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# SOLVING FRACTIONAL VARIABLE-ORDER DIFFERENTIAL EQUATIONS FOR GENERALIZED CAPUTO-TYPE WITH TIME-VARYING DELAY BY USING THE OPERATION MATRIX METHOD

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**Abstract.** This study introduces novel derivative and integral operators derived from newly formulated generalized Caputo fractional derivatives (GCFDs). A numerical approach is developed utilizing these operators to solve variable-order fractional differential equations with time-varying delay (VOFDETVD). The approach entails approximating the solutions of VOFDETVD by employing shifted Legendre polynomials (SLPs) as basis vectors. To assess the efficiency of the numerical method, tests are conducted on various examples, with different values of $\varrho$ specified for the fractional differential operator of the new generalized Caputo. Moreover, the present method is comprehensively evaluated for robustness and effectiveness through comparisons with existing approaches.

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## 1. Introduction

The subject of fractional calculus (FC) studies arbitrary-order differentials and integral operators. Initially, FC was considered an abstract mathematical concept with minimal practical implications. However, over the past few decades, significant developments have been observed in both the applied and pure areas of mathematics [9, 26].

Several fractional operators exist in the literature; however, the Caputo operator has been frequently employed to solve fractional-order initial value problems (IVP) [1, 13, 14].

Furthermore, Caputo fractional differential equations (CFDEs) have been extensively studied for their ability to model a variety of real-life processes mathematically. In [11], the authors used CFDE to study the behaviour of current in electrical circuits.

One of the fractional computations is the widely used fractional integrated operator, which was motivated by the values of the parameters $\alpha(\varsigma)$ and $\varrho$. The generalised Caputo derivative and the Caputo derivative [3, 15, 18] have the same properties. It also provides a valuable approach for organising and creating fractionally measured mathematical models. The generalized Caputo fractional derivative has an additional feature that is not present in other fractional derivatives, such as Caputo, Caputo-Fabrizio, and Atangana-Baleanu [17]. They must be solved numerically because FDEs are difficult to solve analytically. Several numerical techniques exist in the literature for numerical solutions to fractional differential equations FDEs.

This paper develops the spectral method and the concept of an Operational Matrix (OM) by combining the generalized Caputo-type variable-order fractional derivative. The motivation for solving generalized CFDEs stems from the kernel properties of the generalized Caputo fractional derivative, as they govern memory and nonlocal effects. In addition to the fractional order parameter, the $\varrho$ parameter is also effective in constructing graphs for the acquired data.

Fractional Legendre functions were employed in this paper to solve various VOFDETVD problems. Fractional Legendre functions are constructed from Legendre polynomials, which have been applied in different papers (see [10, 20, 21, 24] ).

The numerical scheme converts the proposed equations into a system of algebraic equations, which can then be solved using standard methods [4, 5, 22, 23]. The computational approach has many advantages. One of the most important functions is to calculate OM accurately. In addition, error analysis is thoroughly explored. Finally, to evaluate the accuracy of the suggested method, we consider the results in tables and graphs. The numerical solutions of the suggested method agree well with those obtained using other established methods.

1.1. **Problem Statement.** In this section, we will explore the following two types of VOFDETVD.

$$\begin{cases} {}^{C}D^{\alpha(\varsigma),\varrho}y(\varsigma) = F(\varsigma, y(\varsigma), y(\varsigma-\delta(\varsigma))) & \varsigma \in [0,1], \quad \alpha(\varsigma) \in [0,1], \\ y(0) = \gamma(\varsigma), \end{cases}$$

and

$$\begin{cases} {}^{C}D^{\alpha(\varsigma),\varrho}y(\varsigma) = G(\varsigma, y(\varsigma), y(\delta\varsigma)) & \varsigma \in [0,1], \quad \alpha(\varsigma) \in [0,1], \\ y(0) = \beta, \end{cases}$$

where $F$ and $G$ are smooth functions and $\delta$ denote the delay elements.

1.2. **Structure of the paper.** This paper aims to modify the spectral method and OM to derive solutions of the various FDEs included via the generalized Caputo variable-order fractional derivative.

The manuscript is organized as follows. In Section 2, we recall basic concepts for fractional operators. Section 3 presents the suggested OM and the solution method. Section 4 presents the numerical results, comparison with other numerical methods, and the performance indicators obtained. Finally, Sect. 5 is devoted to our conclusions.

## 2. Basic concepts

**Definition 2.1.** ([18]) The generalized Riemann-type fractional derivative with order $\alpha > 0$ is defined as follows:

$$ {}^{R}D^{\alpha,\varrho}_{\sigma+}y(\varsigma) = \frac{\varrho^{\alpha-\gamma+1}}{\Gamma(\gamma-\alpha)}(\varsigma^{1-\varrho}\frac{d}{d\varsigma})^{\gamma}\int_{\sigma}^{\varsigma} s^{\varrho-1}(\varsigma^{\varrho}-s^{\varrho})^{\gamma-\alpha-1}y(s)ds, \quad \varsigma > \sigma, \tag{2.1}$$

thus $\sigma \geq 0$, $\varrho > 0$ and $\gamma = \lceil \alpha \rceil$.

**Definition 2.2.** ([26]) Caputo's fractional derivative of order $\alpha > 0$ is defined as:

$$ {}^{C}D^{\alpha}_{\sigma+}y(\varsigma) = \frac{1}{\Gamma(\gamma-\alpha)}\int_{\sigma}^{\varsigma}(\varsigma-s)^{\gamma-\alpha-1}y^{(\gamma)}(s)ds, \quad \varsigma > \sigma, \tag{2.2}$$

where $\gamma - 1 < \alpha \leq \gamma$ for $\gamma \in \mathbb{N}$.

**Definition 2.3.** ([15]) A fractional integral of order $\alpha > 0$ of a function $y(\varsigma)$, $I^{\alpha,\varrho}_{\sigma+}y(\varsigma)$, is defined by:

$$I^{\alpha,\varrho}_{\sigma+}y(\varsigma) = \frac{\varrho^{1-\alpha}}{\Gamma(\alpha)} \int_{\sigma}^{\varsigma} s^{\varrho-1}(\varsigma^{\varrho} - s^{\varrho})^{\alpha-1} y(s)ds, \quad \alpha > 0, \quad \varsigma > \sigma, \tag{2.3}$$

thus $\varsigma > \sigma$ and $\varrho > 0$.

**Definition 2.4.** ([18]) The generalized Caputo-type fractional derivative of order $\alpha > 0$ is defined as:

$${}^{C}D^{\alpha,\varrho}_{\sigma+}y(\varsigma) = \big( {}^{R}D^{\alpha,\varrho}_{\sigma+}\big[y(\varsigma) - \sum_{v=0}^{\gamma-1} \frac{y^{(v)}(\varsigma)}{v!}(x-\varsigma)^{v}\big]\big)(\varsigma), \quad \varsigma > \sigma, \tag{2.4}$$

where $\sigma \geq 0$, $\varrho > 0$ and $\gamma = \lceil \alpha \rceil$.

**Definition 2.5.** ([15]) The novel generalized Caputo-type fractional derivative of order $\alpha > 0$ is defined as:

$${}^{C}D^{\alpha,\varrho}_{\sigma+}y(\varsigma) = \frac{\varrho^{\alpha-\gamma+1}}{\Gamma(\gamma-\alpha)} \int_{\sigma}^{\varsigma} s^{\varrho-1}(\varsigma^{\varrho}-s^{\varrho})^{\gamma-\alpha-1}\big(s^{1-\varrho}\frac{d}{ds}\big)^{\gamma} y(s)ds, \quad \varsigma > \sigma, \tag{2.5}$$

where $\sigma \geq 0$, $\varrho > 0$, $\gamma - 1 < \alpha < \gamma$, $\gamma = \lceil \alpha \rceil$, and $y(\varsigma) \in C^{\gamma}[\sigma, \rho]$.

Furthermore, the following is included in the new generalized Caputo fractional derivative [3, 15, 18]:

${}^{C}D^{\alpha,\varrho}C = 0$, and $C$ is a constant. Moreover, if $\gamma - 1 < \alpha < \gamma$, $k > \gamma - 1$ and $k \notin \mathbb{N}$,

$${}^{C}D^{\alpha,\varrho}_{\sigma+}(\varsigma^{\varrho} - \sigma^{\varrho})^{\tau} = \begin{cases} \varrho^{\alpha}\frac{\Gamma(\tau+1)}{\Gamma(\tau-\alpha+1)}(\varsigma^{\varrho}-\sigma^{\varrho})^{\tau-\alpha}, \tau \in \mathbb{N}_0, and \quad \tau \geq \lceil \alpha \rceil, \\ \qquad or \quad \tau \in \mathbb{N}, and \quad \tau > \lfloor \alpha \rfloor, \\ 0 \qquad , \tau \in \mathbb{N}_0 \quad and \quad \tau < \lceil \alpha \rceil. \end{cases} \tag{2.6}$$

**Definition 2.6.** ([15]) The new type of generalized Caputo variable order fractional derivative of order $\alpha(\varsigma) > 0$ is defined as:

$${}^{C}D^{\alpha(\varsigma),\varrho}_{\sigma+}y(\varsigma) = \frac{\varrho^{\alpha(\varsigma)-\gamma+1}}{\Gamma(\gamma-\alpha(\varsigma))} \int_{\sigma}^{\varsigma} s^{\varrho-1}(t^{\varrho}-s^{\varrho})^{\gamma-\alpha(\varsigma)-1}\big(s^{1-\varrho}\frac{d}{ds}\big)^{\gamma} y(s)ds, \quad \varsigma > \sigma, \tag{2.7}$$

where $\sigma \geq 0$, $\varrho > 0$, $\gamma - 1 < \alpha < \gamma$, $\gamma = \lceil \alpha \rceil$ and $y(\varsigma) \in C^{\gamma}[\sigma, \rho]$.

2.1. **Properties of SLPs.** LPs are represented in the interval $[-1,1]$, are orthogonal polynomials, as well as specified by a recurrence formula relation [10], as expressed by

$$\bar{L}_{v+1}(z) = \frac{2v+1}{v+1} z\bar{L}_v(z) - \frac{v}{v+1}\bar{L}_{v-1}(z), \quad v = 1, 2, ... \tag{2.8}$$

in which $P_0(\varsigma) = 1$ and $P_1(\varsigma) = 2\varsigma - 1$.

Moreover, the analytical form of SLPs $P_v(\varsigma)$ of degree $v$ may be obtained by means of a

$$P_v(\varsigma) = \sum_{\tau=0}^{v} (-1)^{v+\tau} \frac{(v+\tau)!}{(v-\tau)!} \frac{\varsigma^\tau}{(\tau!)^2}. \tag{2.9}$$

Observe that $P_v(0) = (-1)^v$ and $P_v(1) = 1$. Moreover, the orthogonal condition is given by

$$\int_0^1 P_v(\varsigma)P_u(\varsigma)dt = \begin{cases} \frac{1}{2v+1}, & for \quad v = u, \\ 0, & for \quad v \neq u. \end{cases} \tag{2.10}$$

The function $y(\varsigma)$, which is square-integrable over the interval $[0,1]$, can be represented by SLPs as follows:

$$y(\varsigma) = \sum_{u=0}^{\infty} c_u P_u(\varsigma), \tag{2.11}$$

where coefficients $c_u$ are derived from

$$c_u = (2u+1)\int_0^1 g(\varsigma)P_u(\varsigma)dt, \quad u = 1, 2, .... \tag{2.12}$$

The function $y(\varsigma)$, which is square-integrable over the interval $[0,1]$, can be represented by SLPs as follows:

$$y(\varsigma) = \sum_{u=0}^{\infty} c_u P_u(\varsigma), \tag{2.13}$$

where coefficients $c_u$ are derived from

$$c_u = (2u+1)\int_0^1 g(\varsigma)P_u(\varsigma)dt, \quad u = 1, 2, .... \tag{2.14}$$

For the $(\Omega+1)$ terms of SLPs that were considered, we get

$$y(\varsigma) = \sum_{u=0}^{\infty} c_u P_u(\varsigma) = C^T\phi(\varsigma), \tag{2.15}$$

where the expressions for the shifted Legendre vector $\phi(\varsigma)$ and the shifted Legendre coefficient vector $C$ are

$$C^T = [c_0, ..., c_\Omega], \quad \phi(\varsigma) = [P_0(\varsigma), P_1(\varsigma), ..., P_\Omega(\varsigma)]^T \tag{2.16}$$

and assume that

$$\lambda(\varsigma) = [1, \varsigma, \varsigma^2, ..., \varsigma^\gamma]^T. \tag{2.17}$$

The vector $\phi(\varsigma)$ can then be expressed as follows.

$$\phi(\varsigma) = A\lambda(\varsigma) \tag{2.18}$$

in which $A$ is a square matrix $(\Omega + 1) \times (\Omega + 1)$ expressed as

$$(\sigma_{v,u})_{0 \le v,u \le \gamma} = \begin{cases} (-1)^{\gamma - v} \frac{\Gamma(\gamma+1)\Gamma(\gamma+v+1)}{\Gamma(v+1)\Gamma(\gamma+1)\Gamma(\gamma-v+1)\Gamma(v+1)^{lv}}, & \text{for} \quad v \ge u, \\ 0, & \text{for} \quad \text{otherwise}. \end{cases} \tag{2.19}$$

For example, if $\gamma = 4$, then the square matrix $A$ is obtained by

$$A = \frac{1}{l^v} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 \\ 1 & -6 & 6 & 0 & 0 \\ -1 & 12 & -30 & 20 & 0 \\ 1 & -20 & 90 & -140 & 70 \end{pmatrix}.$$

Therefore, using Eq.(2.16), we claim

$$\lambda(\varsigma) = A^{-1}\phi(\varsigma). \tag{2.20}$$

2.2. **Operational Delay Matrix.** The delay shifted Legendre orthogonal vector $y(\varsigma - \delta(\varsigma))$ or $y(\delta(\varsigma)\varsigma)$ can be defined such that

$$y(\varsigma - \delta(\varsigma)) = R\phi(\varsigma) \tag{2.21}$$

in which $R$ resembles the $\Omega \times \Omega$ operational delay matrix expressed by

$$R = A_1 A^{-1},$$

where $A_1 = \int_0^1 \phi(\varsigma - \delta(\varsigma))\phi^T(\varsigma)d\gamma$ and $A = \int_0^1 \phi(\varsigma)\phi^T(\varsigma)d\varsigma$.

In addition, delay function $y(\varsigma - \delta(\varsigma))$ or $y(\delta(\varsigma)\varsigma)$ can be defined in terms of SLPs as follows:

$$y(\varsigma - \delta(\varsigma)) = C^T R\phi(\varsigma), \tag{2.22}$$

where $C$ is given in Eq.(2.14).

## 3. Methodology

This section describes the numerical solution of the problem using the VOFDETVD LOM with GCFDs, employing the Legendre OM shifted by a variable order. Thus, the problem will be transformed into an algebraic system of equations that can be solved numerically at the collocation points.

Initially, the shifted Legendre OM of the variable-order fractional differential operator ${}^{C}D^{\alpha(\varsigma),\varrho}\phi(\varsigma)$ will be explored as follows: Since $\phi(\varsigma) = A\lambda(\varsigma)$,, we obtain the following.

$$ {}^{C}D^{\alpha(\varsigma),\varrho}\phi(\varsigma) = {}^{C}D^{\alpha(\varsigma),\varrho}(A\lambda(\varsigma)) = A\ {}^{C}D^{\alpha(\varsigma),\varrho}[1,\varsigma,\varsigma^2,...,\varsigma^{\gamma}]^T. \tag{3.1} $$

Applying the variable-order GCFDs given in Eq.(2.6) to this scenario, we obtain Eq.(3.1) as follows:

$$ \begin{aligned} {}^{C}D^{\alpha(\varsigma),\varrho}\phi(\varsigma) &= \Big[0, \varrho^{\alpha(\varsigma)}\frac{\Gamma(2)}{\Gamma(2-\alpha(\varsigma))}\varsigma^{1-\alpha(\varsigma)\varrho}\ \varrho^{\alpha(\varsigma)}\frac{\Gamma(3)}{\Gamma(3-\alpha(\varsigma))}\varsigma^{2-\alpha(\varsigma)\varrho}, \\ &\quad ..., \varrho^{\alpha(\varsigma)}\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha(\varsigma))}\varsigma^{\gamma-\alpha(\varsigma)\varrho}\Big]^T \\ &= A\begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & P & 0 & \dots & 0 \\ 0 & 0 & Q & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & R \end{pmatrix}\begin{pmatrix} 1 \\ \varsigma \\ \varsigma^2 \\ \vdots \\ \varsigma^{\gamma} \end{pmatrix} \\ &= A \quad B(\varsigma)\lambda(\varsigma), \end{aligned} \tag{3.2} $$

for

$$ P = \varrho^{\alpha(\varsigma)}\frac{\Gamma(2)}{\Gamma(2-\alpha(\varsigma))}\varsigma^{-\alpha(\varsigma)\varrho}, $$

$$ Q = \varrho^{\alpha(\varsigma)}\frac{\Gamma(3)}{\Gamma(3-\alpha(\varsigma))}\varsigma^{-\alpha(\varsigma)\varrho} $$

and

$$ R = \varrho^{\alpha(\varsigma)}\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha(\varsigma))}\varsigma^{-\alpha(\varsigma)\varrho}, $$

where

$$ B(\varsigma) = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & P & 0 & \dots & 0 \\ 0 & 0 & Q & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & R \end{pmatrix}. \tag{3.3} $$

Using Eq.(2.18), we have

$$ {}^{C}D^{\alpha(\varsigma),\varrho}\phi(\varsigma) = AB(\varsigma)A^{-1}\phi(t). \tag{3.4} $$

Thus, we can use the variable-order OM to get the approximate solution shown in Eq.(2.13) as follows:

$$ {}^{C}D^{\alpha(\varsigma),\varrho}y(\varsigma) = {}^{C}D^{\alpha(\varsigma),\varrho}(C^T\phi(\varsigma)) = C^T \, {}^{C}D^{\alpha(\varsigma),\varrho}\phi(\varsigma) = C^T AB(\varsigma)A^{-1}\phi(\varsigma). \tag{3.5}$$

Second, using the same techniques used to obtain the OM of ${}^{C}D^{\alpha(\varsigma),\varrho}\phi(\varsigma)$, the OM of the SLPs can be obtained for ${}^{C}D^{\beta_q(\varsigma),\varrho}\phi(\varsigma)$, $q = 1, 2, ..., k$. Hence, we claim that

$$ {}^{C}D^{\beta_q(\varsigma),\varrho}\phi(\varsigma) = (AG_q(\varsigma)A^{-1})\phi(\varsigma), \tag{3.6}$$

where

$$G_q(\varsigma) = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & U & 0 & \dots & 0 \\ 0 & 0 & V & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & W \end{pmatrix} \tag{3.7}$$

for

$$U = \varrho^{\beta_q(\varsigma)} \frac{\Gamma(2)}{\Gamma(2-\beta_q(\varsigma))} \varsigma^{-\beta_q(\varsigma)\varrho},$$

$$V = \varrho^{\beta_q(\varsigma)} \frac{\Gamma(3)}{\Gamma(3-\beta_q(\varsigma))} \varsigma^{-\beta_q(\varsigma)\varrho}$$

and

$$W = \varrho^{\beta_q(\varsigma)} \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\beta_q(\varsigma))} \varsigma^{-\beta_q(\varsigma)\varrho}.$$

The square matrix $(AG_q(\varsigma)A^{-1})$ is the OM of ${}^{C}D^{\beta_q(\varsigma),\varrho}\phi(\varsigma)$. Therefore,

$$ {}^{C}D^{\beta_q,\varrho}y(\varsigma) = {}^{C}D^{\beta_q(\varsigma),\varrho}(C^T\phi(\varsigma)) = C^T \, {}^{C}D^{\beta_q(\varsigma),\varrho}\phi(\varsigma) = C^T(AG_q(\varsigma)A^{-1})\phi(\varsigma). \tag{3.8}$$

Eqs.(3.7) and (3.10), the main problem that was given in ${}^{C}D^{\alpha(\varsigma)}y(\varsigma) = F(\varsigma, y(\varsigma), y(\varsigma-\delta), ...)$ is transformed into the following:

$$C^T AB(\varsigma)A^{-1}\phi(\varsigma) = F[\varsigma, C^T\phi(\varsigma), C^T R\phi(\varsigma), ...], \quad 0 \le \varsigma \le 1. \tag{3.9}$$

Here, using the collocation points, $\varsigma_q = \frac{2q+1}{2\gamma+2}$, $q = 0, 1, 2, ..., n$, to convert the system of equations in (3.9) into a system of algebraic equations has the following form:

$$C^T AB(\varsigma_q)A^{-1}\phi(\varsigma_q) = F[\varsigma_q, C^T\phi(\varsigma_q), C^T R\phi(\varsigma_q), ...], \quad 0 \le \varsigma \le 1. \tag{3.10}$$

Finally, the unknown vector $C$ in the numerical solution presented in Eq.(2.13) of the given problem can be calculated by numerically solving the system of algebraic equations given in Eq.(3.10).

## 4. Error Bound and Convergence

4.1. **Estimation Error.** Let nonlinear FDEs be given as follows:

$$D^{\alpha(\varsigma),\varrho}y(\varsigma) = F(\varsigma, y(\varsigma), D^{q_1(\varsigma)}y(\varsigma), ...). \tag{4.1}$$

The constitution of the residual correction procedure uses [6] to estimate the absolute error using Eq.(4.1).

$$F = g(y(\varsigma), D^{q_1,\varrho}y(\varsigma), ...) + h(\varsigma).$$

Next, we subtract and adder

$$D^{(\alpha(\varsigma),\varrho)}y_\Omega(\varsigma) - g\big(y_\Omega(\varsigma), D^{(q_1(\varsigma),\varrho)}y_\Omega(\varsigma), ...y_\Omega(\varsigma)\big) \tag{4.2}$$

in Eq.(4.1) and obtaining $e_\Omega := y(\varsigma) - y_\Omega(\varsigma)$ gives

$$\begin{aligned} &e_\Omega^{(\alpha(\varsigma),\varrho)} - g\big(e_\Omega(\varsigma), e_\Omega^{(q_1(\varsigma),\varrho)}(\varsigma), ...\big) \\ &\quad = h(\varsigma) + g\big(y_\Omega(\varsigma), y_\Omega^{(q_1(\varsigma),\varrho)}(\varsigma), ...\big) + R \end{aligned} \tag{4.3}$$

in which $R$ may be gained from the non-linear $g$ terms. Moreover, the answer for Eq.(4.1) is gained similarly to the section 'Applications of OM for Legendre polynomials', which depends on the initial conditions given below.

$$e(0) = 0.$$

Let $\hat{e}_\gamma$ be similar to the approximate solution for Eq.(4.1), which is solved with this specific approach. If

$$||e - \hat{e}_\gamma|| < \epsilon, \tag{4.4}$$

the absolute error $e$ may be obtained by applying $\hat{e}_\gamma$.

4.2. **Error Bound.** This section presents the error bound for the OM of GCFDs. For this purpose, the following theorem [8] is given.

**Theorem 4.1.** *The error $|\triangle_\Omega| = |\,{}^{C}D^{\alpha(\varsigma),\varrho}y(\varsigma) - {}^{C}D^{\alpha(\varsigma)}y_\Omega(\varsigma)|$ in approximating ${}^{C}D^{\alpha(\varsigma),\varrho}y(\varsigma)$ having the OM of the fractional derivative is bounded as expressed below:*

$$|\triangle_\Omega| \le \sum_{v=\Omega+1}^{\infty} |c_v| \sum_{u=1}^{\Omega} |D_{v,u}| \sum_{\tau=1}^{u} \Big|\frac{(-1)^{u+\tau}(u+\tau)!}{(u-\tau)!(\tau!)^2}\Big|$$

*in which $y_\Omega$ resembles the $h$ function estimate based on the SLPs, $c_v$. Here, $v = 1, 2, 3, ..., \Omega$ resemble the coefficients of this approximation in which*

$$D_{i,j} = \begin{cases} \sum_{\tau=\lceil\alpha\rceil}^{v} \theta_{v,u,\tau}, & for \quad v = \lceil\alpha\rceil, ..., \Omega, \quad u = 1, ..., \Omega, \\ 0, & for \quad v = 1, ..., \lceil\alpha\rceil, \quad u = 1, ..., \Omega. \end{cases}$$

*Proof.* By using Eq.(2.11) can get

$$y(\varsigma) = \sum_{v=1}^{\infty} c_v P_v(\varsigma),$$

$${}^{C}D^{\alpha(\varsigma),\varrho} y(\varsigma) = \sum_{v=1}^{\infty} c_v \sum_{u=1}^{\Omega} D_{v,u} P_u.$$

By considering only the first $\Omega$ terms of the infinite series given above, the following can now be obtained

$${}^{C}D^{\alpha(\varsigma),\varrho} y(\varsigma) - \sum_{v=1}^{\Omega} c_v \sum_{u=1}^{\Omega} D_{v,u} P_u(\varsigma) = \sum_{v=\Omega+1}^{\infty} c_v \sum_{u=1}^{\Omega} D_{v,u} P_u(\varsigma). \tag{4.5}$$

Employing Eq.(2.13) and Eq.(4.5) can be illustrated to have a matrix form as given below:

$${}^{C}D^{\alpha(\varsigma),\varrho} y(\varsigma) - C^T D^{(\alpha(\varsigma),\varrho)} \phi(\varsigma) = \sum_{v=\Omega+1}^{\infty} c_v \sum_{u=1}^{\Omega} D_{v,u} P_u(\varsigma).$$

Now, it can get as:

$$\begin{aligned} |\ {}^{C}D^{\alpha(\varsigma),\varrho} y(\varsigma) - C^T D^{(\alpha(\varsigma),\varrho)} \phi(\varsigma)| &= \Big| \sum_{v=\Omega+1}^{\infty} c_v \sum_{u=1}^{\Omega} D_{v,u} P_u(\varsigma) \Big| \\ &\leq \sum_{v=\Omega+1}^{\infty} |c_v| \sum_{u=1}^{\Omega} |D_{v,u}||P_u(\varsigma)|. \end{aligned} \tag{4.6}$$

The following is an upper bound for SLPs:

$$\begin{aligned} |P_u(\varsigma)| &= \Big| \sum_{\tau=1}^{u} \frac{(-1)^{u+\tau}(u+\tau)!}{(u-\tau)!(\tau!)^2} \varsigma^{\tau} \Big| \\ &\leq \sum_{\tau=1}^{u} \Big| \frac{(-1)^{u+\tau}(u+\tau)!}{(u-\tau)!(\tau!)^2} \Big| |\varsigma^{\tau}| \\ &\leq \sum_{\tau=1}^{u} \Big| \frac{(-1)^{u+\tau}(u+\tau)!}{(u-\tau)!(\tau!)^2} \Big|. \end{aligned} \tag{4.7}$$

Thus, by substituting Eq.(4.6) in to Eq.(4.7) yields:

$$|\ {}^{C}D^{\alpha(\varsigma),\varrho} y(\varsigma) - C^T D^{(\alpha(\varsigma),\varrho)} \phi(\varsigma)| \leq \sum_{v=\Omega+1}^{\infty} |c_v| \sum_{u=1}^{\Omega} |D_{v,u}| \sum_{\tau=1}^{u} \Big| \frac{(-1)^{u+\tau}(u+\tau)!}{(u-\tau)!(\tau!)^2} \Big|.$$

Hence, the obtained result is

$$| \ ^{C}D^{\alpha(\varsigma),\varrho}y(\varsigma) - \ ^{C}D^{\alpha(\varsigma),\varrho}y_{\Omega}(\varsigma)| \leq \sum_{v=\Omega+1}^{\infty} |c_v| \sum_{u=1}^{\Omega} |D_{v,u}| \sum_{\tau=1}^{u} |\frac{(-1)^{u+\tau}(u+\tau)!}{(u-\tau)!(\tau!)^2}|.$$

Thus, the proof is complete. □

4.3. **Convergence Analysis.** In this section, a theorem on the convergence of the proposed method is introduced and proven.

**Theorem 4.2.** *The series solutions of Eq.*(4.1) *converge to exact solutions.*

*Proof.* Let

$$y(\varsigma) = \sum_{u=0}^{\infty} c_u p_u(\varsigma),$$

$$y_{\Omega}(\varsigma) = \sum_{u=0}^{\Omega} c_u p_u(\varsigma)$$

and

$$y_{\gamma}(\varsigma) = \sum_{u=0}^{\gamma} c_u p_u(\varsigma),$$

the exact and approximate solutions (partial sums) to Eq.(4.1) with $\gamma > \Omega$. Then, the following holds

$$\begin{aligned}\big(y(\varsigma), y_{\gamma}(\varsigma)\big)_{w(\varsigma)} &= \big(y(\varsigma), \sum_{u=0}^{\gamma} c_u p_u(\varsigma)\big)_{w(\varsigma)} \\ &= \sum_{u=0}^{\gamma} c_u \big(y(\varsigma), p_u(\varsigma)\big)_{w(\varsigma)} \\ &= \sum_{u=0}^{\gamma} c_u c_u = \sum_{u=0}^{\gamma} |c_u|^2 .\end{aligned}$$

The sequence $\{y_{\gamma}(\varsigma)\}$ is a Cauchy sequence in the complete Hilbert space $L^2[\sigma, \rho]$, which implies convergence.

$$\|y_{\gamma}(\varsigma) - y_{\Omega}(\varsigma)\|^2_{w(\varsigma)} = \sum_{u=\Omega+1}^{\gamma} |c_u|^2 .$$

By Bessel's inequality, the series $\sum_{u=0}^{\infty} |c_u|^2$ is convergent, which yields $\|y_{\gamma}(\varsigma) - y_{\Omega}(\varsigma)\|^2_{w(\varsigma)} \to 0$ and $\Omega, \gamma \to \infty$ and hence $y_{\gamma}(\varsigma)$ to a limit, denoted as $\rho(\varsigma)$. Proving that $\rho(\varsigma) = y(\varsigma)$,

$$\begin{aligned}\big(\rho(\varsigma) - y(\varsigma), p_u(\varsigma)\big)_{w(\varsigma)} &= \big(\rho(\varsigma), p_u(\varsigma)\big)_{w(\varsigma)} - \big(y(\varsigma), p_u(\varsigma)\big)_{w(\varsigma)} \\ &= \big(\lim_{n\to\infty} y_\gamma, p_u(\varsigma)\big)_{w(\varsigma)} - c_u \\ &= \lim_{n\to\infty} \big(y_\gamma, p_u(\varsigma)\big)_{w(\varsigma)} - c_u \\ &= 0.\end{aligned}$$

This proves $\sum_{u=0}^{\infty} c_u p_u(\varsigma)$ converges to $y(\varsigma)$. □

## 5. Numerical examples

This section will use numerical experiments to demonstrate the proposed method. The absolute error will be used in our computational results to quantify the difference between the exact and approximate solutions. All the numerical programs are coded and run in MATLAB R2020b software.

- **LOM** Legendre Operational Matrix method derived in this study.
- **COM** Chebyshev of first kind Operational Matrix [7].
- **JOM** Jacobi Operational Matrix [2].
- **CWM** Chebyshev Wavelet Method [16].
- **SCM** Spectral Collocation Method [12].
- **BWOM** Bernoulli Wavelet Operational Matrix [25].

**Example 5.1.** Consider the VOFDETVD with g-Caputo [7]

$$D^{\alpha(\varsigma),\varrho} y(\varsigma) + y(\varsigma) - y(\varsigma - \delta(\varsigma)) = 2\varsigma + \frac{\Gamma(3)}{\Gamma(1.5)}\varsigma^{1.5} - 1.$$

The initial condition of the subject is $y(0) = 0$, $\alpha(\varsigma) = 0.5$, $\delta(\varsigma) = 1$ and the exact solution is $y(\varsigma) = \varsigma^2$.

Table 1. Comparison of the values of exact and approximate obtained by the LOM and other methods at $\Omega = 6$ and $\varrho = 1$ for Example 1

| $\varsigma$ | Exact | LOM | COM | JOM |
|---|---|---|---|---|
| 0 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.2 | 0.4000 | 0.4000 | 0.4000 | 0.4000 |
| 0.4 | 0.1600 | 0.1600 | 0.1600 | 0.1600 |
| 0.6 | 0.3600 | 0.3600 | 0.3600 | 0.3600 |
| 0.8 | 0.6400 | 0.6400 | 0.6400 | 0.6400 |
| 1 | 1.0000 | 1.0000 | 1.0000 | 1.0279 |

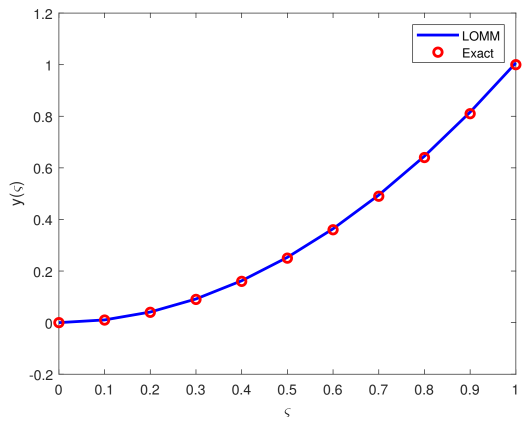


FIGURE 1. The exact and approximate solutions obtained by the LOM method at $\Omega = 6$ and $\varrho = 0.98$ for Example 1

**Example 5.2.** ([16]) Consider the linear VOFDETVD

$$D^{\alpha(\varsigma),\varrho}y(\varsigma) + y(\varsigma) - y(\varsigma - \delta(\varsigma)) = \frac{2}{\Gamma(3-\alpha(\varsigma))}\varsigma^{2-\alpha(\varsigma)} - \frac{1}{\Gamma(2-\alpha(\varsigma))}\varsigma^{1-\delta(\varsigma)} + 2\varsigma\delta(\varsigma) - (\delta(\varsigma))^2 - \delta(\varsigma)$$

for $\varsigma > 0$, $0 < \alpha(\varsigma) < 1$.

The initial condition of the subject is $y(0) = 0$ and the exact solution is $y(\varsigma) = \varsigma^2 - \varsigma$, where $\Omega = 6$.

TABLE 2. Comparison of the absolute error obtained by the LOM with different methods at $\delta = 0.01e^{-\varsigma}$ and $\varrho = 1$ for Example 2

| $\varsigma$ | LOM($\alpha = 0.5$) | CWM($\alpha = 0.5$) | LOM($\alpha = 0.7$) | CWM($\alpha = 0.7$) |
|---|---|---|---|---|
| 0 | 1.2000e-14 | 1.0000e-12 | 2.4000e-12 | 2.1000e-12 |
| 0.2 | 1.3000e-10 | 1.8000e-1 | 1.2000e-10 | 3.1000e-2 |
| 0.4 | 4.4000e-10 | 6.9000e-2 | 3.7000e-10 | 2.3000e-2 |
| 0.6 | 1.2000e-10 | 1.9000e-2 | 1.2000e-10 | 6.2000e-4 |
| 0.8 | 4.2000e-10 | 1.8000e-1 | 3.2000e-10 | 2.6000e-2 |
| 1 | 2.1000e-8 | 1.7000e-1 | 8.8000e-9 | 5.5000e-2 |

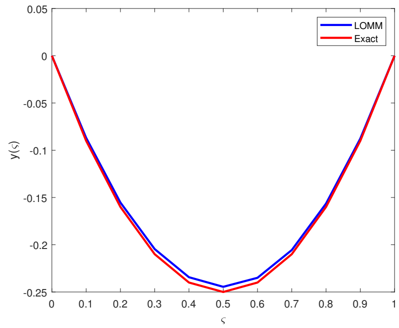


FIGURE 2. The exact and approximate solutions obtained by the LOM method at $\delta = 0.01e^{-\varsigma}$, $\alpha = 0.5$ and $\varrho = 1.02$ for Example 2

TABLE 3. Comparison of the absolute error obtained by the LOM with different methods at $\delta = 0.01$, $\alpha = 1$ and $\varrho = 1$ for Example 2

| $\varsigma$ | LOM | COM | SCM | BWM |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 0.2 | 8.5775e-15 | 1.1963e-14 | 7.5856e-14 | 8.2183e-14 |
| 0.4 | 2.6196e-14 | 1.6376e-15 | 3.9079e-14 | 1.1129e-16 |
| 0.6 | 3.0074e-15 | 7.2997e-15 | 1.4516e-14 | 5.1566e-14 |
| 0.8 | 3.6872e-14 | 4.9405e-15 | 7.9603e-14 | 3.2345e-14 |
| 1 | 0 | 0 | 0 | 0 |

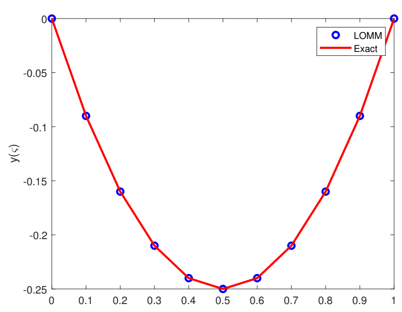


FIGURE 3. The exact and approximate solutions obtained by the LOM method at $\delta = 0.01e^{-\varsigma}$, $\alpha(\varsigma) = tanh(\varsigma+1)$ and $\varrho = 1$ for Example 2

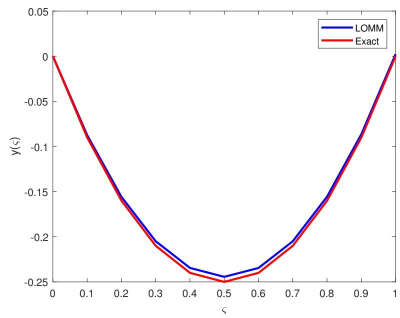


FIGURE 4. The exact and approximate solutions obtained by the LOM method at $\delta = 0.01e^{-\varsigma}$, $\alpha(\varsigma) = tanh(\varsigma + 1)$ and $\varrho = 1.01$ for Example 2

TABLE 4. The absolute error obtained by the LOM with different values of $\delta(\varsigma)$ with $\alpha = 1$ and $\varrho = 1$ for Example 2

| $\delta(\varsigma)$ | $0.3e^{-\varsigma}$ | $0.2e^{-\varsigma}$ | $0.01e^{-\varsigma}$ | $0.001e^{-\varsigma}$ |
|---|---|---|---|---|
| 0 | 9.9680e-11 | 4.2101e-11 | 5.8857e-11 | 5.6542e-11 |
| 0.2 | 5.0119e-8 | 6.5958e-9 | 7.5860e-10 | 3.4353e-10 |
| 0.4 | 3.9362e-8 | 2.9786e-9 | 1.0259e-9 | 5.7404e-10 |
| 0.6 | 4.2135e-8 | 4.1935e-9 | 7.1883e-10 | 2.7247e-10 |
| 0.8 | 4.2147e-8 | 4.3708e-9 | 1.1320e-9 | 7.2496e-10 |
| 1 | 7.8288e-8 | 3.9253e-8 | 1.7041e-8 | 1.7056e-8 |

**Example 5.3.** ([25]) Consider the fractional nonlinear pantograph of VOFDETVD

$$D^{\alpha(\varsigma),\varrho}y(\varsigma) = 1 - 2y^2(0.5\varsigma),$$
$$0 \leq \varsigma \leq 1, \quad 1 < \alpha \leq 2.$$

The initial condition of the subject is $y(0) = 1$, $y^{'}(0) = 0$ and $\delta(\varsigma) = 0$. The exact solution, when $\alpha(\varsigma) = 2$ is $y(\varsigma) = cos(\varsigma)$.

TABLE 5. Comparison of the absolute error with other methods and various values of $\Omega$ at $\alpha(\varsigma) = 2$ and $\varrho = 1$ for Example 3

| $\varsigma$ | LOM($\Omega = 5$) | BWOM($\Omega = 5$) | LOM($\Omega = 7$) | BWOM($\Omega = 7$) |
|---|---|---|---|---|
| 0 | 5.9300e-11 | 2.3300e-7 | 6.9500e-11 | 1.0500e-10 |
| 0.2 | 1.0300e-6 | 8.0100e-8 | 2.9700e-9 | 3.2100e-11 |
| 0.4 | 1.7700e-6 | 3.7800e-8 | 5.4900e-9 | 3.8100e-11 |
| 0.6 | 2.3100e-6 | 1.0100e-4 | 7.5800e-9 | 1.3100e-6 |
| 0.8 | 7.2700e-6 | 1.4200e-4 | 1.1000e-10 | 1.8200e-6 |

TABLE 6. Comparison of the absolute error with other methods and various values of $\alpha(\varsigma)$ at $\Omega = 8$ and $\varrho = 1$ for Example 3

| $\varsigma$ | $\alpha(\varsigma) = 1.97$ | $\alpha(\varsigma) = 1.98$ | $\alpha(\varsigma) = 1.99$ | $\alpha(\varsigma) = 2$ |
|---|---|---|---|---|
| 0.1 | 4.1516e-4 | 2.7296e-4 | 1.3461e-4 | 1.6081e-9 |
| 0.3 | 2.6228e-3 | 1.7326e-3 | 8.5846e-4 | 4.2617e-9 |
| 0.5 | 5.3487e-3 | 3.5454e-3 | 1.7625e-3 | 6.5591e-9 |
| 0.7 | 7.6429e-3 | 5.0826e-3 | 2.5348e-3 | 8.1353e-9 |
| 0.9 | 8.6932e-3 | 5.8074e-3 | 2.9092e-3 | 4.7964e-8 |

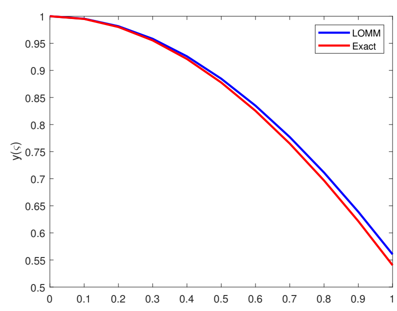


FIGURE 5. The exact and approximate solutions obtained by the LOM method at $\Omega = 7$, $\alpha(\varsigma) = 2$ and $\varrho = 1.01$ for Example 3

## 6. Conclusion

This study introduces an OM based on SLPs for solving variable-order generalized Caputo-type fractional time-varying delay equations. The recent development of this derivative has led to the emergence of a new class of differential equations. To demonstrate the efficacy and precision of the proposed approach, several illustrative examples were presented and solved. The results were compared with those obtained from other established numerical methods, revealing that the current methodology is both convenient and effective. When behavior and memory effects are incorporated into the formulations of the studied models, numerical solutions can reveal additional dynamic aspects.

For many variable orders, the systems exhibit stable periodic orbits, while other orders display rich and complex dynamics. The proposed technique has the potential to be applied to a wide range of biological systems, such as the mathematical modeling of infectious disease dynamics, as well as other important disciplines, including finance, engineering, and economics. This article opens up numerous new avenues for studying the modeling of real-world situations.

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