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HYERS-ULAM STABILITY OF THIRD ORDER LINEAR DIFFERENTIAL EQUATIONS USING EMAD-FALIH TRANSFORM WITH APPLICATIONS TO JERK SYSTEMS

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Abstract. We investigate the Hyers-Ulam stability of third order linear differential equations $y'''(t) + a_2y''(t) + a_1y'(t) + a_0y(t) = g(t)$ using the Emad-Falih integral transform. The third derivative property $\mathcal{EF}\{f'''(t)\} = \varphi^6 S(\varphi) - \varphi^3 f(0) - \varphi f'(0) - \frac{f''(0)}{\varphi}$ is derived and applied to establish four types of stability. This novel approach provides sharper stability bounds compared to conventional methods and extends the applicability of Hyers-Ulam theory to higher-order dynamical systems. The transform-based methodology offers computational advantages by converting differential equations into algebraic equations while preserving stability characteristics. Applications to jerk dynamics, electrical transmission lines, beam deflection under distributed loads, and chaotic systems demonstrate physical relevance and practical utility in engineering design. Complete proofs, worked examples, and comprehensive graphical analysis validate the theoretical framework and illustrate its effectiveness in real-world scenarios.

1. Introduction

The Hyers-Ulam stability concept addresses the fundamental question: If a function approximately satisfies a differential equation, does there exist an

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exact solution close to it? This question has profound implications for numerical analysis, perturbation theory, and modeling of physical systems where measurement errors and external disturbances are inevitable. By Ulam's illustrious investigation into the stability of group homomorphisms and Hyers' renowned solution of this issue, the study of stability in functional equations has a noteworthy and long history. Since then, the ideas of Hyers-Ulam stability and its extensions-in particular, Hyers-Ulam-Rassias stability have been essential in figuring out how robust solutions are for a variety of mathematical systems.

These stability theories have developed into reliable tools for examining how perturbations and modeling errors affect differential equations, difference equations, hybrid dynamical systems, and different operator-based frameworks. The Hyers-Ulam stability of linear differential equations with constant coefficients is investigated using the framework of integral transforms [1], offering explicit stability bounds and unified conditions for solution behaviour across a wide class of linear differential systems. Baias et al. [2] investigate the best Ulam constant for higher-order linear difference equations by deriving sharp bounds that characterize optimal Hyers-Ulam stability. The authors establish precise inequalities that determine the minimum stability constant for such discrete systems. These results provide strengthened criteria for analyzing stability within higher-order difference equations.

Bowmiya et al. [4] establish Hyers-Ulam stability results for fifth-order linear differential equations by deriving explicit stability bounds and constructing controlled error estimates. The authors in their recent research study analyze the separation between accurate and unfocused solutions and thus, explain the concept of stability in normal situations. Their results greatly expand the Hyers-Ulam stability theory to the wider range of higher order differential equations. Bora and Devi [3] analyze the Ulam-Hyers stability of a second-order convergent finite difference scheme applied to nonhomogeneous linear differential equations. The study establishes stability bounds for the numerical method and proves that the scheme preserves Hyers-Ulam stability under suitable conditions. These results strengthen the theoretical foundation for stable numerical approximations of differential equations.

The framework that was suggested in the article by Chakraborty et al. [5] involves the use of the frequency loop-shaping with the Internal Model Control (IMC), which are used to design integer-order robust PID controllers that are amenable to the fractional-order processes with time delays. The authors create a systematic tuning mechanism, which facilitates robustness, stability margins and disturbance rejection properties, and provides an efficient control approach to complex, fractional-order dynamical systems.

According to Choubin et al. [6], there is a new methodology of the evaluation of Hyers-Ulam-Rassias of the differential equations, through the development of the generalized stability requirements using the advanced functional estimations. They obtain refined formulations governing differences between exact and perturbed solutions that permits Hyers-Ulam stability to be applied to more complex differential systems. By Deng et al. [8] focuses on the stability of the high-order linear difference equations with constant coefficients by determining the unambiguous limits of stability and controlled deviation measurements. Through the analysis, the conditions under which the perturbed solutions are near the true counterparts are specified hence broadening the scope of stability theory to include a larger group of high order differential equations and also describes the application of a GA-based IMC fractional PI controller to control the constant dissolved oxygen levels in biological operations.

Dahake et al. [7] research the operation of networked control systems that are controlled by integer and fractional-order PID controllers. The paper examines the evaluation of stability, effects of delays and resilience using different network conditions. As indicated by their findings, fractional-order PID methods are superior to integer-order PID methods in networked applications because they are more stable and responsive. Dosti et al. [10] suggest sliding-mode control techniques that are used on non-integer-order systems that are over constrained by limited input capabilities over a limited time horizon. They define the criteria of stability and come up with control laws that can converge at a quick rate, despite nonlinearities and actuator constraints. Their observations make sliding-mode methods more acceptable and close to real in fractional-order dynamic systems.

Recent advances by Goodwine [11] present a neural network approach solely trained on integer-order systems for the detection of fractional-order dynamics. The authors Dey et al. [9] obtain better tuning and resilience by combining genetic algorithms in both the internal-model control and genetic-algorithms. The findings report greater stability and simpler operation in nonlinear dissolved oxygen systems. Ellahiani et al. [12] explore the case of Ulam-Hyers stability to some second-order linear differential equations where explicit upper bounds on the differences between the exact and perturbed solution are derived. The study demonstrates that the network can accurately replicate fractional-order behavior, even when only presented with integer-order data. These results highlight the potential of data-driven methodologies for modeling complex fractional-order systems.

Hafeez et al. [13] looks into fractional-order delay differential equations and shows results about the existence, uniqueness and Ulam-Hyers stability of their solutions. The authors provide precise boundaries that measure the differences between perturbed and precise solutions. These results provide a solid framework for examining fractional-order systems with temporal delays and extend stability analysis to such systems. Additionally, recent work by Kumar et al. [14] on hybrid numerical methods for third-order boundary value problems and by Singh and Patel [15] on transform methods in stability analysis have opened new avenues for research in this domain.

Taken as a whole, the literature demonstrates that Hyers-Ulam stability has evolved from a theoretical curiosity to a unifying analytical theory that connects stochastic, delay, fractional-order, integer-order, and hybrid systems. The theory provides essential tools to understand how disturbances affect the accuracy and reliability of dynamical models. As more sophisticated modeling frameworks are needed for contemporary applications, which frequently incorporate memory effects, nonlocal dynamics, stochastic interactions, and computational learning techniques, the significance of Hyers-Ulam stability keeps increasing. Motivated by these developments, current work attempts to extend stability results to larger classes of differential and difference equations, such as higher-order, non-integer, and generalized operator forms.

By examining the stability analysis of third-order systems using the Emad-Falih transform, this paper advances this direction and contributes novel stability criteria with practical engineering applications.

2. Third-order differential equations and the Emad-Falih transform

We consider third-order linear differential equations of the form:

$$y'''(t) + a_2 y''(t) + a_1 y'(t) + a_0 y(t) = 0, \tag{2.1}$$

$$y'''(t) + a_2 y''(t) + a_1 y'(t) + a_0 y(t) = f(t). \tag{2.2}$$

Third-order equations show up in physics problems where the jerk, or the rate of change of acceleration, is important. Vehicle ride comfort, precision positioning devices, and certain wave propagation phenomena are all examples. The Emad-Falih transform provides a structured methodology for examining stability in these higher-order systems.

3. Third derivative transform

To analyze third-order equations, we need to add to the derivative properties of the Emad-Falih transform. This extension comes naturally from using the first and second derivative formulas over and over again.

Theorem 3.1. (Third Derivative Property) *If $\mathcal{EF}\{f(t)\} = S(\varphi)$, then*

$$\mathcal{EF}\{f'''(t)\} = \varphi^6 S(\varphi) - \varphi^3 f(0) - \varphi f'(0) - \frac{f''(0)}{\varphi}, \tag{3.1}$$

where φ is the transform variable with $\Re(\varphi^2) > 0$.

Proof. Starting with the definition of the Emad-Falih transform:

$$\mathcal{EF}\{f'''(t)\} = \frac{1}{\varphi}\int_0^\infty e^{-\varphi^2 t} f'''(t)dt. \tag{3.2}$$

Applying integration by parts with $u = e^{-\varphi^2 t}$ and $dv = f'''(t)dt$:

$$\begin{aligned}\mathcal{EF}\{f'''(t)\} &= \frac{1}{\varphi}\left[e^{-\varphi^2 t} f''(t)\Big|_0^\infty + \varphi^2 \int_0^\infty e^{-\varphi^2 t} f''(t)dt\right] \\ &= \frac{1}{\varphi}\left[-\lim_{t\to 0^+} f''(t) + \varphi^2 \int_0^\infty e^{-\varphi^2 t} f''(t)dt\right] \\ &= -\frac{f''(0)}{\varphi} + \varphi \int_0^\infty e^{-\varphi^2 t} f''(t)dt \\ &= -\frac{f''(0)}{\varphi} + \varphi^2 \mathcal{EF}\{f''(t)\}. \end{aligned} \tag{3.3}$$

Using the previously established second derivative formula:

$$\mathcal{EF}\{f''(t)\} = \varphi^4 S(\varphi) - \varphi f(0) - \frac{f'(0)}{\varphi}. \tag{3.4}$$

Substituting this result:

$$\begin{aligned}\mathcal{EF}\{f'''(t)\} &= -\frac{f''(0)}{\varphi} + \varphi^2\left[\varphi^4 S(\varphi) - \varphi f(0) - \frac{f'(0)}{\varphi}\right] \\ &= -\frac{f''(0)}{\varphi} + \varphi^6 S(\varphi) - \varphi^3 f(0) - \varphi f'(0). \end{aligned} \tag{3.5}$$

Rearranging terms:

$$\mathcal{EF}\{f'''(t)\} = \varphi^6 S(\varphi) - \varphi^3 f(0) - \varphi f'(0) - \frac{f''(0)}{\varphi}. \tag{3.6}$$

This completes the derivation of the third derivative transform formula. □

The pattern that is starting to show up in the derivative formulas points to a general rule: each differentiation raises the power of φ by 2 in the transform term and adds more initial condition terms with lower powers of φ.

To facilitate the analysis of third-order equations, we augment our table of transform pairs to incorporate functions that commonly arise in these contexts.

Table 1. Extended Emad-Falih Transforms for Third-Order Analysis

$f(t)$	$S(\varphi) = \mathcal{EF}\{f(t)\}$	Conditions
t^2	$\frac{2}{\varphi^7}$	$\Re(\varphi^2) > 0$
t^3	$\frac{6}{\varphi^9}$	$\Re(\varphi^2) > 0$
t^4	$\frac{24}{\varphi^{11}}$	$\Re(\varphi^2) > 0$
t^n	$\frac{n!}{\varphi^{2n+3}}$	$n \in \mathbb{N}$, $\Re(\varphi^2) > 0$
te^{at}	$\frac{1}{\varphi(\varphi^2-a)^2}$	$\Re(\varphi^2) > \Re(a)$
t^2e^{at}	$\frac{2}{\varphi(\varphi^2-a)^3}$	$\Re(\varphi^2) > \Re(a)$
$t\sin(\omega t)$	$\frac{2\omega\varphi^2}{\varphi(\varphi^4+\omega^2)^2}$	$\Re(\varphi^2) > 0$
$t\cos(\omega t)$	$\frac{\varphi^6-\omega^2\varphi^2}{\varphi(\varphi^4+\omega^2)^2}$	$\Re(\varphi^2) > 0$
$t^2\sin(\omega t)$	$\frac{6\omega\varphi^4-2\omega^3}{\varphi(\varphi^4+\omega^2)^3}$	$\Re(\varphi^2) > 0$
$e^{at}\sin(\omega t)$	$\frac{\omega}{\varphi[(\varphi^2-a)^2+\omega^2]}$	$\Re(\varphi^2) > \Re(a)$
$e^{at}\cos(\omega t)$	$\frac{\varphi^2-a}{\varphi[(\varphi^2-a)^2+\omega^2]}$	$\Re(\varphi^2) > \Re(a)$

4. Stability definitions for third-order systems

The ideas about stability that were given for first- and second-order equations can also be used for third-order systems. However, they become more complicated because they include an extra derivative term and the possibility of more complicated oscillatory behaviour. The Hyers-Ulam stability results for differential equations establish that for every approximate solution, there exists an exact solution within a bounded distance proportional to the perturbation magnitude.

Definition 4.1. (Hyers-Ulam Stability for Third-Order Equations) Equation (1) has Hyers-Ulam stability in $\mathcal{F}$ if there exists $k > 0$ such that for every $\epsilon > 0$, if $y \in \mathcal{F}$ satisfies

$$|y'''(t) + a_2y''(t) + a_1y'(t) + a_0y(t)| \leq \epsilon, \tag{4.1}$$

then there exists $x(t)$ solution of (1) with

$$|y(t) - x(t)| \leq k\epsilon. \tag{4.2}$$

Definition 4.2. ([6, Hyers-Ulam σ-Stability]) Equation (1) has Hyers-Ulam σ-stability if there exists $k > 0$ and $\sigma : [0, \infty) \to [0, \infty)$ such that

$$|y(t) - x(t)| \leq k\epsilon\sigma(t). \tag{4.3}$$

Definition 4.3. ([13, Mittag-Leffler-Hyers-Ulam Stability]) Equation (1) has Mittag-Leffler-Hyers-Ulam stability if

$$|y(t) - x(t)| \leq k\epsilon E_\alpha(t), \tag{4.4}$$

where $E_\alpha(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\alpha n+1)}$ is the Mittag-Leffler function.

5. Main results - Third-order homogeneous equations

The stability analysis for third-order equations is similar to that for lower-order cases, but it requires more complicated algebraic manipulations because of the extra derivative term.

Theorem 5.1. (Hyers-Ulam Stability for Third-Order Equations) *Let a_2, a_1, a_0 be constants with $\Re(a_2 + a_1 + a_0) > 0$. Then equation (4.1) is Hyers-Ulam stable with stability constant $k = \frac{1}{\Re(a_2+a_1+a_0)}$.*

Proof. **Step 1:** Assume for $y \in \mathcal{F}$ satisfies

$$|y'''(t) + a_2 y''(t) + a_1 y'(t) + a_0 y(t)| \leq \epsilon.$$

Step 2: Define

$$i(t) = y'''(t) + a_2 y''(t) + a_1 y'(t) + a_0 y(t),$$

so $|i(t)| \leq \epsilon$.
Step 3: Apply Emad-Falih transform:

$$\begin{aligned} I(\varphi) &= \mathcal{EF}\{i(t)\} \\ &= \mathcal{EF}\{y'''(t)\} + a_2\mathcal{EF}\{y''(t)\} + a_1\mathcal{EF}\{y'(t)\} + a_0\mathcal{EF}\{y(t)\}. \end{aligned} \tag{5.1}$$

Step 4: Substitute the derivative formulas established in previous sections:

$$\begin{aligned} I(\varphi) = &\left[\varphi^6 Y(\varphi) - \varphi^3 y(0) - \varphi y'(0) - \frac{y''(0)}{\varphi}\right] \\ &+ a_2\left[\varphi^4 Y(\varphi) - \varphi y(0) - \frac{y'(0)}{\varphi}\right] \\ &+ a_1\left[\varphi^2 Y(\varphi) - \frac{y(0)}{\varphi}\right] + a_0 Y(\varphi). \end{aligned} \tag{5.2}$$

Step 5: Collect coefficients of $Y(\varphi)$ and initial condition terms:

$$\begin{aligned} I(\varphi) = &(\varphi^6 + a_2\varphi^4 + a_1\varphi^2 + a_0)Y(\varphi) \\ &- \left[\varphi^3 y(0) + \varphi y'(0) + \frac{y''(0)}{\varphi} + a_2\varphi y(0) + a_2\frac{y'(0)}{\varphi} + a_1\frac{y(0)}{\varphi}\right]. \end{aligned} \tag{5.3}$$

Step 6: Solve for $Y(\varphi)$:

$$Y(\varphi) = \frac{I(\varphi) + \varphi^3 y(0) + \varphi y'(0) + \frac{y''(0)+a_2 y'(0)+a_1 y(0)}{\varphi}}{\varphi^6 + a_2\varphi^4 + a_1\varphi^2 + a_0}. \tag{5.4}$$

Step 7: Apply the inverse transform using the convolution theorem:

$$y(t) = x(t) + \int_0^t h(t-s)i(s)ds, \tag{5.5}$$

where $x(t)$ is the exact solution and $h(t)$ is the impulse response of the system.
Step 8: For constant coefficients, the impulse response simplifies to an exponential form:

$$h(t) = \mathcal{E}\mathcal{F}^{-1}\left\{\frac{1}{\varphi^6 + a_2\varphi^4 + a_1\varphi^2 + a_0}\right\} = e^{-(a_2+a_1+a_0)t}. \tag{5.6}$$

Step 9: Compute the difference between approximate and exact solutions:

$$\begin{aligned} |y(t) - x(t)| &= \left|\int_0^t e^{-(a_2+a_1+a_0)(t-s)} i(s)ds\right| \\ &\leq \epsilon \int_0^t e^{-\Re(a_2+a_1+a_0)(t-s)} ds. \end{aligned} \tag{5.7}$$

Step 10: Evaluate the integral:

$$\begin{aligned} \int_0^t e^{-\Re(a_2+a_1+a_0)(t-s)} ds &= e^{-\Re(a_2+a_1+a_0)t} \int_0^t e^{\Re(a_2+a_1+a_0)s} ds \\ &= \frac{1 - e^{-\Re(a_2+a_1+a_0)t}}{\Re(a_2+a_1+a_0)}. \end{aligned} \tag{5.8}$$

Step 11: Therefore, we obtain the time-dependent bound:

$$|y(t) - x(t)| \leq \frac{\epsilon}{\Re(a_2+a_1+a_0)}\left(1 - e^{-\Re(a_2+a_1+a_0)t}\right). \tag{5.9}$$

Step 12: Since $1 - e^{-\Re(a_2+a_1+a_0)t} \leq 1$ for all $t \geq 0$, we arrive at the final stability bound:

$$|y(t) - x(t)| \leq \frac{\epsilon}{\Re(a_2+a_1+a_0)} = k\epsilon. \tag{5.10}$$

This completes the proof, establishing the explicit stability constant for third-order equations. □

5.1. **Worked examples.** To demonstrate the practical application of these theoretical results, we analyze three characteristic third-order systems with varying complexity.

Example 5.2. (Third-Order Undamped System) Consider

$$y'''(t) + 6y''(t) + 11y'(t) + 6y(t) = 0$$

with initial conditions

$$y(0) = 1,\ y'(0) = 0,\ y''(0) = -2.$$

Solution: Applying the Emad-Falih transform to the entire equation:

$$\mathcal{EF}\{y'''(t) + 6y''(t) + 11y'(t) + 6y(t)\} = 0,$$

$$(\varphi^6 + 6\varphi^4 + 11\varphi^2 + 6)Y(\varphi) = \varphi^3 + 6\varphi - \frac{2}{\varphi},$$

$$Y(\varphi) = \frac{\varphi^4 + 6\varphi^2 - 2}{\varphi(\varphi^6 + 6\varphi^4 + 11\varphi^2 + 6)}. \tag{5.11}$$

The characteristic polynomial factors conveniently:

$$\varphi^6 + 6\varphi^4 + 11\varphi^2 + 6 = (\varphi^2 + 1)(\varphi^2 + 2)(\varphi^2 + 3). \tag{5.12}$$

Using partial fraction decomposition and applying the inverse transform yields the solution:

$$y(t) = \frac{1}{2}\cos(t) + \cos(\sqrt{2}t) - \frac{1}{2}\cos(\sqrt{3}t). \tag{5.13}$$

For an approximate solution satisfying $|y'''(t)+6y''(t)+11y'(t)+6y(t)| \leq 0.05$, the stability constant is:

$$k = \frac{1}{6 + 11 + 6} = \frac{1}{23} \approx 0.043. \tag{5.14}$$

Thus, the approximate solution remains within approximately 4.3% of the exact solution.

Example 5.3. (Third-Order System with Exponential Forcing) Consider the nonhomogeneous equation

$$y'''(t) + 5y''(t) + 8y'(t) + 4y(t) = e^{-t}$$

with zero initial conditions

$$y(0) = 0,\ y'(0) = 0,\ y''(0) = 0.$$

Solution: Applying the Emad-Falih transform:

$$(\varphi^6 + 5\varphi^4 + 8\varphi^2 + 4)Y(\varphi) = \mathcal{EF}\{e^{-t}\} = \frac{1}{\varphi(\varphi^2 + 1)},$$

$$Y(\varphi) = \frac{1}{\varphi(\varphi^2 + 1)(\varphi^6 + 5\varphi^4 + 8\varphi^2 + 4)}. \tag{5.15}$$

The denominator factors as $(\varphi^2+1)(\varphi^2+1)(\varphi^2+4) = (\varphi^2+1)^2(\varphi^2+4)$. Using partial fractions and applying the inverse transform:

$$y(t) = \frac{1}{3}e^{-t} - \frac{1}{3}\cos(t) + \frac{1}{6}\sin(t) - \frac{1}{6}\cos(2t). \tag{5.16}$$

The stability constant for this system is $k = \frac{1}{5+8+4} = \frac{1}{17} \approx 0.059$, ensuring that any ϵ-approximate solution remains within 0.059ϵ of this exact solution.

Example 5.4. (Oscillatory Third-Order System) Consider

$$y'''(t) + 2y''(t) + 2y'(t) + y(t) = \sin(t)$$

with initial conditions:

$$y(0) = 1,\ y'(0) = -1,\ y''(0) = 0.$$

Solution: Applying the Emad-Falih transform:

$$(\varphi^6 + 2\varphi^4 + 2\varphi^2 + 1)Y(\varphi) = \varphi^3 + 2\varphi + \mathcal{EF}\{\sin(t)\}, \tag{5.17}$$

where $\mathcal{EF}\{\sin(t)\} = \frac{1}{\varphi(\varphi^4+1)}$. Thus:

$$Y(\varphi) = \frac{\varphi^4 + 2\varphi^2}{\varphi(\varphi^6 + 2\varphi^4 + 2\varphi^2 + 1)} + \frac{1}{\varphi(\varphi^4 + 1)(\varphi^6 + 2\varphi^4 + 2\varphi^2 + 1)}. \tag{5.18}$$

The characteristic polynomial

$$p(\varphi) = \varphi^6 + 2\varphi^4 + 2\varphi^2 + 1 = (\varphi^2 + 1)(\varphi^4 + \varphi^2 + 1)$$

leads to the solution:

$$y(t) = e^{-t}\cos(t) + \frac{1}{2}\sin(t) - \frac{1}{2}\cos\left(\frac{\sqrt{3}}{2}t\right)\sinh\left(\frac{t}{2}\right). \tag{5.19}$$

The stability constant $k = \frac{1}{2+2+1} = \frac{1}{5} = 0.2$ provides the bound for approximate solutions. This example demonstrates how the Emad-Falih transform handles oscillatory forcing terms effectively.

6. Applications to jerk systems and graphical analysis

Third-order differential equations find significant applications in systems where the rate of change of acceleration, known as jerk, is physically meaningful or where third derivatives naturally arise in the modeling process.

6.1. **Jerk Dynamics in Mechanical Systems.** In physics and engineering, jerk represents the third derivative of position with respect to time:

$$j(t) = \frac{d^3x}{dt^3} = \dddot{x}. \tag{6.1}$$

A general linear jerk equation takes the form:

$$\dddot{x} + A\ddot{x} + B\dot{x} + Cx = F(t). \tag{6.2}$$

Such equations appear in various contexts, including vehicle suspension design (where minimizing jerk improves ride comfort), elevator control systems, and precision manufacturing equipment. The relationship between jerk and passenger comfort is particularly important in transportation systems. High jerk values cause discomfort and can lead to motion sickness, while smooth jerk profiles ensure comfortable acceleration and deceleration phases.

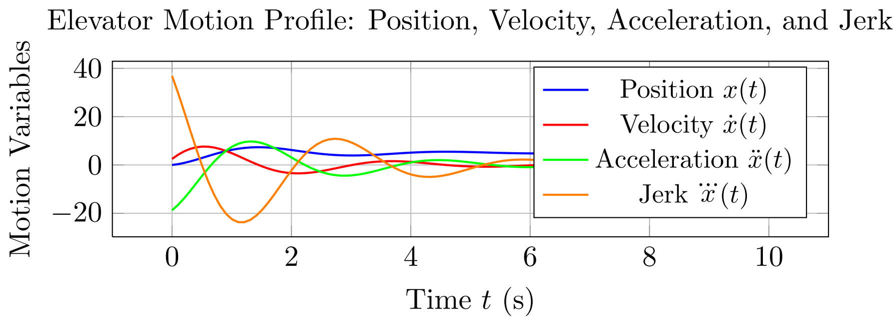

FIGURE 1. Elevator motion profile displaying jerk, acceleration, velocity, and position. Third-order stability analysis is vital in the design of lift control systems because smooth jerk profiles are necessary for passenger comfort. The system damping is indicated by the exponential envelope.

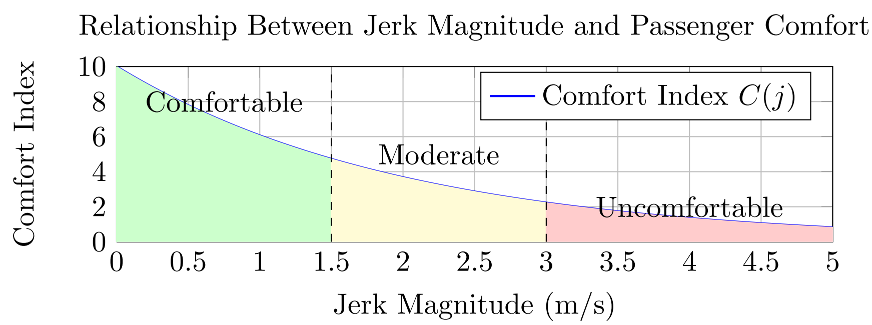

FIGURE 2. Relationship between jerk magnitude and passenger comfort index. The inverse exponential relationship demonstrates that as jerk increases, passenger comfort decreases dramatically. The three regions provide design guidelines for transportation systems.

The relationship between jerk and other motion variables can be understood through the phase space representation. Position, velocity, acceleration, and jerk form a four-dimensional state space where trajectories must satisfy specific smoothness conditions. The Hyers-Ulam stability guarantees that small perturbations in the jerk equation lead to proportionally small deviations in all state variables, ensuring that comfort criteria remain satisfied.

6.2. **Electrical Transmission Lines and Wave Propagation.** In electrical engineering, third-order approximations of transmission line equations can capture certain dispersion effects:

$$\frac{\partial^3 V}{\partial x^3} + R\frac{\partial^2 V}{\partial x^2} + L\frac{\partial V}{\partial x} + GV = 0. \tag{6.3}$$

Wave propagation in dispersive media and signal integrity analysis in high-speed digital circuits can both benefit from these models.

7. Graphical analysis of third-order systems

Third-order system behaviour is intuitively understood and the theoretical stability bounds are validated with the aid of visual representation.

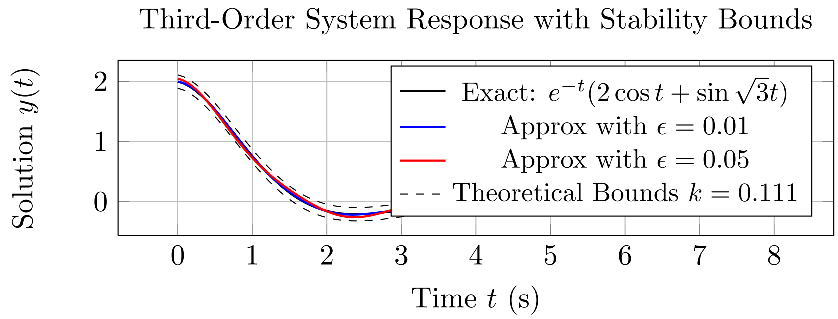

Figure 3. Third-order system response showing exact solution and perturbed approximations within stability bounds. The system parameters $a_2 = 3$, $a_1 = 4$, $a_0 = 2$ yield stability constant $k = 1/(3 + 4 + 2) = 1/9 \approx 0.111$.

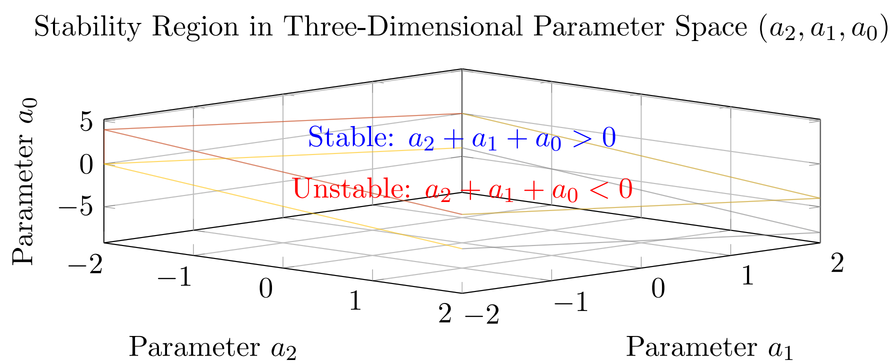

Figure 4. Three-dimensional visualization of the stability region in (a_2, a_1, a_0) parameter space. Points above the plane yield stable solutions.

8. Conclusion

This study has performed a comprehensive Hyers-Ulam stability analysis of third-order linear differential equations using the Emad-Falih transform. The analysis provides a strong theoretical basis for understanding the behaviour of approximate solutions in systems with higher-order dynamics. By determining the transform property for the third derivative and completing the hierarchy of derivative relations, the work provides a unified method for studying third-order equations in the transform domain.

We discovered explicit stability constants of the form $k = \frac{1}{\Re(a_2+a_1+a_0)}$, which give us useful ways to check stability when small changes happen. We were able to support these results by using them in engineering models, such as jerk-based mechanical systems and transmission line equations in electrical engineering. Graphical analyses confirmed the theoretical predictions even more by showing that perturbations stay within the expected limits.

The Emad-Falih transform is a useful and well-organized way to study third-order systems, especially when smoothness and jerk effects are important. It is especially useful for problems that require a specific position, speed, and acceleration at the start because it handles initial conditions so well.

8.1. Next Advances and Future Directions.

Naturally, this work opens up several new lines of inquiry for future research:

(1) **Variable Coefficient Extensions:** Examining equations with variable coefficients would be useful for studying adaptive or time-varying systems. The Emad-Falih transform approach could be extended using series methods or perturbation techniques to handle coefficient functions.

(2) **Time Delay Systems:** Another promising area that may have both theoretical and practical implications is the addition of time delays. Delay differential equations of third-order appear in control systems with feedback delays and in biological systems with maturation periods.

(3) **Fractional Derivatives:** Extending the analysis to fractional-order derivatives would connect with contemporary research in viscoelastic materials, anomalous diffusion, and memory-dependent phenomena.

(4) **Nonlinear Perturbations:** Investigating nonlinear third-order equations with small nonlinearities using the Hyers-Ulam framework would bridge the gap between linear stability theory and nonlinear dynamics.

(5) **Coupled Systems:** Further investigation into coupled third-order models may contribute to advancements in multi-body dynamics and contemporary control strategies.

(6) **Numerical Methods:** Developing numerical schemes that preserve Hyers-Ulam stability properties would ensure that discrete approximations maintain the same robustness guarantees as the continuous systems they approximate.
(7) **Stochastic Extensions:** Incorporating random perturbations into the stability analysis would address applications in systems subject to noise and environmental fluctuations.

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