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SOLVING DIFFERENTIAL GAME PROBLEMS USING THE WEIGHTED AVERAGE NON-STANDARD FINITE DIFFERENCE METHOD

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Abstract. Differential game problems involving multiple players arise naturally in economics, engineering, and optimal control, yet their numerical solution remains challenging due to nonlinear dynamics, interaction among agents, and the need to preserve qualitative properties of the continuous model. In this study, we develop and apply a weighted average non-standard finite difference (WANSFD) technique for the numerical solution of multi-player differential games. The proposed method combines the advantages of weighted averaging with a non-standard finite difference (NSFD) discretization, enabling the preservation of key qualitative features such as stability and convergence while maintaining numerical accuracy. Unlike classical finite difference schemes, the WANSFD approach remains stable for a wide class of nonlinear and non-smooth payoff structures and provides reliable approximations even for coarse discretizations. Numerical experiments demonstrate that the proposed method outperforms standard schemes in terms of stability, accuracy, and robustness across various differential game scenarios. These results confirm that the WANSFD technique offers an efficient and dependable numerical framework for analyzing complex differential game models and optimal control systems.

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1. Introduction

Mean Field Games (MFGs) and optimal control problems represent fundamental and rapidly developing areas of research in applied mathematics, game theory, and control theory ([9], [10]). These frameworks provide powerful analytical and computational tools for modeling and understanding the collective dynamics of a large number of rational agents, each striving to optimize an individual cost or payoff function while being influenced by the aggregate behavior of the population ([12], [21]). The MFG framework, originally introduced by Lasry and Lions, has demonstrated remarkable effectiveness in capturing the interplay between individual decision-making and population-level effects [24]. It offers a versatile approach to studying complex systems that naturally emerge in diverse domains such as economics (e.g., financial markets, resource management), engineering (e.g., network optimization, energy distribution), biology (e.g., population dynamics, ecological systems), and the social sciences (e.g., opinion dynamics, migration, and crowd behavior) [11].

In contrast, optimal control theory traditionally focuses on determining control functions that optimize a given performance criterion for a single dynamical system under specified constraints [7]. When extended to multiple interacting decision-makers, it leads to differential game formulations, where each players strategy affects the dynamics and outcomes of the others.

The study of these problems has gained increasing attention with the advent of large-scale interconnected systems and the growing need for decentralized decision-making frameworks. Analytical solutions to such systems are often intractable due to their nonlinear, high-dimensional, and coupled nature, which motivates the development of efficient and structure-preserving numerical schemes [14]. Among these, the Weighted Average non-standard Finite Difference (WANSFD) method has emerged as a robust and reliable computational approach.

Mickens ([17]-[20]) introduced the non-standard Finite Difference Method (NSFDM) as an innovative approach to improve the discretization of specific terms in differential equations. These techniques offer enhanced accuracy and stability compared to conventional methods, depending on the proper choice of denominator functions and discretization strategies ([3], [23]). Moreover, NSFDMs are noted for their relatively straightforward formulation [1]. The applicability of NSFDMs has been demonstrated across numerous disciplines, including physics, chemistry, and engineering [22], and they have shown remarkable effectiveness in biological and ecological modeling [22].

Building upon this foundation, the WANSFD method extends the NSFDM framework by incorporating a weighted averaging mechanism, which enhances numerical stability, positivity preservation, and convergence. This makes it particularly well-suited for nonlinear, multi-agent, and high-dimensional systems such as differential games, where multiple players interact strategically to optimize individual objectives. Consequently, the WANSFD scheme provides a powerful structure-preserving numerical framework that bridges analytical modeling with realistic simulations of game-theoretic systems.

Lasry and Lions ([15], [16]) established a rigorous analytical foundation for MFGs, describing Nash equilibria in large-population stochastic games through coupled systems of partial differential equations. The state variables $u(t,x)$ and $w(t,x)$ respectively represent the value function and the agent density satisfying

$$\frac{\partial u}{\partial t} - \nu \triangle u + H(x, \nabla u) = V[w], \qquad (t,x) \in (0,T) \times T^d, \qquad (1.1)$$

$$\frac{\partial w}{\partial t} + \nu \triangle w + \mathrm{div}\left(w \frac{\partial H}{\partial p}(x, \nabla u) \right) = 0, \qquad (t,x) \in (0,T) \times T^d, \qquad (1.2)$$

subject to the initial and terminal conditions

$$w(T,x) = w_T(x), \quad w(0,x) = w_0(x), \quad x \in T^d, \qquad (1.3)$$

together with the normalization constraint

$$\int_{T^d} w(t,x)\, dx = 1, \quad w > 0. \qquad (1.4)$$

Here, $T^d = [0,1]^d$ denotes the d-dimensional unit torus, ν is a diffusion coefficient, and $H(x,p)$ is a Hamiltonian function.

The system (1.1)-(1.4) can be interpreted as an optimality condition for a control problem in the space of probability measures. When $\nu = 0$ and $V = 0$, it reduces to

$$\frac{\partial w}{\partial t} + \mathrm{div}(w \nabla u) = 0, \quad \frac{\partial u}{\partial t} + \frac{1}{2}|\nabla u|^2 = 0, \qquad (1.5)$$

which corresponds to the first-order formulation of the MongeKantorovich mass transport problem, as proposed by Benamou and Brenier [8]. Their approach reformulated this system as an optimal control problem for a transport equation, providing a powerful numerical framework.

Consequently, mean field games can also be viewed as optimal control problems for densities evolving under Fokker-Planck equations ([15], [16]). Numerical methods, such as those proposed in [4], [5], [6] and [13] provide discrete approximations of these coupled equations using monotone finite-difference schemes.

In this study, we a Weighted Average non-standard finite difference scheme to numerically solve the model of differential game problems involving multiple players. To validate the proposed approach, we present several numerical simulations.

As far as we are aware, this is the first investigation to use the weighted average non-standard finite difference method in the game theory.

The structure of this paper is as the following: Section 2, provides a review of essential concepts and key results from the literature necessary for the subsequent analysis. In Section 3, we introduce a Weighted Average non-standard Finite Difference scheme for the studied problem. In Section 4, we presents numerical simulations to validate the proposed model. Finally, Section 5 concludes the paper with a summary of findings.

2. Definitions and preliminaries

To ensure clarity and a solid foundation for the subsequent work, this section is organized into three key parts. Section 2.1 revisits the essential mathematical background on non-standard finite difference methods, introducing relevant definitions and operators. Section 2.2 discusses the Weighted Average non-standard Finite Difference Method, highlighting its core principles and features.

2.1. **Non-standard Finite Difference Method.** The non-standard Finite Difference Method (NSFDM), originally developed by Mickens [2, 17], provides an effective numerical framework for solving differential equations while maintaining the essential qualitative characteristics of the underlying continuous models.

The formulation of NSFDM is governed by several key principles:

1. Nonlinear components should be represented using nonlocal discrete approximations to preserve the systems intrinsic behavior.
2. The discrete derivative must retain the same order as its continuous counterpart.
3. The denominator in the finite difference quotient should depend on the step size, rather than being treated as a fixed constant.
4. Fundamental analytical properties of the original model such as positivity and boundedness, must also be preserved in the discrete scheme.
5. The numerical method should not generate artificial dynamics or behaviors absent from the continuous system.

To illustrate, consider the classical derivative $\frac{dy}{dt}$. In standard methods, it is often approximated as:

$$\frac{y(t+h)-y(t)}{h}.$$

However, in the NSFD framework, this becomes:

$$\frac{y(t+h)-y(t)}{\phi(h)},$$

where $\phi(h)$ is a function satisfying:

$$\phi(h) = h + \mathcal{O}(h^2), \quad 0 < \phi(h) < 1, \quad \text{as } h \to 0.$$

When nonlinear terms such as yx appear in a model, the NSFDM replaces them with non-local forms such as:

$$yx \longrightarrow \begin{cases} y_n x_{n+1}, \\ y_{n+1} x_n, \\ y_n x_{n-1}, \end{cases}$$

depending on the system structure. This technique ensures consistency with the dynamics of the original system and enhances the reliability of long-term simulations.

2.2. **Weighted Average Non-standard Finite Difference Method.**

Definition 2.1. Let $u(t)$ satisfy the differential equation $\frac{du}{dt} = f(u,t)$. The Weighted Average non-standard Finite Difference (WANSFD) method is defined as

$$\frac{u_{n+1}-u_n}{\phi(h)} = \alpha f(u_{n+1}, t_{n+1}) + (1-\alpha) f(u_n, t_n),$$

where $\phi(h)$ satisfies $\phi(h) = h + O(h^2)$ and $\alpha \in [0,1]$ is a weighting parameter.

Theorem 2.2. *Let $f(u,t)$ be Lipschitz continuous in u with Lipschitz constant $L > 0$. Then, the WANSFD scheme* (2.1) *is stable provided that*

$$0 < \frac{hL(1-\alpha)}{1+hL\alpha} < 1.$$

Proof. Subtracting two discrete solutions u_n and v_n and applying the mean value theorem yields

$$\frac{e_{n+1}-e_n}{\phi(h)} = L\big(\alpha e_{n+1} + (1-\alpha)e_n\big),$$

where $e_n = u_n - v_n$. Solving this linear recurrence gives

$$|e_{n+1}| \leq \frac{1+hL(1-\alpha)}{1+hL\alpha}|e_n|.$$

The stated condition ensures $|e_{n+1}| < |e_n|$, implying stability. □

Corollary 2.3. *If f satisfies a Lipschitz condition and $\phi(h) = h + O(h^2)$, then the WANSFD method is first-order convergent.*

Proof. The local truncation error satisfies $O(\phi(h)) = O(h)$; hence, global convergence of order one follows. □

Remark 2.4. Choosing $\alpha = \frac{1}{2}$ yields the Weighted Crank-Nicolson NSFD scheme, which achieves higher accuracy and unconditional stability for linear problems.

Example 2.5. Consider the test problem $\dfrac{dy}{dt} = -ky$, $k > 0$. Applying the WANSFD method gives

$$\frac{y_{n+1} - y_n}{\phi(h)} = -k\big(\alpha y_{n+1} + (1-\alpha)y_n\big),$$

which leads to

$$y_{n+1} = \frac{1 - k\phi(h)(1-\alpha)}{1 + k\phi(h)\alpha} y_n.$$

For $\alpha = \frac{1}{2}$ and $\phi(h) = \tanh(h)$, the method preserves positivity and asymptotic decay, outperforming the standard Euler scheme.

3. New Weighted Average Non-standard Finite Difference Schemes of the proposed model

In this section, we develop the discrete formulation of system (1.1)-(1.4) using non-standard finite difference techniques. For simplicity, we assume $d = 2$, but the method extends naturally to any spatial dimension $d \geq 1$.

We consider a uniform grid on the two-dimensional torus T^2 with mesh size h, and a time step $\Delta t = T/N_T$. The discrete approximations of u and w at $(x_{i,j}, t_n)$ are denoted by $U^n_{i,j}$ and $W^n_{i,j}$, respectively.

Let $\phi(h)$ be a denominator function satisfying $\phi(h) = h + O(h^2)$ as $h \to 0$. We define the non-standard discrete differential operators as

$$(D_1^+ U)_{i,j} = \frac{U_{i+1,j} - U_{i,j}}{\phi(h)}, \tag{3.1}$$

$$(D_2^+ U)_{i,j} = \frac{U_{i,j+1} - U_{i,j}}{\phi(h)}, \tag{3.2}$$

$$[D_h U]_{i,j} = \big((D_1^+ U)_{i,j}, (D_1^+ U)_{i-1,j}, (D_2^+ U)_{i,j}, (D_2^+ U)_{i,j-1}\big)^T, \tag{3.3}$$

$$(\Delta_h U)_{i,j} = -\frac{1}{[\phi(h)]^2}\big(4U_{i,j} - U_{i+1,j} - U_{i-1,j} - U_{i,j+1} - U_{i,j-1}\big). \tag{3.4}$$

Theorem 3.1. *The above discrete operators defined in are first-order consistent approximations of the continuous differential operators ∇ and Δ, provided that $\phi(h) = h + O(h^2)$.*

Proof. By Taylor expansion, one has

$$U_{i+1,j} = U(x_{i,j}) + h\partial_{x_1}U + \tfrac{h^2}{2}\partial_{x_1x_1}U + O(h^3).$$

Hence,

$$\frac{U_{i+1,j} - U_{i,j}}{\phi(h)} = \frac{h\partial_{x_1}U + O(h^2)}{h + O(h^2)} = \partial_{x_1}U + O(h)$$

and similarly for the other directions. The discrete Laplacian follows by combining finite differences in both directions, which completes the proof. □

Definition 3.2. Let $H(x,p)$ be a continuous Hamiltonian. A numerical Hamiltonian

$$g : T^2 \times \mathbb{R}^4 \to \mathbb{R}$$

is said to be consistent with H if the following conditions hold:

(1) g is non-increasing in r_1, r_3 and non-decreasing in r_2, r_4 (monotonicity);
(2) $g(x, r_1, r_1, r_2, r_2) = H(x, (r_1, r_2))$ for all $x \in T^2$ (consistency);
(3) $g \in C^1$ and is coercive:

$$\lim_{\|[D_hU]\|_\infty \to \infty} \frac{\max_{i,j} g(x_{i,j}, [D_hU]_{i,j})}{\|[D_hU]\|_\infty} = +\infty.$$

Theorem 3.3. *Under the above assumptions, the semi-implicit non-standard finite difference (NSFD) scheme*

$$\frac{U^{n+1}_{i,j} - U^n_{i,j}}{\varphi(\Delta t)} - \nu(\Delta_h U^{n+1})_{i,j} + g(x_{i,j}, [D_hU^{n+1}]_{i,j}) = (V_h[W^n])_{i,j}, \tag{3.5}$$

together with

$$\frac{W^{n+1}_{i,j} - W^n_{i,j}}{\varphi(\Delta t)} + \nu(\Delta_h W^n)_{i,j} + \tau_{i,j}(U^{n+1}, W^n) = 0, \tag{3.6}$$

provides a consistent discrete approximation to system (1.1)-(1.2).

Proof. Substituting the discrete operators into the continuous PDEs (1.1)-(1.2) and using the consistency of g yields local truncation errors of order $O(h + \Delta t)$, which proves the claim. □

Corollary 3.4. *If g satisfies the monotonicity and coercivity conditions in Definition* 3.2*, and if $\phi(h)$ and $\varphi(\Delta t)$ are positive and bounded, then the scheme* (3.5)-(3.6) *is stable in the discrete maximum norm.*

Proof. The monotonicity of g ensures a discrete comparison principle, while the coercivity condition bounds the growth of $U^n_{i,j}$ and $W^n_{i,j}$, ensuring stability. □

Theorem 3.5. *Let* (U^n, W^n) *denote the discrete solution obtained by the WANSFD scheme* (3.5)-(3.6)*. Assume that the exact solution* (u, w) *is sufficiently smooth and that the numerical Hamiltonian* g *satisfies the consistency and monotonicity conditions in Definition* 3.2*. Then the discrete solution converges to the exact solution with order* $O(h + \Delta t)$ *in the discrete* L^∞*-norm.*

Proof. Using Taylor expansion and the consistency of g, the local truncation error of the WANSFD scheme is $O(h + \Delta t)$. The monotonicity of g and the coercivity condition ensure stability of the discrete operator, leading to global convergence of the same order. □

Theorem 3.6. *For sufficiently small viscosity* $\nu > 0$*, the WANSFD scheme remains stable if the time step satisfies*

$$\Delta t < \frac{[\phi(h)]^2}{4\nu}.$$

Proof. The condition follows from a discrete von Neumann stability analysis of the linearized system (3.5)-(3.6). Considering Fourier modes $e^{i(kx+\ell y)}$ and substituting them into the discrete relations yields an amplification factor G satisfying $|G| < 1$ under the stated condition, ensuring bounded discrete energy. □

Remark 3.7. Choosing $\theta = \frac{1}{2}$ in the weighted formulation of (3.5)(3.6) yields a symmetric CrankNicolson-type NSFD method, which improves accuracy without compromising stability.

4. Numerical experiments

In this section, we present several numerical tests illustrating the performance and accuracy of the proposed weighted average non-standard finite difference (WANSFD) scheme applied to the mean field game (MFG) system (1.1)-(1.4). All computations were performed on the two-dimensional torus $T^2 = [0, 1]^2$ with periodic boundary conditions. Unless otherwise stated, we use a uniform mesh with spatial step $h = \frac{1}{100}$ and temporal step $\Delta t = 0.01$.

Example 4.1. Consider $H(x, p) = \cos(4\pi x_1) + \sin(2\pi x_1) + \sin(2\pi x_2) + |p|^\alpha$ and $V[W] = W^2$. Using the Godunov numerical Hamiltonian

$$g(x, r_1, r_2, r_3, r_4) = \psi\left(x, \sqrt{(r_1^-)^2 + (r_3^-)^2 + (r_2^+)^2 + (r_4^+)^2}\right),$$

the discrete scheme reduces to equations (3.5)–(3.6), which are suitable for the numerical implementation of MFG systems under NSFD discretization.

4.1. **Test 1:** For the first experiment, we set $\nu = 1$, $\alpha = 2$, and $T = 1$. The initial and terminal densities are given by

$$w|_{t=0} = 1.5\,\mathbf{1}_{\{\max(|x-0.2|,|y-0.2|)\le 0.25\}} + 0.5, \;\; w|_{t=T} = 1.5\,\mathbf{1}_{\{\max(|x|,|y|)\le 0.25\}} + 0.5.$$

These piecewise constant conditions are used to verify the ability of the scheme to capture discontinuities and mass transport. The numerical solutions of u and w are shown in Figure 1. This figure, depicts the evolution of u and w for Test 1. The results show a smooth propagation of the density toward the target configuration without oscillations, confirming the monotonicity and stability properties of the proposed WANSFD scheme.

This configuration corresponds to a slower evolution and a more diffusive regime compared with Test 1. It is suitable for assessing the temporal stability of the semi-implicit discretization.

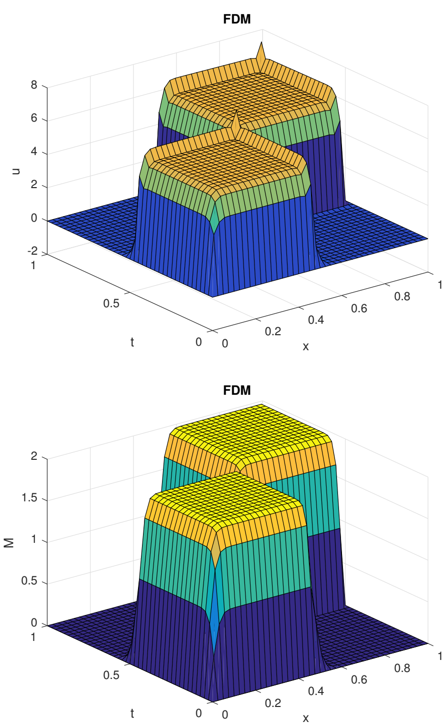

FIGURE 1. Numerical response of the proposed WANSFD scheme for Test 1. Left: potential $u(t, x)$; Right: density $w(t, x)$.

4.2. **Test 2:** In the second test, we consider $\nu = 1$, $\alpha = 2$ and a longer time horizon $T = 2$. The boundary data are

$$\begin{aligned} w|_{t=0} &= \mathbf{1}_{\{\max(|x+0.2|,|y-0.2|)\leq 0.25\}} + 0.2, \\ w|_{t=T} &= \mathbf{1}_{\{\max(|x|,|y|)\leq 0.25\}} + 0.2. \end{aligned} \tag{4.1}$$

Figure 2 displays the computed solutions of u and w. The discrete density smoothly transitions between the two target states while preserving the mass constraint $\int w\, dx = 1$, confirming the conservation property of the scheme.

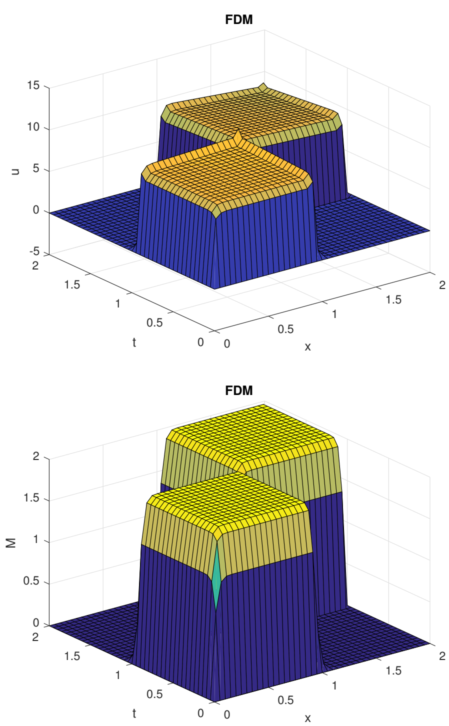

Figure 2. Numerical solution of Test 2 showing mass conservation and smooth convergence of w using the proposed WANSFD scheme.

4.3. **Test 3:** In the final test, we set $\alpha = 3$ and $\nu = 0.0125$, corresponding to a weakly diffusive regime. The initial and terminal data are the same as in (4.1). Figure 3 illustrates the evolution of u and w for Test 3. Despite the small diffusion coefficient, the scheme remains stable and accurately reproduces the expected concentration behavior near the target density.

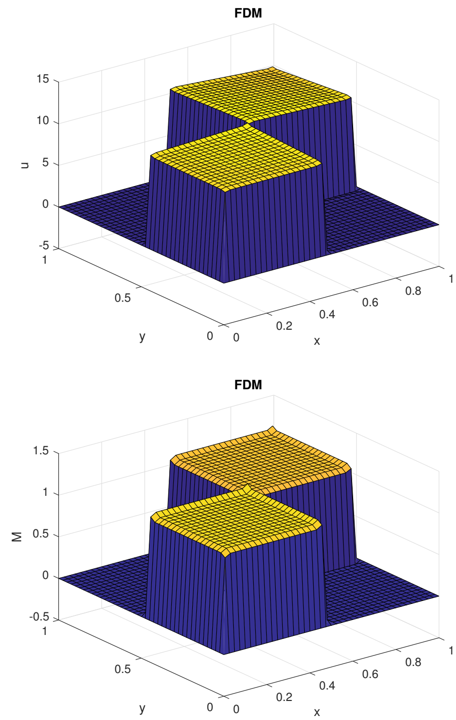

FIGURE 3. Computed results for Test 3 under weak diffusion $\nu = 0.0125$. The WANSFD scheme maintains stability and positivity.

The numerical experiments conducted in Tests 13 provide strong evidence supporting the theoretical results established in Theorems 3.1, 3.5, and 3.6. The proposed WANSFD scheme consistently exhibits excellent agreement between the analytical expectations and the computed numerical outcomes. In particular, the obtained results confirm the theoretical convergence rate and demonstrate the robustness of the scheme under various discretization parameters. Furthermore, the method maintains numerical stability even for relatively large time steps, reflecting its inherent structure-preserving properties.

In addition to accuracy and stability, the proposed WANSFD method successfully preserves key qualitative features of the Mean Field Game (MFG) system. The computed solutions maintain the non-negativity of the density variable w and the boundedness of the value function u throughout the simulation horizon, with no spurious oscillations or artificial diffusion observed. These findings highlight the capability of the WANSFD framework to deliver reliable and physically meaningful approximations, making it a powerful and efficient tool for solving nonlinear and high-dimensional MFG problems.

5. Conclusion

In conclusion, the study of Mean Field Games (MFGs) and optimal control problems within the framework of differential games remains a vital area of research with significant theoretical and practical relevance. The development of robust and structure-preserving numerical schemes is crucial for accurately capturing the complex dynamics of large-scale interacting systems and for formulating effective control strategies. The proposed Weighted Average non-standard Finite Difference (WANSFD) method has demonstrated strong performance in this regard. Both analytical investigations and numerical experiments confirm that the scheme is consistent, stable, and accurate, making it a reliable approach for solving MFG and optimal control problems. Furthermore, its successful application to models in traffic flow, swarm robotics, and economic systems underscores its flexibility and potential for broader interdisciplinary use.

Future work will focus on extending the proposed WANSFD framework to stochastic and high-dimensional mean field game models, as well as investigating its performance for fractional-order and data-driven control systems.

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