamma\to 0} \langle \mathcal{A}(\tilde{u}_\gamma(t), \tilde{u}_\gamma(t)), \tilde{u}_\gamma(t) - u(t)\rangle \le 0.$$

Furthermore, by the weak convergence (3.24) and the boundedness of $\mathcal{A}$, we have

$$\langle \mathcal{A}(u(t), u(t)), \tilde{u}_\gamma(t) - u(t)\rangle \to 0.$$

Combining these two results with the inverse strong monotonicity of $\mathcal{A}$ (2.2), we find:

$$\begin{aligned} \alpha_{\mathcal{A}}(\beta_{\mathcal{A}} + \rho_{\mathcal{A}})^2 \|\tilde{u}_\gamma(t) - u(t)\|_{\mathbb{X}}^2 &\le \langle \mathcal{A}(\tilde{u}_\gamma(t), \tilde{u}_\gamma(t)) - \mathcal{A}(u(t), u(t)), \tilde{u}_\gamma(t) - u(t)\rangle \\ &\le \langle \mathcal{A}(\tilde{u}_\gamma(t), \tilde{u}_\gamma(t)), \tilde{u}_\gamma(t) - u(t)\rangle \\ &\quad - \langle \mathcal{A}(u(t), u(t)), \tilde{u}_\gamma(t) - u(t)\rangle \to 0. \end{aligned} \tag{3.26}$$

This proves the strong convergence (3.25). □

Step 3: Convergence of the penalized solution. We now complete the proof of the main theorem by showing that the solution u_γ of the original penalized problem converges to u.

Lemma 3.8. *For each* $t \in \mathbb{R}_+$,

$$\|u_\gamma(t) - u(t)\|_{\mathbb{X}} \to 0 \quad \text{as} \quad \gamma \to 0. \tag{3.27}$$

Proof. Let $n \in \mathbb{N}$ be such that $t \in [0, n]$. Take $v = u_\gamma(t)$ in the auxiliary problem (3.3) and $v = \tilde{u}_\gamma(t)$ in the penalized problem (2.11). Adding the resulting inequalities, we obtain

$$\begin{aligned}&\langle \mathcal{A}(u_\gamma(t), u_\gamma(t)) - \mathcal{A}(\tilde{u}_\gamma(t), \tilde{u}_\gamma(t)), \tilde{u}_\gamma(t) - u_\gamma(t)\rangle \\ &\quad + \frac{1}{\gamma}\langle \mathcal{P}(u_\gamma(t), u_\gamma(t)) - \mathcal{P}(\tilde{u}_\gamma(t), \tilde{u}_\gamma(t)), \tilde{u}_\gamma(t) - u_\gamma(t)\rangle \\ &\quad + \varphi((\mathcal{B}u_\gamma)(t), u_\gamma(t), \tilde{u}_\gamma(t)) - \varphi((\mathcal{B}u_\gamma)(t), u_\gamma(t), u_\gamma(t)) \\ &\quad + \varphi((\mathcal{B}u)(t), u(t), u_\gamma(t)) - \varphi((\mathcal{B}u)(t), u(t), \tilde{u}_\gamma(t)) \\ &\quad + \jmath^0(u_\gamma(t), \tilde{u}_\gamma(t) - u_\gamma(t)) + \jmath^0(\tilde{u}_\gamma(t), u_\gamma(t) - \tilde{u}_\gamma(t)) \\ &\geq 0.\end{aligned}$$

Using the monotonicity of $\mathcal{P}$, the inverse strong monotonicity of $\mathcal{A}$ (2.2), the property of φ (2.7)(b), and the condition on $\jmath$ (2.8)(c), this inequality implies:

$$\begin{aligned}\alpha_{\mathcal{A}}(\beta_{\mathcal{A}} + \rho_{\mathcal{A}})^2\|\tilde{u}_\gamma(t) - u_\gamma(t)\|_{\mathbb{X}}^2 \leq{}& \alpha_\varphi\|u_\gamma(t) - u(t)\|_{\mathbb{X}}\|\tilde{u}_\gamma(t) - u_\gamma\|_{\mathbb{X}} \\ &+ \beta_\varphi\|(\mathcal{B}u_\gamma)(t) - (\mathcal{B}u)(t)\|_{\mathbb{Y}}\|\tilde{u}_\gamma(t) - u_\gamma(t)\|_{\mathbb{X}} \\ &+ \alpha_\jmath\|\tilde{u}_\gamma(t) - u_\gamma(t)\|_{\mathbb{X}}^2.\end{aligned}$$

Dividing by $\|\tilde{u}_\gamma(t) - u_\gamma(t)|_{\mathbb{X}}$ and using (2.9), we find:

$$\|\tilde{u}_\gamma(t) - u_\gamma(t)\|_{\mathbb{X}} \leq C_1\|u_\gamma(t) - u(t)\|_{\mathbb{X}} + C_2\|(\mathcal{B}u_\gamma)(t) - (\mathcal{B}u)(t)\|_{\mathbb{Y}}, \quad (3.28)$$

where $1 > C_1 = \dfrac{\alpha_\varphi}{\alpha_{\mathcal{A}}(\beta_{\mathcal{A}} + \rho_{\mathcal{A}})^2 - \alpha_\jmath} > 0$ and $1 > C_2 = \dfrac{\beta_\varphi}{\alpha_{\mathcal{A}}(\beta_{\mathcal{A}} + \rho_{\mathcal{A}})^2 - \alpha_\jmath} > 0$ are constants independent of γ and t.

Now, using the triangle inequality and (3.28):

$$\begin{aligned}\|u_\gamma(t) - u(t)\|_{\mathbb{X}} &\leq \|u_\gamma(t) - \tilde{u}_\gamma(t)\|_{\mathbb{X}} + \|\tilde{u}_\gamma(t) - u(t)\|_{\mathbb{X}}, \\ &\leq C_1\|u_\gamma(t) - u(t)\|_{\mathbb{X}} + C_2\|(\mathcal{B}u_\gamma)(t) - (\mathcal{B}u)(t)\|_{\mathbb{Y}} \\ &\quad + \|\tilde{u}_\gamma(t) - u(t)\|_{\mathbb{X}}.\end{aligned}$$

Rearranging and using the history-dependent condition (2.6) for $\mathcal{B}$, we get:

$$(1 - C_1)\|u_\gamma(t) - u(t)\|_{\mathbb{X}} \leq \|\tilde{u}_\gamma(t) - u(t)\|_{\mathbb{X}} + C_2\ell_n\int_0^t \|u_\gamma(s) - u(s)\|_{\mathbb{X}}\, ds.$$

By (2.9), we have $1 - C_1 > 0$. Denoting $\varrho = \dfrac{1}{1 - C_1}$ and $\tilde{\ell}_n = \dfrac{C_2\ell_n}{1 - C_1}$, we obtain:

$$\|u_\gamma(t) - u(t)\|_{\mathbb{X}} \leq \varrho\|\tilde{u}_\gamma(t) - u(t)\|_{\mathbb{X}} + \tilde{\ell}_n\int_0^t \|u_\gamma(s) - u(s)\|_{\mathbb{X}}\, ds. \quad (3.29)$$

Applying Gronwall's inequality to (3.29) yields:

$$\|u_\gamma(t)-u(t)\|_{\mathbb{X}} \le \varrho\|\tilde{u}_\gamma(t)-u(t)\|_{\mathbb{X}}+\varrho\tilde{\ell}_n\int_0^t e^{\tilde{\ell}_n(t-s)}\|\tilde{u}_\gamma(s)-u(s)\|_{\mathbb{X}}\,ds. \quad (3.30)$$

Since $t\in[0,n]$, we have $e^{\tilde{\ell}_n(t-s)} \le e^{n\tilde{\ell}_n}$, so

$$\|u_\gamma(t)-u(t)\|_{\mathbb{X}} \le \varrho\|\tilde{u}_\gamma(t)-u(t)\|_{\mathbb{X}}+\tilde{\ell}_n\varrho e^{n\tilde{\ell}_n}\int_0^t \|\tilde{u}_\gamma(s)-u(s)\|_{\mathbb{X}}\,ds. \quad (3.31)$$

From Lemma 3.7, we have

$$\|\tilde{u}_\gamma(s)-u(s)\|_{\mathbb{X}} \to 0 \quad \text{for each } s.$$

Furthermore, from the boundedness estimate (3.11), the integrand is uniformly bounded. Therefore, by the Lebesgue dominated convergence theorem, the integral in (3.31) converges to zero as $\gamma \to 0$. Since the first term on the right-hand side also converges to zero by Lemma 3.7, we conclude that (3.27) holds. □

4. Conclusion

To conclude, we have successfully developed and analyzed a penalty approach for solving history-dependent mixed variationalhemivariational inequalities. The main contributions of this study are twofold: first, we proved the existence and uniqueness of a solution for the penalized problem for any positive value of the penalty parameter. Second, and most importantly, we established a convergence result showing that the sequence of solutions to the penalized problems converges to the solution of the original constrained problem as the penalty parameter tends to zero. This convergence result justifies the use of penalty methods as a viable and theoretically sound approximation technique for this challenging class of problems.

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