Nonlinear Functional Analysis and Applications
Vol. 17, No. 2 (2012), pp. 187-197

http://nfaa.kyungnam.ac.kr/jour-nfaa.htm

STRONG CONVERGENCE THEOREM FOR A COMMON ZERO OF $m-$ACCRETIVE MAPPINGS IN BANACH SPACES BY VISCOSITY APPROXIMATION METHODS

Truong Minh Tuyen

College of Sciences, Thainguyen University
Thainguyen, Vietnam
e-mail: tm.tuyentm@gmail.com

Abstract. In this paper, we study an iterative algorithm by viscosity method to approximate a common zero of a finite family of $m-$accretive mappings in a reflexive Banach space, which has a weakly continuous duality mapping.

1. Introduction

Let E be a real Banach space and C be a nonempty convex subset of E. Let J denote the normalized duality mapping from E into 2^{E^*} given by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}, \ \forall x \in E,$$

where E^* denotes the dual space of E and $\langle ., .\rangle$ denote the generalized duality pairing. It is well known that if E^* is strictly convex then J is single-valued. In the sequel, we shall denote the single-valued normalized duallity mapping by j. Recall that a self-mapping $f : C \to C$ is contraction on C if there exists a constant $\alpha \in (0, 1)$ such that

$$\|f(x) - f(y)\| \le \alpha\|x - y\|, \ \forall x, y \in C.$$

We use $\prod_C$ to denote collection of all contraction mappings on C. That is, $\prod_C = \{f : f : C \to C$ is a contraction mapping$\}$. Note that each $f \in \prod_C$ has a unique fixed point in C. Also, recall that a mapping $T : C \to C$ is called nonexpansive if

$$\|T(x) - T(y)\| \le \|x - y\|, \ \forall x, y \in C,$$

[0]Received October 8, 2011. Revised May 12, 2012.
[0]2000 Mathematics Subject Classification: 47H06, 47H09, 47H10, 47J25.
[0]Keywords: Accretive operators, weak sequential continuous mapping, contraction mapping.

and T is called pseudocontractive if there exists $j(x-y) \in J(x-y)$ such that

$$\langle T(x) - T(y), j(x-y)\rangle \le \|x-y\|^2, \ \forall x, y \in C.$$

A point $x \in C$ is a fixed point of T provided $T(x) = x$. Denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in C : \ T(x) = x\}$. For a real number $t \in (0,1)$ and a contraction mapping $f \in \prod_C$, we define a mapping $T_t(x) = tf(x) + (1-t)T(x)$ for all $x \in C$, where T is nonexpansive mapping on C. It is obviously that T_t is a contraction mapping on C. Let x_t be the unique fixed point of T_t. That is, x_t is the unique solution of the fixed point equation

$$x_t = tf(x_t) + (1-t)T(x_t). \tag{1.1}$$

A special case has been considered by Browder [2] in a Hilbert space as follow: Fix $u \in C$ and define a contraction mapping S_t on C by

$$S_t(x) = tu + (1-t)T(x), \ \forall x \in C.$$

If z_t is the unique fixed point of S_t, then $z_t = tu + (1-t)T(z_t)$.

In 1967, Browder [3] proved that, in a Hilbert space H, as $t \to 0$, $\{z_t\}$ converges strongly to a fixed point of T which is closets to u, that is, the nearest point projection of u onto $F(T)$.

In 2000, Moudafi [11] proposed a viscosity approximation method which was considered by many authors [5, 6, 11, 12, 13, 14, 17, 19, 20] of selecting a particular fixed point of a given nonexpansive mapping in Hilbert spaces. If H is a Hilbert space, $T : \ C \to C$ is nonexpansive self-mapping on a nonempty closed convex C of H and $f : \ C \to C$ is a contraction mapping, he proved the following results:

(1) The sequence $\{x_n\}$ in C generated by the iterative scheme:

$$x_0 \in C, \ x_n = \frac{1}{1+\varepsilon_n} T(x_n) + \frac{\varepsilon_n}{1+\varepsilon_n} f(x_n), \ \forall n \ge 0,$$

converges strongly to the unique solution of the variational inequality

$$\overline{x} \in F(T) \text{ such that } \langle (I-f)(\overline{x}), \overline{x} - x\rangle \le 0, \ \forall x \in F(T),$$

where $\{\varepsilon_n\}$ is a sequence of positive numbers tending to zero.

(2) With a initial $z_0 \in C$, define the sequence $\{z_n\}$ in C by

$$z_{n+1} = \frac{1}{1+\varepsilon_n} T(z_n) + \frac{\varepsilon_n}{1+\varepsilon_n} f(z_n), \ \forall n \ge 0.$$

Suppose that $\lim_{n\to\infty} \varepsilon_n = 0$ and $\sum_{n=1}^{\infty} \varepsilon_n = +\infty$, and $\lim_{n\to\infty} \left| \frac{1}{\varepsilon_{n+1}} - \frac{1}{\varepsilon_n} \right| = 0$. Then $\{z_n\}$ converges strongly to the unique solution of the variational inequality

$$\overline{x} \in F(T) \text{ such that } \langle (I-f)(\overline{x}), \overline{x} - x\rangle \le 0, \ \forall x \in F(T).$$

Recall that an (possibly multi-valued) operator A with the domain $D(A)$ and range $R(A)$ in E is accretive if, for $x, y \in D(A)$ and $u \in A(x),\ v \in A(y)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle u - v, j(x-y)\rangle \geq 0.$$

An accretive operator is said to be $m-$accretive if $R(I + rA) = E$ for each $r > 0$. The set of zeros of A is denoted by $N(A)$. Hence we have

$$N(A) = \{z \in D(A) :\ 0 \in A(z)\} = A^{-1}(0).$$

For each $r > 0$, we denote by J_r the resolvent of A, i.e., $J_r = (I + rA)^{-1}$. Note that, if A is $m-$accretive operator, then $J_r :\ E \to D(A)$ is a nonexpansive single-valued mapping and $F(J_r) = N(A)$.

Forward, we will assume that J_r is a mapping from E to $C = \overline{D(A)}$ and C is convex.

Kim and Xu [9] and Xu [16] studied the sequence $\{x_n\}$ generated by the following iterative algorithm

$$x_0 \in C,\ x_{n+1} = \alpha_n u + (1-\alpha_n)J_{r_n}(x_n),\ \forall n \geq 0, \tag{1.2}$$

where C is a closed convex subset of a Banach space E and J_{r_n} is a resolvent of a accretive operator. They proved strong convergence theorems of the iterative (1.2) in the framework of uniformly smooth Banach spaces and reflexive Banach spaces, respectively. Xu [17] studied the following iterative scheme by viscosity approximation method introduced by Moudafi [11]:

$$x_0 \in C,\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T(x_n),\ \forall n \geq 0,$$

where $f \in \prod_C$ and T is nonexpansive mapping, and obtained a strong convergence theorem in uniformly smooth Banach spaces.

Chen and Zhu [6] improved the results of Xu [16, 17] and also studied the so-called viscosity approximation methods. More precisely, they considered the following

$$x_t = tf(x_t) + (1-t)T(x_t), \tag{1.3}$$

$$x_0 \in C,\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)J_{r_n}(x_n),\ \forall n \geq 0, \tag{1.4}$$

where C is a closed convex subset of a Banach space E and J_{r_n} is the resolvent of a accretive operator, and obtained the strong convergence theorems for nonexpansive mappings and $m-$accretive mappings in reflexive Banach spaces, respectively.

When A is maximal monotone in Hilbert space H, in 2006, Xu [17]; in 2009, Song and Yang [18] used the technique of nonexpansive mappings to get convergence theorems for $\{x_n\}$ defined by the perturbed version of the proximal point algorithm

$$x_{n+1} = J^A_{r_n}(t_n u + (1-t_n)x_n + e_n),\ u \in H, \tag{1.5}$$

and proved strong convergence of iterative (1.5) to a zero of A.

Zegeye and Shahzed [20] studied the convergence problem of finding a common zero of a finite family of $m-$accretive mappings. More precisely, they proved the following result.

Theorem 1.1. ([20]) *Let E be strictly convex and reflexive Banach space with a uniformly Gateaux differentiable norm, K be a nonempty closed convex subset of E and $A_i: K \to E$ $(i = 1, 2, ..., r)$ be a family of $m-$accretive mappings with $\cap_{i=1}^r N(A_i) \neq \emptyset$. For any $u, x_0 \in K$, let $\{x_n\}$ be a sequence in K generated by the algorithm:*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) S_r(x_n), \ \forall n \geq 0, \tag{1.6}$$

where $S_r := a_0 I + a_1 J_{A_1} + a_2 J_{A_2} + ... + a_r J_{A_r}$ with $J_{A_i} = (I + A_i)^{-1}$ for $0 < a_i < 1$, $i = 0, 1, 2, ..., r$, $\sum_{i=0}^r a_i = 1$ and $\{\alpha_n\}$ is real sequence which satisfies the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$;

(ii) $\sum_{n=1}^\infty |\alpha_n - \alpha_{n-1}| < \infty$ *or* $\lim_{n\to\infty} \dfrac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} = 0.$

If every nonempty closed bounded convex subset of E has the fixed point property for nonexpansive mapping, then $\{x_n\}$ converges strongly to a common solution of the equations $A_i(x) = 0$ for $i = 1, 2, ..., r$.

Motivated by Xu [15] and Zegeye and Shahzed [20], in this paper we introduce an iterative algorithm as follow:

$$x_0 \in C, \ x_{n+1} = S_r(\alpha_n f(x_n) + (1 - \alpha_n) x_n), \ \forall n \geq 0, \tag{1.7}$$

where $S_r := a_0 I + a_1 J_{A_1} + a_2 J_{A_2} + ... + a_r J_{A_r}$ with $a_0, a_1, ..., a_r$ be real numbers in $(0, 1)$ such that $\sum_{i=0}^r a_i = 1$ and $\{\alpha_n\} \subset (0, 1)$ be real sequence of positive numbers.

We prove strong convergence theorems of iterative algorithm (1.7) for a finite family of $m-$accretive mappings in a Banach space E by viscosity approximation method.

2. Preliminaries

Let E be a real Banach space with dual E^*. The norm on E is said to be uniformly Gateaux differentiable if for each $y \in S_E = \{x \in X : \ \|x\| = 1\}$ the limit $\lim_{t\to 0} \dfrac{\|x + ty\| - \|x\|}{t}$ exists uniformly for $x \in S_E$.

Recall that a gauge is a continuous strictly incresing function $\varphi : [0, \infty) \to [0, \infty)$ such that $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$. The duality mapping $J_\varphi : E \to 2^{E^*}$ associated to a gauge φ is defined by

$$J_\varphi(x) = \{f \in E^* : \ \langle x, f\rangle = \|x\|\varphi(\|x\|), \ \|x^*\| = \varphi(\|x\|)\}, \ \forall x \in E.$$

Following Browder [4], we say that a Banach space E has a weakly continuous duality mapping if there exists a gauge φ for which the duality mapping J_φ is single-valued and weak-to-weak* sequentially continuous, i.e., for each $\{x_n\} \subset E$ with $x_n \rightharpoonup x$, $J_\varphi(x_n) \overset{*}{\rightharpoonup} J_\varphi(x)$. it is well known that l^p has a weakly continuous duality mapping for all $1 < p < \infty$. Set $\Phi(t) = \int_0^t \varphi(\tau)d\tau,\ t \geq 0$. Then $J_\varphi(x) = \partial\Phi(\|x\|),\ \forall x \in E$, where ∂ denotes the sub-diffrential in the sense of convex analysis.

A Banach space E is said to be strictly convex if for $a_i \in (0,1),\ i = 1, 1, ..., r$, such that $\sum_{i=1}^r a_i = 1$ we have $\|a_1x_1 + a_2x_2 + ... + a_rx_r\| < 1$ for $x_i \in E,\ i = 1, 2, ..., r$ with $\|x_i\| = 1,\ i = 1, 2, ..., r$ and $x_i \neq x_j$, for some $i \neq j$.

In what follows, we shall make use of the following lemmas and theorems.

Lemma 2.1. ([1, 18]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \sigma_n,\ \forall n \geq 0,$$

where $\{\alpha_n\} \subset (0,1)$ for each $n \geq 0$ such that (i) $\lim_{n\to\infty} \alpha_n = 0$; (ii) $\sum_{n=1}^\infty \alpha_n = \infty$. Suppose either (a) $\sigma_n = o(\alpha_n)$, or (b) $\sum_{n=1}^\infty |\sigma_n| < \infty$, or (c) $\limsup \dfrac{\sigma_n}{\alpha_n} \leq 0$. Then $a_n \to 0$ as $n \to \infty$.

Lemma 2.2. ([10]) *Assume that a Banach space E has a weakly continuous duality mapping J_φ with a gauge φ.*

(i) *For all $x, y \in E$, the following inequality holds*

$$\Phi(\|x + y\|) \leq \Phi(\|x\|) + \langle y, J_\varphi(x + y)\rangle.$$

In particular, for all $x, y \in E$

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle.$$

(ii) *Assume that the sequence $\{x_n\}$ in E converges weakly to a point $x \in E$. Then the following identity holds:*

$$\limsup_{n\to\infty} \Phi(\|x_n - y\|) = \limsup_{n\to\infty} \Phi(\|x_n - x\|) + \Phi(\|y - x\|),\ \forall y \in E.$$

Lemma 2.3. ([20]) *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $A_i : C \to E,\ i = 1, 2, ..., r$, be a family of $m-$accretive mapping with $\cap_{i=1}^r N(A_i) \neq \emptyset$. Let $a_0, a_1, ..., a_r$ be real numbers in $(0,1)$ such that $\sum_{i=0}^r a_i = 1$ and $S_r := a_0I + a_1J_{A_1} + a_2J_{A_2} + ... + a_rJ_{A_r}$, where $J_{A_i} := (I + A_i)^{-1}$. Then S_r is nonexpansive mapping and $F(S_r) = \cap_{i=1}^r N(A_i)$.*

Lemma 2.4. ([6]) *Let E be a real reflexive Banach space and have a weakly continuous duality mapping J_φ with φ. Suppose C is a closed convex subset of E, and $T : C \to C$ is a nonexpansive mapping, let $f : C \to C$ be a fixed contraction mapping. For $t \in (0,1)$, $\{x_t\}$ is defined by (1.3). Then T has a* *fixed point if and only if* $\{x_t\}$ *remains bounded as* $t \to 0^+$*, and in this case,* $\{x_t\}$ *converges strongly to a fixed point of* T *as* $t \to 0^+$.

Let $Q : \prod_C \to F(T)$ by $Q(f) := \lim_{t\to 0^+} x_t,\ f \in \prod_C$. Cho and Qin [7] showed the following inequality

$$\langle (I-f)Q(f), J_\varphi(Q(f)-p)\rangle \le 0,\ \forall p \in F(T). \tag{2.1}$$

Theorem 2.5. ([8]) *Let* A *be a continuous and accretive operator on the real Banach space* E *with* $D(A) = E$*. Then* A *is* $m-$*accretive.*

3. Main results

Now, we give our main results in this paper.

Theorem 3.1. *Let* E *be a strictly convex and reflexive Banach space which has a weakly continuous duality mapping* J_φ *with gauge* φ*. Let* C *be a nonempty closed convex subset of* E *and* $f \in \prod_C$ *with the contractive coefficient* $c \in (0,1)$*. Let* $A_i : C \to E$, $i = 1,2,...,r$*, be a finite family of* $m-$*accretive mappings with* $\cap_{i=1}^r N(A_i) \neq \emptyset$*. Let* $J_{A_i} = (I + A_i)^{-1}$ *for* $i = 1,2,...,r$*. For any* $x_0 \in C$*, let* $\{x_n\}$ *be a sequence generated by algorithm (1.7). If the sequence* $\{\alpha_n\}$ *satisfies the following conditions*

(i) $\lim_{n\to\infty} \alpha_n = 0$,
(ii) $\sum_{n=1}^\infty \alpha_n = \infty$,
(iii) $\sum_{n=1}^\infty |\alpha_n - \alpha_{n-1}| < \infty$ *or* (iii)* $\lim_{n\to\infty} \dfrac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} = 0$,

then $\{x_n\}$ *converges strongly to a common solution of the equations* $A_i(x) = 0$ *for* $i = 1,2,...,r$.

Proof. By Lemma 2.3, we have that $F(S_r) = \cap_{i=1}^r N(A_i) \neq \emptyset$. Now, for each $p \in F(S_r)$, we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|S_r(\alpha_n f(x_n) + (1-\alpha_n)x_n) - S_r(p)\| \\
&\le \|\alpha_n(f(x_n) - f(p)) + (1-\alpha_n)(x_n - p) + \alpha_n(f(p) - p)\| \\
&\le [1 - \alpha_n(1-c)]\|x_n - p\| + \alpha_n(1-c)\frac{\|f(p)-p\|}{1-c} \\
&\le \max\left\{\|x_n - p\|, \frac{\|f(p)-p\|}{1-c}\right\} \qquad (3.1) \\
&\ \vdots \\
&\le \max\left\{\|x_0 - p\|, \frac{\|f(p)-p\|}{1-c}\right\}.
\end{aligned}$$

Hence $\{x_n\}$, $\{f(x_n)\}$ are bounded and suppose that $\max\{\sup\|x_n\|, \sup\|f(x_n)\|\} \leq K$. It follows that

$$\begin{aligned}\|x_{n+1} - S_r(x_n)\| &= \|S_r(\alpha_n f(x_n) + (1-\alpha_n)x_n) - S_r(x_n)\| \\ &\leq \alpha_n\|f(x_n) - x_n\| \to 0, \text{as } n \to \infty.\end{aligned} \tag{3.2}$$

From (1.7) we get that

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|S_r(\alpha_n f(x_n) + (1-\alpha_n)x_n) - S_r(\alpha_{n-1}f(x_{n-1}) + (1-\alpha_{n-1})x_{n-1})\| \\ &\leq \alpha_n\|f(x_n) - f(x_{n-1})\| + |\alpha_n - \alpha_{n-1}|\|f(x_{n-1})\| \\ &\quad + (1-\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|x_{n-1}\| \\ &\leq [1 - \alpha_n(1-c)]\|x_n - x_{n-1}\| + (1-c)\alpha_n\beta_n,\end{aligned}$$

where $\beta_n = 2K\dfrac{|\alpha_n - \alpha_{n-1}|}{\alpha_n(1-c)}$. We consider two cases.

Case 1. Condition (iii) is satisfied. Then

$$\|x_{n+1} - x_n\| \leq [1 - \alpha_n(1-c)]\|x_n - x_{n-1}\| + \sigma_n,$$

where $\sigma_n = 2K|\alpha_n - \alpha_{n-1}|$ so that $\sum_{n=1}^{\infty}\sigma_n < \infty$.

Case 2. Condition (iii)* is satisfied. Then,

$$\|x_{n+1} - x_n\| \leq [1 - \alpha_n(1-c)]\|x_n - x_{n-1}\| + \sigma_n,$$

where $\sigma_n = (1-c)\alpha_n\beta_n$ so that $\sigma_n = o((1-c)\alpha_n)$.

In either case, Lemma 2.1 yields that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$ and hence by (3.2) we obtain that

$$\|x_n - S_r(x_n)\| \leq \|x_{n+1} - x_n\| + \|x_{n+1} - S_r(x_n)\| \to 0 \text{ as } n \to \infty. \tag{3.3}$$

Next, we prove that

$$\limsup_{n\to\infty}\langle (I-f)Q(f), J_\varphi(Q(f) - x_n)\rangle \leq 0, \tag{3.4}$$

where $Q(f)$ is defined by Lemma 2.4. Take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\begin{aligned}&\limsup_{n\to\infty}\langle (I-f)Q(f), J_\varphi(Q(f) - x_n)\rangle \\ &= \lim_{n\to\infty}\langle (I-f)Q(f), J_\varphi(Q(f) - x_{n_k})\rangle.\end{aligned} \tag{3.5}$$

Since Banach space E is reflexive, we may further assume that $x_{n_k} \rightharpoonup \overline{x}$ for some $\overline{x} \in C$. Since the duality mapping J_φ is weakly continuous, we have, by Lemma 2.2,

$$\limsup_{n\to\infty}\Phi(\|x_{n_k} - x\|) = \limsup_{n\to\infty}\Phi(\|x_{n_k} - \overline{x}\|) + \Phi(\|x - \overline{x}\|),\ \forall x \in E.$$

Putting

$$g(x) = \limsup_{n\to\infty}\Phi(\|x_{n_k} - x\|),\ \forall x \in E,$$

then it follows that

$$g(x) = g(\overline{x}) + \Phi(\|x - \overline{x}\|),\ \forall x \in E. \tag{3.6}$$

Thus, from (3.3), we arrive at

$$\begin{aligned} g(S_r(\overline{x})) &= \limsup_{n\to\infty} \Phi(\|x_{n_k} - S_r(\overline{x})\|) \\ &= \limsup_{n\to\infty} \Phi(\|S_r(x_{n_k}) - S_r(\overline{x})\|) \\ &\le \limsup_{n\to\infty} \Phi(\|x_{n_k} - \overline{x}\|) = g(\overline{x}). \end{aligned} \tag{3.7}$$

On the other hand, from (3.6), we have

$$g(S_r(\overline{x})) - g(\overline{x}) = \Phi(\|S_r(\overline{x}) - \overline{x}\|). \tag{3.8}$$

Combining (3.7) and (3.8), we get $\Phi(\|S_r(\overline{x}) - \overline{x}\|) \le 0$. Hence we have $S_r(\overline{x}) = \overline{x}$, that is, $\overline{x} \in F(S_r)$. It follows that

$$\begin{aligned} &\limsup_{n\to\infty} \langle (I - f)Q(f), J_\varphi(Q(f) - x_n)\rangle \\ &= \langle (I - f)Q(f), J_\varphi(Q(f) - \overline{x})\rangle \le 0. \end{aligned}$$

That is, (3.4) holds.

Now, we prove the sequence $\{x_n\}$ converges strongly to $Q(f)$ as $n \to \infty$. By the property of Φ and by Lemma 2.2, we have

$$\begin{aligned} \Phi(\|x_{n+1} - Q(f)\|) &= \Phi(\|S_r(\alpha_n f(x_n) + (1 - \alpha_n)x_n) - S_r(Q(f))\|) \\ &\le \Phi(\|\alpha_n f(x_n) + (1 - \alpha_n)x_n - Q(f)\|) \\ &\le \Phi(\alpha_n\|f(x_n) - f(Q(f))\| + \alpha_n\|f(Q(f)) - Q(f)\| \\ &\quad + (1 - \alpha_n)\|x_n - Q(f)\|) \\ &\le \Phi([1 - \alpha_n(1 - c)]\|x_n - Q(f)\| + \alpha_n\|f(Q(f)) - Q(f)\|) \\ &\le \Phi([1 - \alpha_n(1 - c)]\|x_n - Q(f)\|) \\ &\quad + \alpha_n\langle f(Q(f)) - Q(f), J_\varphi(x_{n+1} - Q(f))\rangle \\ &\le [1 - \alpha_n(1 - c)]\Phi(\|x_n - Q(f)\|) \\ &\quad + \alpha_n\langle f(Q(f)) - Q(f), J_\varphi(x_{n+1} - Q(f))\rangle. \end{aligned}$$

By the condition (i) and (3.4), we know that all the conditions in Lemma 2.1 are satisfied. Therefore, it follows that $\Phi(\|x_{n+1} - Q(f)\|) \to 0$ as $n \to \infty$, that is, $x_n \to Q(f)$. □

Remark 3.2. If we take $r = 1$, then we may take $S_1 := J_A = (I + A)^{-1}$ and that strict convexity of E and real constant $a_i,\ i = 0, 1$, may not be needed.

Corollary 3.3. *Let E be a reflexive Banach space which has a weakly continuous duality mapping J_φ with gauge φ. Let C be a nonempty closed convex subset of E and $f \in \prod_C$ with the contractive coefficient $c \in (0,1)$. Let $A: C \to E$ be an $m-$accretive mapping with $N(A) \neq \emptyset$. For given $x_0 \in C$, let $\{x_n\}$ be generated by*

$$x_{n+1} := J_A(\alpha_n f(x_n) + (1-\alpha_n)x_n), \ \forall n \geq 0, \tag{3.9}$$

where $J_A := (I+A)^{-1}$ and $\{\alpha_n\} \subset (0,1)$. If the sequence $\{\alpha_n\}$ satisfies the following conditions

(i) $\lim_{n\to\infty} \alpha_n = 0$,

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ *or* (iii)* $\lim_{n\to\infty} \dfrac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} = 0$,

then $\{x_n\}$ converges strongly to a common solution of the equations $A_i = 0$ for $i = 1, 2, ..., r$.

Proof. The proof follows as in the proof of Theorem 3.1 with use of Remark 3.2. □

Remark 3.4. The Corollary 3.3 is more general than the result of Xu [15] (Theorem 3.3). The result of Xu [15] is only a particular case of Corollary 3.3, when E is a Hilbert space and $f(x) = u$ for all $x \in C$.

Theorem 3.5. *Let E be a strictly convex and reflexive Banach space which has a weakly continuous duality mapping J_φ with gauge φ. Let C be a nonempty closed convex subset of E and $f \in \prod_C$ with the contractive coefficient $c \in (0,1)$. Let $T_i: E \to E$, $i = 1, 2, ..., r$ be a family of continuous pseudo-contractive mappings on E with $\cap_{i=1}^r F(T_i) \neq \emptyset$. Let $J_{T_i} := (2I - T_i)^{-1}$ for $i = 1, 2, ..., r$. For given $x_0 \in E$, let $\{x_n\}$ be generated by*

$$x_{n+1} := S_r(\alpha_n f(x_n) + (1-\alpha_n)x_n), \ \forall n \geq 0, \tag{3.10}$$

where $S_r = a_0 I + a_1 J_{T_1} + ... + a_r J_{T_r}$, for $0 < a_i < 1$, $i = 1, 2, ..., r$, $\sum_{i=0}^r a_i = 1$. and $\{\alpha_n\} \subset (0,1)$. If the sequence $\{\alpha_n\}$ satisfies the following conditions

(i) $\lim_{n\to\infty} \alpha_n = 0$,

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ *or* (iii)* $\lim_{n\to\infty} \dfrac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} = 0$,

then $\{x_n\}$ converges strongly to a common fixed point of $T_1, T_2, ..., T_r$.

Proof. For each $i = 1, 2, ..., r$, then $A_i = I - T_i$ is continuous accretive with $D(A_i) = E$. Hence, from Theorem 2.5, we deduce A_i is $m-$accretive operator. Apply Theorem 3.1, we obtain the proof of this theorem. □

Now, we consider a single pseudocontractive mapping, we obtain the analogue of Corollary 3.3.

Corollary 3.6. *Let E be a reflexive Banach space which has a weakly continuous duality mapping J_φ with gauge φ. Let C be a nonempty closed convex subset of E and $f \in \prod_C$ with the contractive coefficient $c \in (0,1)$. Let $T: E \to E$ be a continuous pseudocontractive mapping on E with $F(T) \neq \emptyset$. Let $J_T = (2I - T)^{-1}$. For given $x_0 \in E$, let $\{x_n\}$ be generated by the algorithm*

$$x_{n+1} := J_T(\alpha_n f(x_n) + (1-\alpha_n)x_n),\ \forall n \geq 0, \tag{3.11}$$

where $\{\alpha_n\} \subset (0,1)$. If the sequence $\{\alpha_n\}$ satisfies the following conditions

(i) $\lim_{n\to\infty} \alpha_n = 0$,
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(iii) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ *or* (iii)* $\lim_{n\to\infty} \dfrac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} = 0$,

then $\{x_n\}$ converges strongly to a fixed point of T.

References

[1] V. Berinde, *Iterative Approximation of Fixed Points*, (2002), Editura Efemeride, Baia Mare.
[2] F. E. Browder, *Fixed points theorems for noncompact mappings in Hilbert spaces*, Proc. Natl. Acad. Sci. USA **53** (1965), 1272-1276.
[3] F. E. Browder, *Convergence of approximations to fixed points of nonexpansive mappings in Banach spaces*, Arch. Ration. Mech. Anal. **24** (1967), 82-90.
[4] F. E. Browder, *Covergence theorems for sequences of nonlinear operators in Banach space*, Math. Z. **100** (1967), 201-225.
[5] S. S. Chang, *Viscosity approximation methods for a finite family of nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl. **323** (2006), 1402-1416.
[6] R. Chen and Z. Zhu, *Viscosity approximation fixed points for nonexpansive and maccretive operators*, Fixed Point Theory and Appl. **2006**, Article ID 81325, Pages 1-10.
[7] Y. J. Cho and X. Qin, *Viscosity approximation methods for a family of $m-$accretive mappings in reflexive Banach spaces*, Positivity **12** (2008), 483-494.
[8] I. Cioranescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, (1990), Kluwer Academic Publishers, Dordrecht.
[9] T. H. Kim and H. K. Xu, *Strong convergence of modified Mann iterations*, Nonl. Appl. **61** (2005), 51-60.
[10] T. C. Lim and H. K. Xu, *Fixed point theorems for asymptotically nonexpansive mappings*, Nonl. Anal. **22** (1994), 1345-1355.
[11] A. Moudafi, *Vicosity approximation methods for fixed point problems*, J. Math. Anal. Appl. **241** (2000), 45-55.
[12] X. Qin and Y. Su, *Approximation of zero point of accretive operator in Banach spaces*, J. Math. Anal. Appl. **329** (2007), 415-424.
[13] Y. Song, R. Chen and H. Zhou, *Viscosity approximation methods for nonexpansive mapping sequences in Banach spaces*, Nonl. Anal. **66** (2007), 1016-1024.

[14] S. Takahashi and W. Takahashi, *Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces*, J. Math. Anal. Appl. **331** (2007), 506-515.

[15] H.-K. Xu, *A regularization method for the proximal point algorithm,* J. Glob. Optim. **36** (2006), 115-125.

[16] H.-K. Xu, *Strong convergence of an iterative method for nonexpansive and acretive operators,* J. Math. Anal. Appl. **314** (2006), 631-643.

[17] H. K. Xu, *Viscosity approximation methods for nonexpansive mappings*, J. Math. Anal. Appl. **298** (2004), 279-291.

[18] H. K. Xu, *Iterative algorithms for nonlinear operators*, J. London Math. Soc. **66** (2002), 240-256.

[19] H. Yao and M. A. Noor, *On viscosity iterative methods for variational inequalities*, J. Math. Anal. Appl. **325** (2007), 776-787.

[20] H. Zegeye and N. Shahzad, *Strong convergence theorems for a common zero of a finite family of m-accretive mappings*, Nonlinear Anal. **66** (2007), 1161-1169.