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EIGENVALUE LOCALIZATION INEQUALITIES INVOLVING WEIGHTED EIGENVALUES FOR $N \times N$ COMPLEX MATRICES WITH $N \geq 3$

F. Alrimawi

Department of Allied Sciences, Al-Ahliyya Amman University,
Amman, Jordan
e-mail: f.rimawi@ammanu.edu.jo

Abstract. In this paper, we obtain some eigenvalue localization inequalities involving weighted eigenvalues for $n \times n$ complex matrices with $n \geq 3$.

1. Introduction

Let $\mathbb{M}_n(\mathbb{C})$ be the algebra of all $n \times n$ complex matrices. Let $\lambda_1, \lambda_2, ..., \lambda_n$ denote the eigenvalues of the matrix $A \in \mathbb{M}_n(\mathbb{C})$ repeated according to their algebraic multiplicities. For $0 < p < \infty$, define $\|A\|_p$ by $(tr\,|A|^p)^{\frac{1}{p}}$, where $|A| = (A^*A)^{\frac{1}{2}}$. For $1 \leq p < \infty$, this is the Schatten p-norm of A and it is the Hilbert–Schmidt norm when $p = 2$.

A matrix $A \in \mathbb{M}_n(\mathbb{C})$ can be presented by the Cartesian decomposition as $A = \mathrm{Re}(A) + i\,\mathrm{Im}(A)$, where $\mathrm{Re}(A) = \frac{1}{2}(A + A^*)$ and $\mathrm{Im}(A) = \frac{1}{2i}(A - A^*)$ are the real and imaginary parts of A, respectively. For more properties and recent results about the norms, the Cartesian decomposition, and Schatten p-norms, the reader may refer to [1], [3], [4], [5], [6] and [13].

The Gersgorin Disk Theorem (see [6, p. 244]) asserts that for a matrix $A \in \mathbb{M}_n(\mathbb{C})$ (where $A = [a_{ij}], i, j = 1, ..., n$), the eigenvalues of A are included

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by the union of n small disks $\Gamma_i(A)$, that is

$$\Gamma(A) = \bigcup_{i=1}^{n} \Gamma_i(A) = \bigcup_{i=1}^{n} \left\{ z \in \mathbb{C} : |z - a_{ii}| \leq \sum_{i \neq j} |a_{ij}| \right\}.$$

For some improvements and related results of the Gersgorin Disk Theorem in determining the location of eigenvalues, we may refer the reader to [7], [8], [14], [16], [17], [18], [20], [21], [22], [23], [24] and references therein.

Motivated by the work in [15] and [24], the author in [12] proved the following results for any $A \in \mathbb{M}_n(\mathbb{C})$ with eigenvalue $\lambda_l (l = 1, 2, ..., n)$ with $S_l = \{1, 2, ..., n\} \backslash \{l\}$, $l = 1, 2, ..., n$:

$$\left| \operatorname{Re} \lambda_l - \frac{\operatorname{Re}(trA)}{n} \right|^2 \leq \frac{n-1}{n} \left(\|\operatorname{Re} A\|_2^2 - \frac{|\operatorname{Re}(trA)|^2}{n} - \frac{1}{2} k^2(A) \right) \tag{1.1}$$

and

$$\left| \operatorname{Im} \lambda_l - \frac{\operatorname{Im}(trA)}{n} \right|^2 \leq \frac{n-1}{n} \left(\|\operatorname{Im} A\|_2^2 - \frac{|\operatorname{Im}(trA)|^2}{n} - \frac{1}{2} h^2(A) \right), \tag{1.2}$$

where

$$k^2(A) = \min_{1 \leq l \leq n} \max_{j,k \in S_l} |\operatorname{Re} \lambda_j - \operatorname{Re} \lambda_k|^2$$

and

$$h^2(A) = \min_{1 \leq l \leq n} \max_{j,k \in S_l} |\operatorname{Im} \lambda_j - \operatorname{Im} \lambda_k|^2 .$$

Also, he proved that all eigenvalues are located in the following elliptic region:

$$\frac{\left(x - \frac{\operatorname{Re}(trA)}{n}\right)^2}{2M^2} + \frac{\left(y - \frac{\operatorname{Im}(trA)}{n}\right)^2}{2N^2} = c \quad \text{for some } c \in (0, 1], \tag{1.3}$$

where

$$M = \sqrt{\frac{n-1}{n}} \left\| \operatorname{Re} A - \frac{\operatorname{Re}(trA)}{n} I \right\|_2$$

and

$$N = \sqrt{\frac{n-1}{n}} \left\| \operatorname{Im} A - \frac{\operatorname{Im}(trA)}{n} I \right\|_2 .$$

Many authors used to study the estimation of matrix eigenvalues according to the importance of such studies in physics (especially in quantum mechanics, see, e.g., [10], [11] and [19]). In their work, mathematicians consider the case when the order of the matrix A is $n \geq 3$ since the range of the eigenvalues of a 2×2 matrix can be considered by direct calculation.

In Section 2, we give refinements of Theorem 2.2 and Theorem 2.3 given in [24]. Also, we give generalizations of the inequalities (1.1), (1.2) and (1.3) given in [12].

2. Inequalities involving general 2×2 block matrices

In this section, we need the following lemmas. For the first lemma we refer to [2], while we can find the second and third lemmas in [6] and [9], respectively. Throughout this paper, we use the symbol S_l for the set $\{1, 2, ..., n\} \backslash \{l\}$, $l = 1, 2, ..., n$.

Lemma 2.1. *Let $z_1, z_2, ..., z_n$ be complex numbers and let $\alpha_1, \alpha_2, ..., \alpha_n$ be positive real numbers with $\sum_{i=1}^{n} \alpha_j = 1$, where $n \geq 3$. Then*

$$\frac{\alpha_l}{1-\alpha_l}\left|z_l - \sum_{j=1}^{n} \alpha_j z_j\right|^2 + \sum_{j,k \in S_l}^{n} \frac{\alpha_j \alpha_k}{2(1-\alpha_l)} \left|z_j - z_k\right|^2 = \sum_{j=1}^{n} \alpha_j \left|z_j\right|^2 - \left|\sum_{j=1}^{n} \alpha_j z_j\right|^2$$

for $l = 1, 2, ..., n$.

Lemma 2.2. *Let $\lambda_1, \lambda_2, ..., \lambda_n$ be the eigenvalues of $A \in \mathbb{M}_n(\mathbb{C})$. Then*

$$\sum_{j=1}^{n} (\operatorname{Re} \lambda_j)^2 \leq \|\operatorname{Re} A\|_2^2$$

and

$$\sum_{j=1}^{n} (\operatorname{Re} \lambda_j)^2 \leq \|\operatorname{Im} A\|_2^2 .$$

Lemma 2.3. *Let $z_1, z_2, ..., z_n$ be complex numbers. Then*

$$\frac{n}{2} \max_{j,k=1,...,n} \left|z_j - z_k\right|^2 \leq \sum_{1 \leq j < k \leq n} \left|z_j - z_k\right|^2 .$$

Our first result in this section can be as follows:

Theorem 2.4. *Let $A \in \mathbb{M}_n(\mathbb{C})$ with $(n \geq 3)$ with eigenvalues $\lambda_l(l = 1, 2, ..., n)$ and let $\alpha_1, \alpha_2, ..., \alpha_n$ be positive real numbers with $\sum\limits_{i=1}^{n} \alpha_j = 1$. Then*

$$\begin{aligned}&\left| \operatorname{Re} \lambda_l - \sum_{j=1}^{n} \alpha_j (\operatorname{Re} \lambda_j) \right|^2 \\ &\qquad \leq \frac{1-\alpha_l}{\alpha_l} \left(\sum_{j=1}^{n} \alpha_j \left| \operatorname{Re} \lambda_j \right|^2 - \left| \sum_{j=1}^{n} \alpha_j \operatorname{Re} \lambda_j \right|^2 - \frac{\gamma(\alpha) k^2(A)}{2\alpha_l} \right) \end{aligned} \tag{2.1}$$

and

$$\begin{aligned}&\left| \operatorname{Im} \lambda_l - \sum_{j=1}^{n} \alpha_j (\operatorname{Im} \lambda_j) \right|^2 \\ &\qquad \leq \frac{1-\alpha_l}{\alpha_l} \left(\sum_{j=1}^{n} \alpha_j \left| \operatorname{Im} \lambda_j \right|^2 - \left| \sum_{j=1}^{n} \alpha_j \operatorname{Im} \lambda_j \right|^2 - \frac{\gamma(\alpha) h^2(A)}{2\alpha_l} \right), \end{aligned} \tag{2.2}$$

where

$$\gamma(\alpha) = \min_{1 \leq l \leq n} \min_{j,k \in S_l} \{\alpha_j \alpha_k\},$$

$$k^2(A) = \min_{1 \leq l \leq n} \max_{j,k \in S_l} \left| \operatorname{Re} \lambda_j - \operatorname{Re} \lambda_k \right|^2$$

and

$$h^2(A) = \min_{1 \leq l \leq n} \max_{j,k \in S_l} \left| \operatorname{Im} \lambda_j - \operatorname{Im} \lambda_k \right|^2.$$

Proof. Let λ_l be an arbitrary eigenvalue of A. By Lemma 2.1 then applying Lemma 2.3, we have

$$\begin{aligned}\left| \operatorname{Re} \lambda_l - \sum_{j=1}^{n} \alpha_j (\operatorname{Re} \lambda_j) \right|^2 &= \frac{1-\alpha_l}{\alpha_l} \left(\sum_{j=1}^{n} \alpha_j \left| \operatorname{Re} \lambda_j \right|^2 - \left| \sum_{j=1}^{n} \alpha_j \operatorname{Re} \lambda_j \right|^2 \right) \\ &\quad - \sum_{j,k \in S_l}^{n} \frac{\alpha_j \alpha_k}{2\alpha_l} \left| \operatorname{Re} \lambda_j - \operatorname{Re} \lambda_k \right|^2\end{aligned}$$

$$\leq \frac{1-\alpha_l}{\alpha_l}\left(\sum_{j=1}^{n}\alpha_j\left|\operatorname{Re}\lambda_j\right|^2-\left|\sum_{j=1}^{n}\alpha_j\operatorname{Re}\lambda_j\right|^2\right)$$
$$-\frac{\gamma(\alpha)\sum\limits_{j,k\in S_l}^{n}\left|\operatorname{Re}\lambda_j-\operatorname{Re}\lambda_k\right|^2}{2\alpha_l}$$
$$\leq \frac{1-\alpha_l}{\alpha_l}\Big(\sum_{j=1}^{n}\alpha_j\left|\operatorname{Re}\lambda_j\right|^2-\left|\sum_{j=1}^{n}\alpha_j\operatorname{Re}\lambda_j\right|^2$$
$$-\frac{\gamma(\alpha)k^2(A)}{2\alpha_l}\Big),$$

this proves the inequality (2.1). Similarly, we can obtain the second inequality (2.2). □

From Theorem 2.4 and using Lemma 2.2, we have the following corollary:

Corollary 2.5. *Let $A\in\mathbb{M}_n(\mathbb{C})$ with $(n\geq 3)$ with eigenvalues $\lambda_l(l=1,2,...,n)$ and let $\alpha_1,\alpha_2,...,\alpha_n$ be positive real numbers with $\sum\limits_{i=1}^{n}\alpha_j=1$. Then*

$$\left|\operatorname{Re}\lambda_l-\sum_{j=1}^{n}\alpha_j\left(\operatorname{Re}\lambda_j\right)\right|^2$$
$$\leq\frac{1-\alpha_l}{\alpha_l}\left(\left(\max_{1\leq j\leq n}\alpha_j\right)\|\operatorname{Re}A\|_2^2-\left(\min_{1\leq j\leq n}\alpha_j\right)^2\left|\operatorname{Re}\left(trA\right)\right|^2-\frac{\gamma(\alpha)k^2(A)}{2\alpha_l}\right) \tag{2.3}$$

and

$$\left|\operatorname{Im}\lambda_l-\sum_{j=1}^{n}\alpha_j\left(\operatorname{Im}\lambda_j\right)\right|^2$$
$$\leq\frac{1-\alpha_l}{\alpha_l}\left(\left(\max_{1\leq j\leq n}\alpha_j\right)\|\operatorname{Im}A\|_2^2-\left(\min_{1\leq j\leq n}\alpha_j\right)^2\left|\operatorname{Im}\left(trA\right)\right|^2-\frac{\gamma(\alpha)h^2(A)}{2\alpha_l}\right), \tag{2.4}$$

where $\gamma(\alpha)$, $k^2(A)$ and $h^2(A)$ are as defined in Theorem 2.4.

Remark 2.6. The following inequalities can be obtained by direct computations from Theorem 2.4 by taking $\alpha_j = \frac{1}{n}$, $1 \leq j \leq n$:

$$\left|\operatorname{Re}\lambda_l - \frac{\operatorname{Re}(trA)}{n}\right|^2 \leq \frac{n-1}{n}\left(\|\operatorname{Re}A\|_2^2 - \frac{|\operatorname{Re}(trA)|^2}{n} - \frac{1}{2}k^2(A)\right)$$

and

$$\left|\operatorname{Im}\lambda_l - \frac{\operatorname{Im}(trA)}{n}\right|^2 \leq \frac{n-1}{n}\left(\|\operatorname{Im}A\|_2^2 - \frac{|\operatorname{Im}(trA)|^2}{n} - \frac{1}{2}h^2(A)\right),$$

where $k^2(A)$ and $h^2(A)$ are as defined in Theorem 2.4. Which means that our result in Theorem 2.4 gives generalizations of the inequalities (1.1) and (1.2) given in [12]. Also, when $k^2(A) = h^2(A) = 0$, we get

$$\left|\operatorname{Re}\lambda_l - \frac{\operatorname{Re}(trA)}{n}\right|^2 \leq \frac{n-1}{n}\left(\|\operatorname{Re}A\|_2^2 - \frac{|\operatorname{Re}(trA)|^2}{n}\right)$$

and

$$\left|\operatorname{Im}\lambda_l - \frac{\operatorname{Im}(trA)}{n}\right|^2 \leq \frac{n-1}{n}\left(\|\operatorname{Im}A\|_2^2 - \frac{|\operatorname{Im}(trA)|^2}{n}\right),$$

which implies that our result in Theorem 2.4 gives refinements of Theorem 2.2 given in [24].

Our second result in this paper can be stated as follows:

Theorem 2.7. *Let $A \in \mathbb{M}_n(\mathbb{C})$ with $(n \geq 3)$ with eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ and let $\alpha_1, \alpha_2, ..., \alpha_n$ be positive real numbers with $\sum_{i=1}^{n} \alpha_j = 1$. Then all eigenvalues are located in elliptic region*

$$\frac{\left(x - \sum_{j=1}^{n} \alpha_j (\operatorname{Re}\lambda_j)\right)^2}{2M^2} + \frac{\left(y - \sum_{j=1}^{n} \alpha_j (\operatorname{Re}\lambda_j)\right)^2}{2N^2} = c$$

for some $c \in (0, 1]$, where

$$M = \sqrt{\frac{1-\alpha_l}{\alpha_l}\left(\max_{1\leq j\leq n} \alpha_j\right)}\left\|\operatorname{Re}A - \left(\sum_{j=1}^{n} \alpha_j (\operatorname{Re}\lambda_j)\right) I\right\|_2$$

and

$$N = \sqrt{\frac{1-\alpha_l}{\alpha_l}\left(\max_{1\leq j\leq n} \alpha_j\right)}\left\|\operatorname{Im}A - \left(\sum_{j=1}^{n} \alpha_j (\operatorname{Re}\lambda_j)\right) I\right\|_2.$$

Proof. We will discuss the case when $trA = 0$. It follows from the inequalities (2.3) and (2.4) that

$$|\mathrm{Re}\,\lambda_l|^2 + \frac{1-\alpha_l}{2\alpha_l^2}\gamma(\alpha)k^2(A) \le \frac{1-\alpha_l}{\alpha_l}\left(\max_{1\le j\le n}\alpha_j\right)\|\mathrm{Re}\,A\|_2^2 \tag{2.5}$$

and

$$|\mathrm{Im}\,\lambda_l|^2 + \frac{1-\alpha_l}{2\alpha_l^2}\gamma(\alpha)h^2(A) \le \frac{1-\alpha_l}{\alpha_l}\left(\max_{1\le j\le n}\alpha_j\right)\|\mathrm{Im}\,A\|_2^2 . \tag{2.6}$$

It follows from the inequalities (2.5) and (2.6) that

$$\frac{|\mathrm{Re}\,\lambda_l|^2 + \frac{1-\alpha_l}{2\alpha_l^2}\gamma(\alpha)k^2(A)}{2\frac{1-\alpha_l}{\alpha_l}\left(\max\limits_{1\le j\le n}\alpha_j\right)\|\mathrm{Re}\,A\|_2^2} + \frac{|\mathrm{Im}\,\lambda_l|^2 + \frac{1-\alpha_l}{2\alpha_l^2}\gamma(\alpha)h^2(A)}{2\frac{1-\alpha_l}{\alpha_l}\left(\max\limits_{1\le j\le n}\alpha_j\right)\|\mathrm{Im}\,A\|_2^2} \le 1,$$

which means that the real parts and imaginary parts of the eigenvalues $\lambda_l(l = 1, 2, ..., n)$ are satisfying

$$\frac{x^2 + \frac{1-\alpha_l}{2\alpha_l^2}\gamma(\alpha)k^2(A)}{2\frac{1-\alpha_l}{\alpha_l}\left(\max\limits_{1\le j\le n}\alpha_j\right)\|\mathrm{Re}\,A\|_2^2} + \frac{y^2 + \frac{1-\alpha_l}{2\alpha_l^2}\gamma(\alpha)h^2(A)}{2\frac{1-\alpha_l}{\alpha_l}\left(\max\limits_{1\le j\le n}\alpha_j\right)\|\mathrm{Im}\,A\|_2^2} \le 1,$$

which can be written as

$$\frac{x^2}{2\frac{1-\alpha_l}{\alpha_l}\left(\max\limits_{1\le j\le n}\alpha_j\right)\|\mathrm{Re}\,A\|_2^2} + \frac{y^2}{2\frac{1-\alpha_l}{\alpha_l}\left(\max_{1\le j\le n}\alpha_j\right)\|\mathrm{Im}\,A\|_2^2}$$
$$\le 1 - \frac{\gamma(\alpha)k^2(A)}{4\alpha_l\left(\max\limits_{1\le j\le n}\alpha_j\right)\|\mathrm{Re}\,A\|_2^2} - \frac{\gamma(\alpha)h^2(A)}{4\alpha_l\left(\max\limits_{1\le j\le n}\alpha_j\right)\|\mathrm{Im}\,A\|_2^2}.$$

For the general case, we set $A - \left[\left(\sum\limits_{j=1}^{n}\alpha_j\left(\mathrm{Re}\,\lambda_j\right)\right)I\right]$ and repeat the above progress. So, we get

$$\frac{\left(x - \sum\limits_{j=1}^{n}\alpha_j\left(\mathrm{Re}\,\lambda_j\right)\right)^2}{2M^2} + \frac{\left(y - \sum\limits_{j=1}^{n}\alpha_j\left(\mathrm{Re}\,\lambda_j\right)\right)^2}{2N^2}$$
$$\le 1 - \left(\frac{1-\alpha_l}{4\alpha_l^2}\right)\frac{\gamma(\alpha)k^2(A)}{M^2} - \left(\frac{1-\alpha_l}{4\alpha_l^2}\right)\frac{\gamma(\alpha)h^2(A)}{N^2}.$$

□

Remark 2.8. The following inequalities can be obtained by direct computations from Theorem 2.7 by taking $\alpha_j = \frac{1}{n}, 1 \leq j \leq n$:

$$\frac{\left(x - \frac{\mathrm{Re}(trA)}{n}\right)^2}{2M^2} + \frac{\left(y - \frac{\mathrm{Re}(trA)}{n}\right)^2}{2N^2} = c \ \text{ for some } c \in (0,1],$$

where

$$M = \sqrt{\frac{n-1}{n}} \left\| \mathrm{Re}\, A - \frac{\mathrm{Re}\,(trA)}{n} I \right\|_2$$

and

$$N = \sqrt{\frac{n-1}{n}} \left\| \mathrm{Im}\, A - \frac{\mathrm{Im}\,(trA)}{n} I \right\|_2 .$$

Which means that our result in Theorem 2.7 gives generalizations of the inequality (1.3) given in [12]. Also, when $k^2(A) = h^2(A) = 0$, we get

$$\frac{\left(x - \frac{\mathrm{Re}(trA)}{n}\right)^2}{2M^2} + \frac{\left(y - \frac{\mathrm{Re}(trA)}{n}\right)^2}{2N^2} = 1,$$

where M and N are defined as above. Which implies that our result in Theorem 2.4 gives refinements of Theorem 2.3 given in [24].

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