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SOME NEW PROPERTIES OF CERTAIN PATH GRAPHS

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Abstract. Path graphs were proposed as a generalization of line graphs. The 2-path graph denoted by $P_2(G)$, of a graph G has vertex set the set of all paths of length two. Two such vertices are adjacent in the new graph if their union is a path of length three or a cycle of length three. In this paper we characterize when $P_2(G)$ is a path, a cycle or a tree. If G is connected, we investigate when $P_2(G)$ is complete and we study the path graphs of Bipartite graphs. Also we characterize when $P_2(G)$ is isomorphic to the complement of G.

1. Introduction

For any graph G, as a generalization of the line graph Broesma and Hoede, see [3], defined the k-path graphs of G denoted by $P_k(G)$. They studied some properties of these graphs.

Definition 1.1. The k-path graph of a graph G denoted by $P_k(G)$ has vertex set the set of all paths of length k in G. Two such vertices are adjacent in $P_k(G)$ if their union is a path or a cycle of length k+1.

In this paper, we will focus our study on the graph $P_2(G)$.

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A characterization of 2-path graphs has been given by Broersma [3] and Li, [8]. Later on Prisner gave a new characterization of k-path graphs, see [10].

Diameters, centers and distance in path graphs were studied in [2, 6, 7, 9]. Isomorphisims of path graphs were studied in [1, 12]. Paths of length 2 in G as well as vertices of $P_2(G)$ will be represented by triples abc, where b is the middle vertex of the path of length 2 in G from a to c and abc=cba.

In addition to the foundational studies in graph and path graph theory, recent advancements in generalized metric spaces and their applications have provided powerful tools for broader mathematical and computational contexts. Notably, in a series of recently published works.

The following two examples are important for our investigation, see [3].

Example 1.2. Let G be the graph obtained from $K_{1,3}$ by subdividing all of it's edges once, this graph is denoted by $S(K_{1,3})$. Observe that $P_2(S(K_{1,3}))=C_6$.

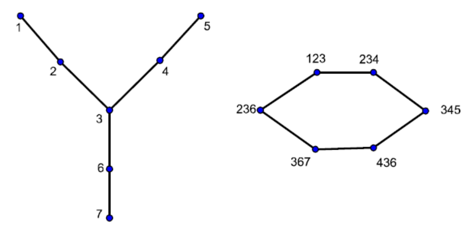

FIGURE 1. The graph $S(K_{1,3})$ and $P_n(S(K_n, 4))$.

Example 1.3. The graph $S(K_{1,3}) - s$, where s is an end vertex, is denoted by Y. Observe that $P_2(Y) = P_5$. Figure 2 shows the graph Y and $P_2(Y)$.

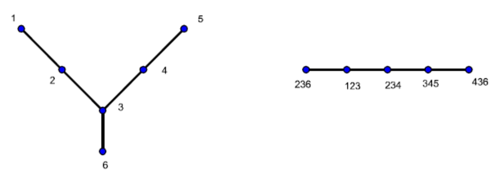

FIGURE 2. The graph Y and $P_2(Y)$.

We need the following two results about path graphs, see [3].

Theorem 1.4. ([3]) *For a vertex abc of $P_2(G)$, $deg(abc) = deg(a)+deg(c)-2$. Note that deg(a) and deg(c) are degrees in G, wheareas deg(abc) is the degree of the vertex abc of $P_2(G)$.*

Theorem 1.5. ([3]) *If H is a subgraph of G, then $P_2(H)$ is an induced subgraph of $P_2(G)$.*

2. Characterizing path graphs that are paths, cycles, or trees

In this section we will characterize when $P_2(G)$ is a path, a cycle or a tree.

Theorem 2.1. *Let G be a graph. Then $P_2(G)$ is a path if and only if G is a path or G is isomorphic to $Y = S(K_{1,3}) - s$. (Y is given in Figure 2).*

Proof. Assume that G is a path graph, say P_n, then $P_2(P_n) = P_{n-2}$. If G is the graph Y, then $P_2(Y) = P_5$.

Assume that $P_2(G)$ is a path. Since $P_2(G)$ is connected and has no cycles, then G is a connected graph that has no cycles. If G has a path of length two, say aub, with deg(a) = deg(b) =1, then the degree of the vertex aub in $P_2(G)$ is equal to deg(a)+deg(b)-2=0. So, $P_2(G)$ has an isolated vertex and hence $P_2(G)$ is disconnected, which is a contradiction. Thus, G is a tree with no subgraph isomorphic to the path aub with deg(a) = deg(b) =1. If G is a path, then $P_2G)$ is a path. So assume that G is not a path. Thus, G has a vertex say s with deg(s)$\geq$ 3. So G has a subgraph H of the form shown in Figure 3.

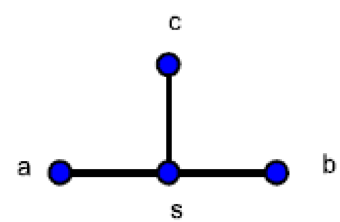

Figure 3. H.

Observe that $G \neq H$ because $P_2(H)$ is disconnected. Now, since $P_2(G)$ is connected there exist at least two vertices adjacent to two of the end vertices of H, say a and b. Thus we get a subgraph H_1 of G of the form shown in Figure 4.

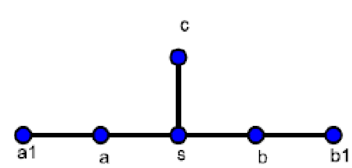

Figure 4. H_1.

Now, $P_2(H_1)$ is the path P_5. (The graph $P_2(H_1)$ is shown in Figure 5.

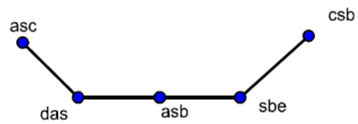

FIGURE 5. $P_2(H_1)$.

So G might be H_1. Observe that H_1 is the graph Y. We want to show that $G = H_1$.

Assume $G \neq H_1$. Since $P_2(G)$ is a path that contains $P_2(H_1)$ properly, then $P_2(G)$ has a vertex that is adjacent to the vertex asc or a vertex that is adjacent to the vertex csb. Thus G contains one of the following subgraphs in Figure 6.

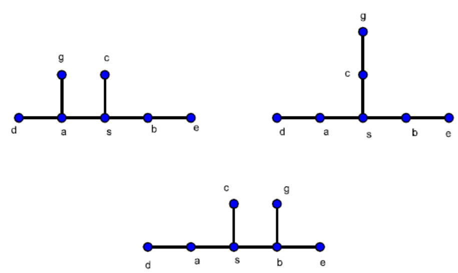

FIGURE 6. The possible subgraphs that contained in G properly if $G \neq H_1$.

But the path graph of any of these subgraphs is not a path. This contradicts the fact that $P_2(G)$ is a path graph. Hence $G = H_1 = Y$ and $P_2(G) = P_5$. □

Theorem 2.2. *Let G be a graph, then $P_2(G)$ is C_n if and only if G is C_n or $S(K_{1,3})$.*

Proof. It has been shown that $P_2(C_n) = C_n$ see [3] and $P_2(S(K_{1,3})) = C_6$.

Now, assume that $P_2(G)$ is C_n, We want to show that G is isomorphic to C_n or $S(K_{1,3})$.

Observe that G does not contain a cycle as a proper subgraph, otherwise $P_2(G)$ will contain this cycle as a proper subgraph. Thus, G is a cycle or G is a tree. If G is a cycle, then $P_2(G)$ is so. If G is a tree, then G is not a path since $P_2(P_n) = P_{n-2}$. Hence, G has a vertex of degree 3. Observe that $G \neq H$ (H is shown in figure 1). Since $P_2(H)$ is disconnected. Since G is connected there exist at least two vertices that are adjacent to the end vertices of H. So, G has H_1 (H_1 and $P_2(H_1)$ are shown in Figure 7) as a subgraph.

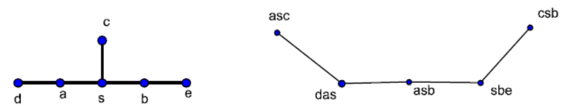

FIGURE 7. H_1 and $P_2(H_1)$.

Note that $G \neq H_1$. Since $P_2(H_1) = P_5$, i.e. H_1 is a proper subgraph of G. Since $P_2(G)$ is a cycle that contains $P_2(H_1)$ properly, then $P_2(G)$ has a vertex that is adjacent to the vertex *asc* or a vertex that is adjacent to the vertex *csb*. So, G contains one of the following subgraphs shown in Figure 8.

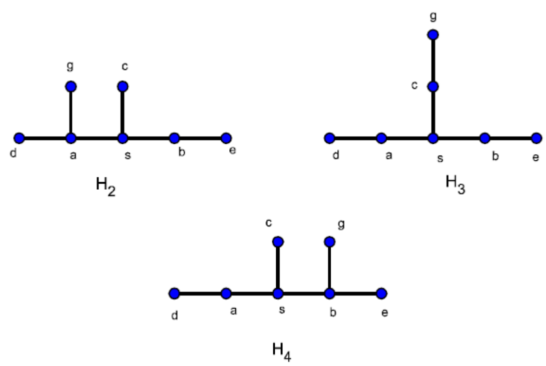

FIGURE 8. H_2, H_3, H_4.

Observe that G can not have H_3 or H_4 as subgraphs and this is because $P_2(G)$ is a cycle and $P_2(H_3)$ and $P_2(H_4)$ have vertices of degree 3.

So, G contains H_2 as a subgraph and $P_2(H_2)=C_6$. Indeed $G=H_2$ (the graph $H_2 \equiv S(K_{1,3})$). Note that if $G \neq H_2$, then G contains H_2 properly and $P_2(G)$ can not be a cycle since $P_2(H_2)$ is a cycle. □

Theorem 2.3. *Let G be a graph. Then $P_2(G)$ is a tree if and only if G is a caterpillar satisfying two conditions:*

(1) *G does not contain a path, aub, with $deg(a) = deg(b) = 1$.*
(2) *G does not contain two consecutive vertices on the backbone path of the caterpillar of degree* 3.

Proof. Let G be a graph such that $P_2(G)$ is a tree. Then G is connected. Otherwise $P_2(G)$ would be a disconnected graph. If G has a path of length two, say the path, aub , with $deg(a) = deg(b) = 1$, then $P_2(G)$ will be a disconnected graph, which is a contradiction. Hence, G is connected with no subgraph isomorphic to the path, aub, with $deg(a) = deg(b) = 1$. If G has a cycle, then $P_2(G)$ will have a cycle and this is impossible, since $P_2(G)$ is a tree. Thus G has no cycles, connected, and has no subgraph of the form, aub, with $deg(a) = deg(b) = 1$. Thus, G is a tree with no subgraph isomorphic to aub, with $deg(a) = deg(b) = 1$. If G has a subgraph isomorphic to $S(K_{1,3})$, then $P_2(G)$ has an induced subgraph isomorphic to $P_2(S(K_{1,3}))$, which is a graph isomorphic to C_6. Therefore, G is a tree with no subgraph isomorphic to $S(K_{1,3})$, that is, G is a caterpillar. Thus, G is a caterpillar that satisfies condition 1.

Now, if G contains two consecutive vertices of degree three, then $P_2(G)$ must contain a subgraph isomorphic to C_4, this gives a contradiction since $P_2(G)$ is a tree. Hence, G is a caterpillar that satisfies both conditions 1 and 2. Let G be the graph that satisfies the two conditions with maximum number of edges. Then G has the form in Figure 9.

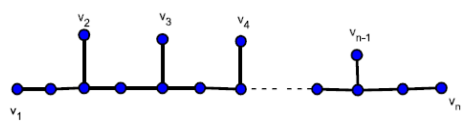

FIGURE 9. G.

The 2-path graph of G is given in Figure 10.

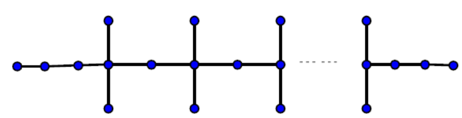

FIGURE 10. $P_2(G)$.

Now, any graph say G^* that satisfies the two conditions in the theorem can be obtained from G by removing some of vertices $v_2, v_3, \cdots, v_{n-1}$. Observe that the removal of any one of these vertices results in removing two vertices of degree one in $P_2(G)$ we will show this more in the following example. The removal of these vertices of degree one from $P_2(G)$ results in a tree, and we get the result. $\square$

Example 2.4. Let G be the graph that satisfies the two conditions in Theorem 2.1 with $| V(G) |= 15$, and has the maximum number of edges. This graph and it's path graph P_2(G) are shown in Figure 11.

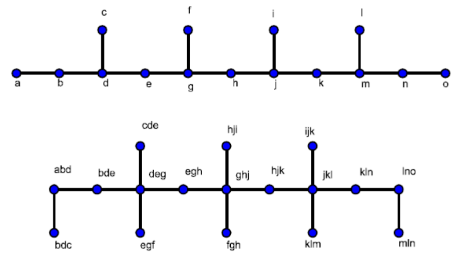

FIGURE 11. G and $P_2(G)$.

Now, by removing some of the vertices c, f, i, or m, from the graph G, we still have a graph that satisfies the two conditions in the Theorem 2.1, say this new graph is G*. Observe that the removal of c, f, i, or m results in removing bdc and cde, egf and fgh, hji and ijk, klm and mln from P_2 (G) respectively. It is clear that P_2 (G*) is tree. In Figure 12, we have the graph G* and P_2(G*) where G* is produced from G by removing the vertex f.

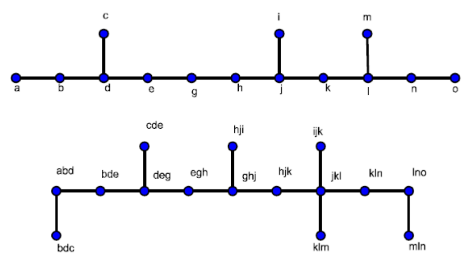

FIGURE 12. G^* and $P_2(G^*)$..

Now, we will characterize all graphs G such that P_2(G) is complete and find the clique number of P_2(G). It was shown in [3] that $P_2(G) = K_3$ if and only if G is a triangle. Also, according to the characterization of P_2-graphs, no vertex of P_2(G) belongs to more than one triangle see [6]. Thus, if G is a graph that has a triangle, then the clique number of P_2(G) is equal to 3. And if G is a graph that has no triangles then the clique number of P_2(G) is equal to 2 or 1. Also, there is no graph G with $P_2(G) = K_n$ for $n \geq 4$.

We summarize the previous results in the following theorem.

Theorem 2.5. *Let G be a graph.*

(1) *There is no graph G with* $P_2(G) = K_m$ *for* $m \geq 4$.
(2) $P_2(G) = K_3$ *if and only if* $G = K_3$.
(3) $P_2(G) = K_1$ *if and only if G is a path of length* 2.
(4) $P_2(G) = K_2$ *if and only if G is a path of length* 3.
(5) *The clique number of* $P_2(G) \leq 3$.
(6) *The clique number of* $G = 3$ *if and only if G contains a triangle.*

Example 2.6. The clique number of P_2 ($P_2 \times P_n$) is equal to two. This is because $P_2 \times P_n$ is a grid, which is a connected graph, that does not contain any triangle.

Example 2.7. The clique number of P_2 (K_n) is equal to three for all n $\geq$ 3.

Now, we look at path graphs of bipartite graphs.

Theorem 2.8. *Let G be a graph. If G is a bipartite graph, then* P_2 *(G) is a bipartite graph.*

Proof. Let {A,B} be a bipartition of the set V(G). Let us define sets C and D as follows: C is the set of all paths of length two in G with the first vertex in A and D is the set of all paths of length two in G with the first vertex in B. Two paths from the set C (or D) can't intersect each other in a path of length one in such a way that their union forms a path or a cycle of length three. Therefore, {C,D} is a bipartition of the graph P_2(G). □

Now, we will investigate when P_2(G) is isomorphic to the complement of G.

Lemma 2.9. *Let G be a graph. If* P_2*(G) is isomorphic to the complement of G, then G must be a connected graph.*

Proof. Assume that G consists of m components, then P_2(G) consists at least of m components. But the complement of the graph G consists of only one component. Hence m must be 1. Thus G is connected. □

Theorem 2.10. *Let G be a graph. Then* P_2*(G) is isomorphic to the complement of G if and only if G is the cycle* C_5.

Proof. Assume that G is C_5, then P_2(G)=C_5 and the complement of G is isomorphic to C_5then P_2(G) is isomorphic to the complement of G.

Now, assume that P_2(G) is isomorphic to the complement of G. To show that G is C_5, we will discuss the following cases:

Case 1: G does not contain any cycle. So G is a tree since G is connected. If G is a path, then G=P_n with n$\geq$ 3. We have |V(P_n)|=n, |V(P_2(P_n))|=n-2.$But|V(\bar{P}_n)| = n$, thus P_2 (G) can not be isomorphic to the complement of

P_n. Therefore, it is enough to consider trees that are not paths. If G is a star, then P $_2$(G) is the empty graph, while the complement of G has a triangle and hence P_2(G) is not isomorphic to the complement of G.

Suppose G is a tree that is neither a path nor a star. So G has a subgraph say H (H and its complement are shown in Figure 13). Observe that the complement of H has a triangle. Since G is a tree, then the graph P_2(G) has no triangles. But the complement of G has a triangle and hence P_2(G) and the complement of G can not be isomorphic.

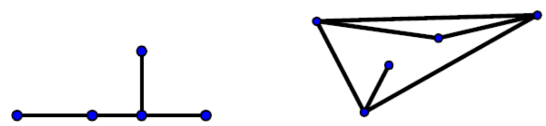

FIGURE 13. H and it's complement.

Case 2: G contains at least one cycle. Then either G itself is a cycle or G contains a cycle as a proper subgraph.

Assume that G is a cycle. If G=C_3, then P_2*(G)=*C_3, but the complement of G is the null graph on three vertices. So P_2 (G) is not isomorphic to the complement of G.

If G=C_4 then P_2*(G)=*C_4. But the complement of G is a disconnected graph. Thus P_2(G) is not isomorphic to the complement of G.

If G=C_5, then P_2*(G)=*C_5. And the complement of G is isomorphic to C_5.

Suppose G=C_n for $n \geq 6$. Then P_2*(G)=*C_n. But P_2 (G) is not isomorphic to the complement of G and this is because the complement of G contains at least one triangle and G contains no triangles. We conclude that the only cycle C_n for which P_2(G) is isomorphic to the complement of G is the cycle C_5.

Suppose G contains a cycle C_n $(n \geq 3)$ as a proper subgraph. The most simple form of G consists of the cycle C_n and a tree T_m that are connected with one vertex. (This form is shown in Figure 14). Observe that $|V(G)| < |V(P_2(G))|$. Furthermore if G contains more edges (between the vertices of C_n or between the vertices of T_m or between the vertices of C_n and the vertices of T_m),then the number of vertices of $P_2(G)$ will be more. So, $|V(G)| < |V(P_2(G))|$. $P_2(G)$ is not isomorphic to the complement of G.

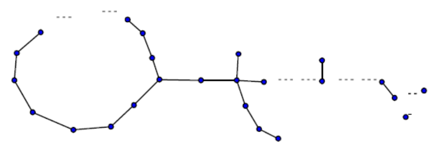

FIGURE 14. A Graph G contains a cycle

□

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