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ITERATIVE METHOD FOR SOLVING QUASI-EQUILIBRIUM PROBLEMS AND ZERO POINT PROBLEM IN HADAMARD SPACES

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Abstract. In this paper, we propose a new iterative algorithm for finding a common element of the set of solutions of the quasi-equilibrium problem and a common zero of a finite family of monotone operators in Hadamard spaces. We obtain weak convergence of the sequence to a solution of the quasi-equilibrium problem and a common zero of a finite family of monotone operators under some mild assumptions. Then we show strong convergence of the generated sequence to a solution of the problem by imposing some additional conditions. Finally, a numerical experiment is reported to illustrate the efficiency of the proposed algorithm.

1. Introduction

Consider a metric space denoted by (X, d). Given any pair of points $x, y \in X$, a mapping $c : [0, l] \to X$ (with $l \geq 0$) is referred to as a geodesic linking x and y provided that $c(0) = x$, $c(l) = y$, and the isometric condition

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$d(c(t), c(t')) = |t - t'|$ holds for all $t, t' \in [0, l]$. Whenever such a geodesic can be established for every pair $x, y \in X$, the structure (X, d) is designated a geodesic metric space. Additionally, if this connecting geodesic is singular for each $x, y \in X$, we characterize (X, d) as uniquely geodesic.

In a uniquely geodesic space X, a subset C is defined as convex if, for any choice of $x, y \in C$, the complete geodesic path between x and y remains inside C. For any $x, y \in X$, the range of the geodesic map c connecting these endpoints is termed the geodesic segment between x and y, which we symbolize as $[x, y]$.

Assume X is a uniquely geodesic metric space. Then, for arbitrary $x, y \in X$ and any parameter $t \in [0, 1]$, there is exactly one point $z \in [x, y]$ that satisfies the distance relations $d(x, z) = (1 - t)d(x, y)$ and $d(y, z) = td(x, y)$. We adopt the notation $tx \oplus (1 - t)y$ to represent this uniquely determined point z.

Definition 1.1. ([8]) A geodesic space X is termed a $CAT(0)$ space if, for every triple $x, y, z \in X$ and parameter $t \in [0, 1]$, the following inequality is satisfied:

$$d^2(tx \oplus (1 - t)y, z) \leq td^2(x, z) + (1 - t)d^2(y, z) - t(1-t)d^2(x, y).$$

Whenever such a structure is additionally complete, it is referred to as an Hadamard space.

Berg and Nikolaev in [2, 3] proposed the framework of quasi-linearization, which can be described as follows:

For any ordered pair $(a, b) \in X \times X$, we formally assign the notation $\overrightarrow{ab}$ and refer to it as a vector. Within this setting, quasi-linearization is established as a mapping $\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ given by the rule:

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}\Big(d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d)\Big), \quad a, b, c, d \in X. \quad (1.1)$$

Direct verification confirms that the relations $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle$ and $\langle \overrightarrow{ax}, \overrightarrow{cd} \rangle + \langle \overrightarrow{xb}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$ hold universally for all $a, b, c, d, x \in X$. The space X is said to fulfill the Cauchy-Schwarz inequality whenever the bound $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a, b)d(c, d)$ is valid across all $a, b, c, d \in X$. As established in Corollary 3 of [3], a geodesically connected metric space qualifies as a $CAT(0)$ space precisely when it adheres to this Cauchy-Schwarz condition.

Suppose (X, d) represents an Hadamard space and $\{x_n\}$ denotes a bounded sequence within X. For an arbitrary point $x \in X$, define the functional

$$r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n).$$

The asymptotic radius associated with $\{x_n\}$ is then defined as:

$$r(\{x_n\}){=}\inf\{r(x,\{x_n\}):x{\in}X\},$$

while the asymptotic center of $\{x_n\}$ corresponds to the collection

$$A(\{x_n\}){=}\{x{\in}X:r(x,\{x_n\}){=}r(\{x_n\})\}.$$

Within Hadamard spaces, it is a well-established result that $A(\{x_n\})$ consists of exactly one point.

Definition 1.2. ([21]) Within an Hadamard space (X,d), a sequence $\{x_n\}$ is said to $\triangle$-converge to a limit $x{\in}X$ provided that $A(\{x_{n_k}\}){=}\{x\}$ holds for every subsequence $\{x_{n_k}\}$ extracted from $\{x_n\}$.

We shall employ the symbol $\overset{\triangle}{\longrightarrow}$ to indicate $\triangle$-convergence within X, whereas ordinary metric convergence will be represented by $\rightarrow$.

The following established property concerning $\triangle$-convergence is recalled next.

Lemma 1.3. ([21]) *Let X be an Hadamard space. Then, each bounded, closed, and convex subset of X possesses $\triangle$-compactness; specifically, any bounded sequence contained therein admits a $\triangle$-convergent subsequence.*

Lemma 1.4. ([8]) *Let (X,d) be a $CAT(0)$ space. Then, for arbitrary points $x,y,z{\in}X$ and any parameter $t{\in}[0,1]$, the inequality*

$$d(tx\oplus(1-t)y,z)\leq td(x,z)+(1-t)d(y,z)$$

is satisfied.

Definition 1.5. Consider an Hadamard space (X,d) and a subset $C{\subset}X$. We define the metric projection onto C as the mapping $P_C{:}X{\to}C$, where $P_C(x)$ designates the element $u{\in}C$ fulfilling

$$d(u,x)=\inf\{d(z,x):z\in C\}. \tag{1.2}$$

It is a standard fact that whenever $C{\subset}X$ is nonempty, closed, and convex, each $x{\in}X$ admits precisely one point $u{\in}C$ that meets condition (1.2).

The following result provides an alternative characterization for this projection operator.

Proposition 1.6. ([7]) *Let C represent a nonempty convex subset within a $CAT(0)$ space X, and let $x{\in}X$ and $u{\in}C$ be given. Then the equality $u{=}P_C(x)$ holds precisely when the condition*

$$\langle\overrightarrow{yu},\overrightarrow{xu}\rangle\leq 0,\quad\forall y\in C$$

is fulfilled.

Assume that X is an Hadamard space containing a nonempty, closed, and convex subset $C \subset X$. Consider a set-valued operator $K : C \to 2^C$ with the property that $K(x)$ constitutes a nonempty, closed, and convex subset of C for every $x \in C$. Let $f : X \times X \to \mathbb{R}$ denote a given bifunction. We formulate the quasi-equilibrium problem $(QEP(f, K))$ as the task of determining a point $x^* \in K(x^*)$ satisfying the condition

$$f(x^*, y) \geq 0, \quad \forall y \in K(x^*). \tag{1.3}$$

We represent the complete collection of solutions to $QEP(f, K)$ by $S(f, K)$. Furthermore, $F(K)$ stands for the fixed point set of the mapping K. The corresponding Minty-type quasi-equilibrium problem requires identifying an element $x^* \in K(x^*)$ for which $f(y, x^*) \leq 0$ holds across all $y \in K(x^*)$. In the special scenario where $K(x) = C$ for every $x \in C$, the formulation $QEP(f, K)$ simplifies to the standard equilibrium problem $EP(f, C)$, and its Minty counterpart correspondingly reduces to the traditional Minty equilibrium problem (refer to [9]). A prominent instance of a quasi-equilibrium problem is provided by quasi-variational inequality problems.

Building upon the foundational work of Berg and Nikolaev [3], Kakavandi and Amini [18] formulated the notion of a dual space for an Hadamard space X in the following manner.

Define the mapping $\Theta : \mathbb{R} \times X \times X \to C(X, \mathbb{R})$ through the relation:

$$\Theta(t, a, b)(x) = t\langle \overrightarrow{ab}, \overrightarrow{ax} \rangle, \quad (a, b, x \in X, t \in \mathbb{R}),$$

wherein $C(X, \mathbb{R})$ represents the set of all continuous real-valued functions on $\mathbb{R} \times X \times X$. An application of the Cauchy-Schwarz inequality ensures that each $\Theta(t, a, b)$ is Lipschitz continuous, possessing a Lipschitz semi-norm given by $L(\Theta(t, a, b)) = |t| d(a, b)$, valid for every $t \in \mathbb{R}$ and $a, b \in X$. Here, the Lipschitz semi-norm for an arbitrary function $\varphi : X \to \mathbb{R}$ is expressed as

$$L(\varphi) = \sup \left\{ \frac{\varphi(x) - \varphi(y)}{d(x, y)}; x, y \in X, \, x \neq y \right\}.$$

We then introduce a pseudometric D over $\mathbb{R} \times X \times X$ via the formula:

$$D((t, a, b), (s, c, d)) = L(\Theta(t, a, b) - \Theta(s, c, d)), \quad (a, b, c, d \in X, \; t, s \in \mathbb{R}).$$

Within the context of an Hadamard space (X, d), the resulting pseudometric space $(\mathbb{R} \times X \times X, D)$ may be viewed as embedded within the pseudometric space comprising all real-valued Lipschitz functions, denoted by $(Lip(X, \mathbb{R}), L)$.

According to Lemma 2.1 in [18], the condition $D((t, a, b), (s, c, d)) = 0$ is equivalent to the requirement that $t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle = s\langle \overrightarrow{cd}, \overrightarrow{xy} \rangle$ for every pair $x, y \in X$.

Consequently, D generates an equivalence relation on $\mathbb{R}\times X\times X$, yielding the equivalence class associated with (t,a,b) as

$$[t\overrightarrow{ab}]=\{s\overrightarrow{cd};\ t\langle\overrightarrow{ab},\overrightarrow{xy}\rangle=s\langle\overrightarrow{cd},\overrightarrow{xy}\rangle,\ \forall x,y\in X\}.$$

The collection $X^*:=\{[t\overrightarrow{ab}];\ (t,a,b)\in\mathbb{R}\times X\times X\}$, equipped with the distance $D([t\overrightarrow{ab}],[s\overrightarrow{cd}]):=D((t,a,b),(s,c,d))$, forms a metric space known as the dual space of (X,d). Observe that $[\overrightarrow{aa}]=[\overrightarrow{bb}]$ holds universally for any $a,b\in X$. By selecting a reference point $o\in X$, we designate $\mathbf{0}=[\overrightarrow{oo}]$ to serve as the zero element of this dual space. Furthermore, X^* operates on $X\times X$ according to the rule:

$$\langle x^*,\overrightarrow{xy}\rangle=t\langle\overrightarrow{ab},\overrightarrow{xy}\rangle,\ (x^*=[t\overrightarrow{ab}]\in X^*,\ x,y\in X).$$

Suppose $K(\cdot)$ represents a set-valued operator mapping C into itself, with the property that $K(x)$ constitutes a nonempty, closed, and convex subset of C for each $x{\in}C$. Let $T{:}X{\to}X^*$ be a given operator and define

$$f(x,y){=}\langle T(x),y{-}x\rangle,$$

where $\langle\cdot,\cdot\rangle{:}X^*{\times}X{\to}\mathbb{R}$ signifies the duality pairing, specifically satisfying $\langle z,x\rangle=z(x)$. Under this framework, $QEP(f,K)$ coincides precisely with the quasi-variational inequality problem $QVIP(T,K)$, which seeks a point $x^*{\in}K(x^*)$ fulfilling

$$\langle T(x^*),x{-}x^*\rangle{\geq}0,\quad\forall x\in K(x^*).$$

This theoretical framework unifies several fundamental mathematical models, including convex optimization, variational inequalities, fixed point theory, Nash equilibrium formulations, and various other applied challenges. A substantial body of literature addresses equilibrium problems across Hilbert spaces, Banach spaces, and more general topological vector spaces; see, for instance, the works cited in ([4, 5, 6, 12, 13, 14, 15, 16, 23, 26]).

In the setting of Hadamard spaces, Reich and Salinas [25] proved metric convergence results for infinite compositions of operators that may lack continuity; related developments can be found in [26]. Furthermore, numerous researchers have investigated quasi-equilibrium problems governed by monotone or pseudo-monotone bifunctions within Hilbert, Banach, and topological vector space environments (consult, e.g., [10], [28]).

The present work introduces a novel iterative scheme designed to locate a point that simultaneously belongs to the solution set of a quasi-equilibrium problem and the common zero set of a finite collection of monotone operators within Hadamard spaces. Under relatively relaxed hypotheses, we establish the weak convergence of the generated iterates toward a solution that addresses both the quasi-equilibrium formulation and the common zero condition for the monotone operators. Subsequently, by introducing further structural requirements, we demonstrate the strong convergence of the iterative sequence to a solution of the underlying problem (refer also to [22]). Finally, a numerical experiment is reported to illustrate the efficiency of the proposed algorithm.

2. PRELIMINARIES

Assume that X is an Hadamard space possessing a dual space X^*. Consider a set-valued operator $A{:}X\rightrightarrows X^*$ characterized by its domain

$$D(A){:=}\{x{\in}X, Ax{\neq}\emptyset\},$$

its range $R(A){:=}\bigcup_{x\in X} Ax$, the inverse mapping $A^{-1}(x^*){=}\{x{\in}X, x^*{\in}Ax\}$, and its graph $gra(A){:=}\{(x,x^*){\in}X\times X^*, x{\in}D(A), x^*{\in}Ax\}$.

Definition 2.1. Suppose X is an Hadamard space with dual X^*. A set-valued mapping $A{:}X\rightrightarrows X^*$ is termed monotone provided that the condition $\langle x^*-y^*, \overrightarrow{yx}\rangle{\geq}0$ is satisfied for all pairs $(x,x^*),(y,y^*){\in}gra(A)$.

Definition 2.2. Consider an Hadamard space X with dual X^*, a parameter $\lambda{>}0$, and a set-valued operator $A{:}X\rightrightarrows X^*$. The resolvent of A corresponding to λ is given by the set-valued map $J_\lambda^A{:}X\rightrightarrows X$, specified through the relation

$$J_\lambda^A(x){:=}\big\{z{\in}X, [\frac{1}{\lambda}\overrightarrow{zx}]{\in}Az\big\}.$$

Equivalently, one may write

$$J_\lambda^A=(\overrightarrow{oI}+\lambda A)^{-1}\circ\overrightarrow{oI},$$

where o denotes any fixed point in X and $\overrightarrow{oI}(x){:=}[\overrightarrow{ox}]$. It is readily verified that this formulation does not depend on the specific selection of o.

Let C represent a nonempty subset within an Hadamard space X, and let $T{:}C{\to}C$ be a single-valued map. We denote the collection of fixed points associated with T by $F(T)$, which is explicitly given by

$$F(T){=}\{x{\in}C : x{=}Tx\}.$$

Definition 2.3. A function $T{:}C{\to}C$ is classified as quasi-nonexpansive if $F(T){\neq}\emptyset$ and the distance inequality $d(p,Tx){\leqslant}d(p,x)$ remains valid across all $(p,x){\in}F(T)\times C$.

Definition 2.4. Assume that $K{:}C{\to}2^C$ represents a set-valued map with the property that $K(x)$ forms a nonempty, closed, and convex set for each $x{\in}C$. The operator K is termed quasi-nonexpansive if the associated composition $T(\cdot){:=}P_{K(\cdot)}(\cdot)$ exhibits quasi-nonexpansiveness, with P standing for the metric projection.

Theorem 2.5. ([20]) *Consider a $CAT(0)$ space X equipped with a dual X^*, and let $A{:}X\rightrightarrows X^*$ denote a set-valued operator. Under these conditions, the following properties hold:*

(i) *For every $\lambda{>}0$, we have $R(J_\lambda^A){\subset}D(A)$ and $F(J_\lambda^A){=}A^{-1}(0)$.*

(ii) *Provided that A is monotone, J_λ^A reduces to a single-valued function over its domain, satisfying*

$$d^2(J_\lambda^A x, J_\lambda^A y){\leq}\langle\overrightarrow{J_\lambda^A x J_\lambda^A y}, \overrightarrow{xy}\rangle, \quad \forall x, y{\in}D(J_\lambda^A),$$

which specifically ensures that J_λ^A acts as a nonexpansive operator.

(iii) *Assuming A is monotone and $0{<}\lambda{\leq}\mu$, the estimate*

$$d^2(J_\lambda^A x, J_\mu^A x){\leq}\frac{\mu{-}\lambda}{\mu{+}\lambda}d^2(x, J_\mu^A x)$$

is valid, consequently yielding the bound $d(x, J_\lambda^A x){\leq}2d(x, J_\mu^A x)$.

A standard result asserts that when T acts as a nonexpansive map on a subset C within a $CAT(0)$ space X, the fixed point set $F(T)$ necessarily forms a closed and convex set. Consequently, whenever A is a monotone operator defined on a $CAT(0)$ space X, applying items (i) and (ii) from Theorem 2.5 guarantees that $A^{-1}(0)$ inherits closedness and convexity. Furthermore, by invoking item (ii) of the same theorem for arbitrary $u{\in}F(J_\lambda^A)$ and $x{\in}D(J_\lambda^A)$, we obtain the relation

$$d^2(J_\lambda^A x, x) \leq d^2(u, x) - d^2(u, J_\lambda^A x). \tag{2.1}$$

Definition 2.6. A set-valued operator $K{:}C{\to}2^C$ is termed demiclosed if the conditions $x_k \xrightarrow{\triangle} \overline{x}$ and $\lim\limits_{k\to\infty} d(x_k, K(x_k)){=}0$, jointly imply that $\overline{x}{\in}F(K)$.

Lemma 2.7. ([8]) *Suppose C forms a closed and convex subset within an Hadamard space X, $T{:}C{\to}C$ acts as a nonexpansive operator, and $\{x_n\}$ represents a bounded sequence in C satisfying $\lim\limits_{n\to\infty} d(x_n, Tx_n){=}0$ alongside $x_n \overset{\Delta}{\to} x$. Under these premises, it follows that $x{=}Tx$.*

Lemma 2.8. ([24]) *Consider an Hadamard space (X, d) together with a sequence $\{x_n\}$ in X. Assume that a nonempty subset C of X can be identified such that:*

(i) *The limit $\lim\limits_{n\to\infty} d(x_n, z)$ is well-defined for each $z{\in}C$.*

(ii) *Whenever a subsequence $\{x_{n_j}\}$ extracted from $\{x_n\}$ $\triangle$-converges to some $x{\in}X$, it necessarily follows that $x{\in}C$.*

Under these hypotheses, one can guarantee the existence of an element $p\in C$ to which $\{x_n\}$ $\triangle$-converges within X. The convergence analysis relies on the following assumptions imposed on the bifunction f and the set-valued mapping K:

B1: *The condition $f(x,x)=0$ holds universally for each $x\in X$.*

B2: *The function $f(x,\cdot):X\to\mathbb{R}$ exhibits convexity and lower semicontinuity across every $x\in X$.*

B3: *The mapping $f(\cdot,y)$ demonstrates $\triangle$-upper semicontinuity for any $y\in X$.*

B4: *The bifunction f possesses Lipschitz-type continuity, meaning that one can find strictly positive constants c_1 and c_2 such that*

$$f(x,y)+f(y,z)\geq f(x,z)-c_1d^2(x,y)-c_2d^2(y,z),\quad \forall x,y,z\in X.$$

B5: *The function f is pseudo-monotone, which implies that the condition $f(x,y)\geq 0$ for any $x,y\in X$, necessarily yields $f(y,x)\leq 0$.*

B6: *Each operator $K_j:C\to 2^C$, where $1\leq j\leq M$, constitutes a quasi-nonexpansive and demiclosed map whose values are consistently nonempty, closed, and convex.*

To ensure the proper definition and bounded nature of the iterative sequences produced by our proposed scheme, we postulate that

$$S^*=\left\{x\in\bigcap_{j=1}^{M}K_j(x)\cap\bigcap_{i=1}^{N}A_i^{-1}(0):f(x,y)\geq 0,\ \ \forall y\in C\right\}\neq\emptyset.$$

Observe that, given assumptions $B1-B6$, the set S^* necessarily forms a closed and convex collection. Furthermore, a straightforward verification confirms that $S^*\subset\Omega$, where $\Omega:=\bigcap_{i=1}^{N}A_i^{-1}(0)\cap S(f,\bigcap_{j=1}^{M}K_j)$.

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3. △-CONVERGENCE AND STRONG CONVERGENCE

Throughout this section, let $C \subset X$ denote a nonempty, closed, and convex subset within an Hadamard space X. Suppose the bifunction $f : X \times X \to \mathbb{R}$ fulfills assumptions $B1-B5$, and that each set-valued operator $K_j : C \to 2^C$ $(1 \leq j \leq M)$ is quasi-nonexpansive while adhering to condition $B6$. Furthermore, let $A_i : X \rightrightarrows X^*$ $(1 \leq i \leq N)$ represent a finite collection of set-valued monotone operators. The iterative procedure outlined below is designed to locate a point that simultaneously belongs to the solution set of the quasi-equilibrium problem and the common zero set of the specified family of monotone operators.

Algorithm 3.1. Initialization: Choose $z_0 \in C$, $0 < \alpha \leq \lambda_n \leq \beta < \min\left\{ \frac{1}{2c_1}, \frac{1}{2c_2} \right\}$, $0 < \gamma < \beta_n < \sigma < 1$, $\gamma_n^i \subset (0, \infty)$ and $\liminf_{n \to \infty} \gamma_n^i > 0$. Set $n=0$ and go to Step 1.

Step 1: Compute

$$t_n = J_{\gamma_n^N}^{A_N} o \cdots o J_{\gamma_n^1}^{A_1}(z_n).$$

Step 2: Compute

$$w_n = P_{K_M(t_n)} o \cdots o P_{K_1(t_n)}(t_n).$$

Step 3: Compute

$$x_n = \beta_n z_n \oplus (1 - \beta_n) w_n.$$

Step 4: Solve the following minimization problem and let y_n be the solution of it, that is,

$$y_n = argmin_{y \in C} \left\{ f(x_n, y) + \frac{1}{2\lambda_n} d^2(x_n, y) \right\}.$$

Step 5: Solve the following minimization problem and let z_{n+1} be the solution of it, that is,

$$z_{n+1} = argmin_{y\in C} \left\{ f(y_n, y) + \frac{1}{2\lambda_n} d^2(x_n, y) \right\}.$$

Step 6: Put $n := n + 1$ and go to Step 1.

Remark 3.2. It is straightforward to verify that the iterative sequences produced by Algorithm 3.1 are properly defined.

The subsequent auxiliary result plays a crucial role in establishing our primary convergence theorem.

Proposition 3.3. *Suppose that* $\{t_n\}$, $\{w_n\}$, $\{x_n\}$, $\{y_n\}$, *and* $\{z_n\}$ *denote the iterates produced by Algorithm* 3.1, *and let* $x^*\in S^*$ *be an arbitrary solution. Under these conditions, the following estimate holds:*

$$\begin{aligned} d^2(z_{n+1}, x^*) \leq & d^2(z_n, x^*) - \beta_n(1-\beta_n)d^2(z_n, w_n) \\ & - (1 - 2c_1\lambda_n)d^2(x_n, y_n) - (1 - 2c_2\lambda_n)d^2(y_n, z_{n+1}). \end{aligned}$$

Proof. From the construction of z_{n+1}, it follows directly that

$$f(y_n, z_{n+1}) + \frac{1}{2\lambda_n} d^2(x_n, z_{n+1}) \leq f(y_n, y) + \frac{1}{2\lambda_n} d^2(x_n, y), \quad \forall y \in C.$$

Inserting the convex combination $y = tz_{n+1}\oplus(1-t)x^*$ into this relation and invoking assumption $B2$ yields

$$\begin{aligned} & f(y_n, z_{n+1}) + \frac{1}{2\lambda_n} d^2(x_n, z_{n+1}) \\ & \leq f(y_n, tz_{n+1}\oplus(1-t)x^*) + \frac{1}{2\lambda_n} d^2(x_n, tz_{n+1}\oplus(1-t)x^*) \\ & \leq tf(y_n, z_{n+1}) + (1-t)f(y_n, x^*) \\ & \quad + \frac{1}{2\lambda_n}\big\{td^2(x_n, z_{n+1}) + (1-t)d^2(x_n, x^*) - t(1-t)d^2(z_{n+1}, x^*)\big\}. \end{aligned}$$

The pseudo-monotonicity of f guarantees that $f(y_n, x^*)\leq 0$. Consequently, we deduce

$$\begin{aligned} f(y_n, z_{n+1}) + \frac{1}{2\lambda_n} d^2(x_n, z_{n+1}) \leq & tf(y_n, z_{n+1}) \\ & + \frac{1}{2\lambda_n}\big\{td^2(x_n, z_{n+1}) + (1-t)d^2(x_n, x^*) - t(1-t)d^2(z_{n+1}, x^*)\big\}, \end{aligned}$$

which subsequently leads to

$$(1-t)f(y_n, z_{n+1}) \leq \frac{1}{2\lambda_n}\big\{(1-t)d^2(x_n, x^*) - t(1-t)d^2(z_{n+1}), x^*) - (1-t)d^2(x_n, z_{n+1})\big\}.$$

From this, we obtain

$$f(y_n, z_{n+1}) \leq \frac{1}{2\lambda_n}\big\{d^2(x_n, x^*) - d^2(x_n, z_{n+1}) - td^2(z_{n+1}, x^*)\big\}.$$

Taking the limit as $t \to 1^-$ in the preceding bound produces

$$f(y_n, z_{n+1}) \leq \frac{1}{2\lambda_n}\big\{d^2(x_n, x^*) - d^2(x_n, z_{n+1}) - d^2(z_{n+1}, x^*)\big\}. \tag{3.1}$$

Analogously, the defining property of y_n implies

$$f(x_n, y_n) + \frac{1}{2\lambda_n} d^2(x_n, y_n) \leq f(x_n, y) + \frac{1}{2\lambda_n} d^2(x_n, y), \quad \forall y \in C.$$

Replacing y with the convex combination $y = ty_n \oplus (1-t)z_{n+1}$ and applying $B2$ gives

$$\begin{aligned}
&f(x_n, y_n) + \frac{1}{2\lambda_n} d^2(x_n, y_n) \\
&\leq f(x_n, ty_n \oplus (1-t)z_{n+1}) + \frac{1}{2\lambda_n} d^2(x_n, ty_n \oplus (1-t)z_{n+1}) \\
&\leq tf(x_n, y_n) + (1-t)f(x_n, z_{n+1}) \\
&\quad + \frac{1}{2\lambda_n}\big\{td^2(x_n, y_n) + (1-t)d^2(x_n, z_{n+1}) - t(1-t)d^2(y_n, z_{n+1})\big\}.
\end{aligned}$$

Passing to the limit as $t \to 1^-$ in this expression yields

$$f(x_n, y_n) - f(x_n, z_{n+1}) \leq \frac{1}{2\lambda_n}\big\{d^2(x_n, z_{n+1}) - d^2(x_n, y_n) - d^2(y_n, z_{n+1})\big\}. \tag{3.2}$$

Owing to the Lipschitz-type continuity of f with parameters c_1 and c_2, the relation

$$f(x_n, y_n) + f(y_n, z_{n+1}) \geq f(x_n, z_{n+1}) - c_1 d^2(x_n, y_n) - c_2 d^2(y_n, z_{n+1})$$

holds, which immediately implies

$$f(y_n, z_{n+1}) \geq f(x_n, z_{n+1}) - f(x_n, y_n) - c_1 d^2(x_n, y_n) - c_2 d^2(y_n, z_{n+1}). \tag{3.3}$$

Merging the estimates (3.1), (3.2), and (3.3) results in

$$d^2(x_n, x^*) - d^2(z_{n+1}, x^*) \geq (1 - 2c_1\lambda_n)d^2(x_n, y_n) + (1 - 2c_2\lambda_n)d^2(y_n, z_{n+1}). \tag{3.4}$$

Recalling the construction $x_n := \beta_n z_n \oplus (1-\beta_n)w_n$, we can expand

$$\begin{aligned}
d^2(x_n, x^*) &= d^2(\beta_n z_n \oplus (1-\beta_n)w_n, x^*) \\
&\leq \beta_n d^2(z_n, x^*) + (1-\beta_n)d^2(w_n, x^*) - \beta_n(1-\beta_n)d^2(z_n, w_n).
\end{aligned} \tag{3.5}$$

Given that $x^* \in S^*$, it necessarily belongs to $\bigcap_{j=1}^{M} K_j(x^*)$.

Observing that $w_n{=}P_{K_M(t_n)}o\cdots oP_{K_1(t_n)}(t_n)$ and recognizing that each K_j $(1{\le}j{\le}M)$ acts as a quasi-nonexpansive operator, we deduce

$$d(w_n,x^*)\le d(t_n,x^*). \tag{3.6}$$

Coupling (3.5) with (3.6) provides

$$d^2(x_n,x^*)\le \beta_n d^2(z_n,x^*)+(1{-}\beta_n)d^2(t_n,x^*)-\beta_n(1-\beta_n)d^2(z_n,w_n). \tag{3.7}$$

Integrating (3.4) and (3.7) leads to

$$\begin{aligned} d^2(z_{n+1},x^*)\le&\beta_n d^2(z_n,x^*)+(1{-}\beta_n)d^2(t_n,x^*)-\beta_n(1-\beta_n)d^2(z_n,w_n)\\ &-(1-2c_1\lambda_n)d^2(x_n,y_n)-(1-2c_2\lambda_n)d^2(y_n,z_{n+1}). \end{aligned} \tag{3.8}$$

Because $x^*{\in}S^*$, we also have $x^*{\in}\bigcap_{i=1}^{N}A_i^{-1}(0)$. Keeping in mind that $t_n = J^{A_N}_{\gamma_n^N}o\cdots oJ^{A_1}_{\gamma_n^1}(z_n)$, along with the monotonicity of each A_i and the nonexpansive nature of the resolvents $J^{A_i}_{\gamma_n^i}$ $(1{\le}i{\le}N)$, we conclude

$$d(t_n,x^*)\le d(z_n,x^*). \tag{3.9}$$

In conjunction with (3.8), this inequality finalizes the demonstration. □

Remark 3.4. Proposition 3.3 establishes that the sequence $\{d^2(z_n,x^*)\}$ is monotonically nonincreasing. Consequently, the limit $\lim_{n\to\infty} d(z_n,x^*)$ is well-defined, which immediately guarantees the boundedness of $\{z_n\}$. Given that $\liminf_{n\to\infty}\beta_n(1{-}\beta_n)>0$ and $\liminf_{n\to\infty}(1{-}2c_i\lambda_n)>0$ for $i{=}1,2$, it follows that

$$\lim_{n\to\infty} d(y_n,z_{n+1})=\lim_{n\to\infty} d(z_n,w_n)=\lim_{n\to\infty} d(x_n,y_n)=0. \tag{3.10}$$

Consequently, the sequences $\{w_n\}$, $\{x_n\}$, and $\{y_n\}$ are all bounded. In conjunction with (3.9), this ensures that $\{t_n\}$ is likewise bounded. Passing to the limit in (3.3) as $n{\to}\infty$ yields

$$\lim_{n\to\infty} f(y_n,z_{n+1})=0. \tag{3.11}$$

We are now ready to establish the primary $\triangle$-convergence theorem.

Theorem 3.5. *Assuming that assumptions $B1{-}B6$ are satisfied, the iterative sequence $\{x_n\}$ produced by Algorithm 3.1 $\triangle$-converges to an element belonging to*

$$\Omega{:=}\bigcap_{i=1}^{N}A_i^{-1}(0){\cap}S(f,\bigcap_{j=1}^{M}K_j).$$

Proof. Since $\{x_n\}$ is bounded, we can extract a subsequence $\{x_{n_k}\}$ that $\triangle$-converges to some element $p \in C$, i.e., $x_{n_k} \overset{\Delta}{\rightarrow} p$. Utilizing the construction of x_n alongside (3.10), we observe that $x_{n_k} = \beta_{n_k} z_{n_k} \oplus (1-\beta_{n_k}) w_{n_k}$ with

$$\lim_{k\to\infty} d(z_{n_k}, w_{n_k}) = 0.$$

This readily implies $w_{n_k} \overset{\Delta}{\rightarrow} p$ and $z_{n_k} \overset{\Delta}{\rightarrow} p$. Owing to the demiclosedness of each K_j $(1 \leq j \leq M)$, we conclude that $p \in \bigcap_{j=1}^{M} K_j(p)$. The remaining task is to verify that $p \in \bigcap_{i=1}^{N} A_i^{-1}(0)$. Proposition 3.3 ensures that $d^2(z_{n+1}, x^*) \leq d^2(z_n, x^*)$. In conjunction with (3.8), this yields

$$\begin{aligned} d^2(z_{n+1}, x^*) - d^2(z_n, x^*) &\leq \beta_n d^2(z_n, x^*) + (1-\beta_n) d^2(t_n, x^*) - d^2(z_n, x^*) \\ &= (1-\beta_n) d^2(t_n, x^*) - (1-\beta_n) d^2(z_n, x^*) \\ &\leq (1-\beta_n) \left(d^2(t_n, x^*) - d^2(z_n, x^*) \right) \\ &\leq d^2(t_n, x^*) - d^2(z_n, x^*). \end{aligned}$$

Consequently, invoking (3.9) leads to

$$d^2(z_{n+1}, x^*) \leq d^2(t_n, x^*) \leq d^2(z_n, x^*), \tag{3.12}$$

which guarantees that $t_n \overset{\Delta}{\rightarrow} p$.

For each index $1 \leq i \leq N$, let us introduce the composite operator

$$S_n^i := J_{\gamma_n^i}^{A_i} o \ldots o J_{\gamma_n^1}^{A_1}.$$

Accordingly, $t_n = S_n^N z_n$, and we set $S_0 = I$ with I representing the identity mapping. Given that each resolvent $J_{\gamma_n^i}^{A_i}$ $(1 \leq i \leq N)$ is nonexpansive, estimate (3.12) provides

$$\begin{aligned} d^2(t_n, x^*) - d^2(z_n, x^*) &= d^2(S_n^N z_n, x^*) - d^2(z_n, x^*) \\ &\leq d^2(S_n^i z_n, x^*) - d^2(z_n, x^*) \\ &\leq d^2(z_n, x^*) - d^2(z_n, x^*) = 0. \end{aligned}$$

Hence, we infer

$$\limsup_{n\to\infty} \left(d^2(S_n^i z_n, x^*) - d^2(z_n, x^*) \right) \leq 0, \quad i = 1, 2, \ldots N. \tag{3.13}$$

Coupling this with (3.12) reveals that

$$\begin{aligned} d^2(z_{n+1}, x^*) - d^2(z_n, x^*) &\leq d^2(t_n, x^*) - d^2(z_n, x^*) \\ &\leq d^2(S_n^i z_n, x^*) - d^2(z_n, x^*), \end{aligned}$$

which subsequently gives

$$0 \leq \liminf_{n\to\infty} \left(d^2(S_n^i z_n, x^*) - d^2(z_n, x^*)\right). \tag{3.14}$$

Merging (3.13) and (3.14) establishes

$$\lim_{n\to\infty} \left(d^2(S_n^i z_n, x^*) - d^2(z_n, x^*)\right) = 0, \quad i = 1, 2, \dots, N. \tag{3.15}$$

Applying (2.1), we derive

$$d^2\left(J_{\gamma_n^i}^{A_i}\left(S_n^{i-1} z_n\right), S_n^{i-1} z_n\right) \leq d^2\left(x^*, S_n^{i-1} z_n\right) - d^2\left(x^*, S_n^i z_n\right).$$

It follows that

$$d^2\left(S_n^i z_n, S_n^{i-1} z_n\right) \leq d^2\left(x^*, S_n^{i-1} z_n\right) - d^2\left(x^*, S_n^i z_n\right).$$

Taking the limit as $n \to \infty$ in the preceding inequality results in

$$\lim_{n\to\infty} d^2\left(S_n^i z_n, S_n^{i-1} z_n\right) = 0, \quad i = 1, 2, \dots, N. \tag{3.16}$$

Consequently, for every $1 \leq i \leq N$, the triangle inequality yields

$$d(z_n, S_n^{i-1} z_n) \leq d(z_n, S_n^1 z_1) + \ldots + d(S_n^{i-1} z_n, S_n^i z_n) \longrightarrow 0.$$

The condition $\liminf_{n\to\infty} \gamma_n^i > 0$ ensures the existence of a positive constant $\gamma \in \mathbb{R}$ satisfying $\gamma_n^i \geq \gamma > 0$ for all $n \in \mathbb{N}$ and $1 \leq i \leq N$. Combining this fact with (2.1) leads to

$$\begin{aligned} d\left(J_\gamma^{A_i}(S_n^{i-1} z_n), S_n^i z_n\right) &\leq d\left(J_\gamma^{A_i}(S_n^{i-1} z_n), S_n^{i-1} z_n\right) + d\left(S_n^{i-1} z_n, S_n^i z_n\right) \\ &\leq 2d\left(J_{\gamma_n^i}^{A_i}(S_n^{i-1} z_n), S_n^{i-1} z_n\right) + d\left(S_n^{i-1} z_n, S_n^i z_n\right) \\ &= 3d\left(S_n^i z_n, S_n^{i-1} z_n\right). \end{aligned}$$

Finally, passing to the limit as $n \to \infty$ and utilizing (3.16) produces

$$\lim_{n\to\infty} d\left(J_\gamma^{A_i}(S_n^{i-1} z_n), S_n^i z_n\right) = 0. \tag{3.17}$$

Conversely, we observe that

$$\begin{aligned} d\left(z_n, J_\gamma^{A_i}(z_n)\right) &\leq d\left(J_\gamma^{A_i}(z_n), J_\gamma^{A_i}(S_n^{i-1} z_n)\right) \\ &\quad + d\left(J_\gamma^{A_i}(S_n^{i-1} z_n), S_n^i z_n\right) + d(S_n^i z_n, z_n) \\ &\leq d(z_n, S_n^{i-1} z_n) + d\left(J_\gamma^{A_i}(S_n^{i-1} z_n), S_n^i z_n\right) + d(S_n^i z_n, z_n). \end{aligned}$$

Taking the limit as $n \to \infty$ in the preceding estimate and invoking (3.15) along with (3.16) yields

$$\lim d(z_n, J_\gamma^{A_i} z_n) = 0, \quad i = 1, 2, \dots, N.$$

Combining this outcome with the fact that $z_{n_k} \overset{\Delta}{\to} p$ and applying Lemma 2.7 confirms that $p\in \bigcap_{i=1}^{N} A_i^{-1}(0)$.

The next objective is to demonstrate that $p\in S(f, \bigcap_{j=1}^{M} K_j)$. Utilizing the defining property of z_{n+1}, we obtain

$$f(y_n, z_{n+1}) + \frac{1}{2\lambda_n} d^2(x_n, z_{n+1}) \le f(y_n, y) + \frac{1}{2\lambda_n} d^2(x_n, y), \quad \forall y \in C.$$

By replacing y with the convex combination $y = tz_{n+1} \oplus (1-t)y$ in this relation and applying condition B2, we deduce

$$\begin{aligned} f(y_n, z_{n+1}) + \frac{1}{2\lambda_n} d^2(x_n, z_{n+1}) \le & f\left(y_n, tz_{n+1} \oplus (1-t)y\right) \\ & + \frac{1}{2\lambda_n} d^2(x_n, tz_{n+1} \oplus (1-t)y) \\ \le & tf(y_n, z_{n+1}) + (1-t) f(y_n, y) \\ & + \frac{1}{2\lambda_n}\Big(td^2(x_n, z_{n+1}) + (1-t)d^2(x_n, y) \\ & - t(1-t)d^2(z_{n+1}, y)\Big), \end{aligned}$$

which subsequently leads to

$$f(y, z_{n+1}) - f(y_n, y) \le \frac{1}{2\lambda_n}\Big(d^2(x_n, y) - d^2(x_n, z_{n+1}) - td^2(z_{n+1}, y)\Big).$$

Passing to the limit as $t \to 1^-$ in the preceding bound yields

$$\begin{aligned} f(y_n, y) - f(y_n, z_{n+1}) & \ge \frac{1}{2\lambda_n}\Big(d^2(x_n, z_{n+1}) + d^2(z_{n+1}, y) - d^2(x_n, y)\Big) \\ & \ge \frac{1}{2\lambda_n}\Big(d^2(z_{n+1}, y) - d^2(x_n, y)\Big) \\ & = \frac{1}{2\lambda_n}\left(d(z_{n+1}, y) - d(x_n, y)\right)\left(d(z_{n+1}, y) + d(x_n, y)\right) \\ & \ge \frac{-1}{2\lambda_n} d(x_n, z_{n+1})\Big(d(z_{n+1}, y) + d(x_n, y)\Big). \end{aligned} \tag{3.18}$$

Conversely, the Δ-convergence $x_{n_k} \overset{\Delta}{\to} p$ together with (3.10) ensures that $y_{n_k} \overset{\Delta}{\to} p$. Substituting n by n_k in (3.18), taking the upper limit, and invoking (3.11) produces

$$0 \le \limsup f(y_{n_k}, y), \quad \forall y \in C.$$

Given that $f(\cdot, y)$ exhibits $\triangle$-upper semicontinuity, it follows that

$$f(p, y) \geq 0, \quad \forall y \in C. \tag{3.19}$$

Consequently, $p \in S_{QEP}(f, \bigcap_{j=1}^{M} K_j)$. Employing Lemma 2.8 ultimately guarantees that the sequence $\{x_n\}$ is $\triangle$-convergent to an element within Ω, thereby finalizing the demonstration. ☐

We now investigate the strong convergence of the iterates produced by Algorithm 3.1 toward a point in Ω, subject to an extra structural condition.

Definition 3.6. A bifunction f is termed strongly monotone provided one can find a constant $\alpha>0$ satisfying

$$f(x, y) + f(y, x) \leq -\alpha d^2(x, y)$$

across all $x, y \in X$. Similarly, f is designated as strongly pseudo-monotone if a constant $\beta > 0$ exists such that the condition $f(x, y) \geq 0$ necessarily forces

$$f(y, x) \leq -\beta d^2(x, y)$$

for every pair $x, y \in X$.

Theorem 3.7. *Suppose that all hypotheses of Theorem* 3.5 *hold, and additionally assume that the bifunction f is strongly pseudo-monotone. Under these conditions, the sequence $\{x_n\}$ produced by Algorithm* 3.1 *converges strongly to a point in Ω.*

Proof. Theorem 3.5 guarantees that the sequence $\{x_n\}$ generated by Algorithm 3.1 is $\triangle$-convergent to some element of Ω. Let us denote this limit by p, so that $x_n \overset{\triangle}{\to} p$. In view of (3.10), it follows immediately that $y_n \overset{\triangle}{\to} p$. Substituting $y = p$ into (3.18) yields

$$-\frac{1}{2\lambda_n} d(x_n, z_{n+1}) \Big(d(z_{n+1}, p) + d(x_n, p) \Big) \leq f(y_n, p) - f(y_n, z_{n+1}). \tag{3.20}$$

Taking the lower limit as $n \to \infty$ in the preceding inequality gives

$$\liminf_{n \to \infty} f(y_n, p) \geq 0. \tag{3.21}$$

Combining this with the fact that $f(y_n, p) \leq 0$ leads to

$$\lim_{n \to \infty} f(y_n, p) = 0. \tag{3.22}$$

Given that $f(p, y_n) \geq 0$ and f satisfies the strong pseudo-monotonicity condition, there exists $\beta > 0$ such that

$$f(y_n, p) \leq -\beta d^2(y_n, p).$$

Passing to the lower limit as $n \to \infty$ in this relation and employing (3.21), we deduce

$$\begin{aligned} 0 &\leq \liminf_{n\to\infty} f(y_n, p) \\ &\leq \liminf_{n\to\infty} (-\beta d^2(y_n, p)) \\ &\leq -\beta \limsup_{n\to\infty} d^2(y_n, p) \\ &\leq 0, \end{aligned}$$

which consequently forces

$$\lim_{n\to\infty} d^2(y_n, p) = 0.$$

This establishes that $y_n \to p$ in the metric sense, and it readily follows that $x_n \to p$. □

4. APPLICATION

We begin by presenting the definition along with several fundamental properties of the subdifferential operator.

Definition 4.1. Consider an Hadamard space X equipped with its dual space X^*, and let $g{:}X{\to}(-\infty,+\infty]$ denote a proper function whose effective domain is given by $D(g){:=}\{x : g(x){<}+\infty\}$. The subdifferential associated with g corresponds to the set-valued mapping $\partial g{:}X{\rightrightarrows}X^*$ specified as follows:

$$\partial g(x) = \big\{x^*{\in}X^* : \; g(z) - g(x) \geq \langle x^*, \overrightarrow{xz}\rangle, \quad z \in X\big\}$$

for any $x{\in}D(g)$, whereas $\partial g(x){=}\emptyset$ in all other cases.

Theorem 4.2. ([18]) *Assume that $g{:}X{\to}(-\infty,+\infty]$ represents a proper, convex, and lower semicontinuous functional defined on an Hadamard space X with dual X^*. Under these conditions, the following properties hold:*

- (i) *g attains its minimum at $x{\in}X$ if and only if $\mathbf{0} \in \partial g(x)$,*
- (ii) *$\partial g{:}X{\rightrightarrows}X^*$ is a monotone operator,*
- (iii) *for any $y{\in}X$ and $\alpha{>}0$, there exists a unique point $x{\in}X$ such that $[\alpha\overrightarrow{xy}] \in \partial g(x)$.*

The assertion in part (*iii*) of Theorem 4.2 demonstrates that the subdifferential operator corresponding to a proper, convex, and lower semicontinuous functional fulfills the range condition.

Lemma 4.3. ([20]) *Suppose* $g{:}X{\to}(-\infty,+\infty]$ *is a proper, convex, and lower semicontinuous mapping on an Hadamard space* X *with dual* X^**. Then, for every* $\lambda{>}0$ *and any point* $x{\in}X$*, the resolvent satisfies the identity*

$$J_\lambda^{\partial g}(x) = argmin_{y\in X}\left\{g(y)+\frac{1}{2\lambda}d^2(y,x)\right\}.$$

In this section, let $C{\subset}X$ denote a nonempty, closed, and convex subset within an Hadamard space X. Assume the bifunction $f{:}X{\times}X{\to}\mathbb{R}$ fulfills assumptions $B1{-}B5$, while each set-valued operator $K_j{:}C{\to}2^C$ $(1{\leq}j{\leq}M)$ is quasi-nonexpansive and adheres to condition $B6$. Furthermore, consider a family of proper functions $g_i{:}X{\rightrightarrows}X^*$ indexed by $1{\leq}i{\leq}N$. Based on this framework, we present the subsequent iterative scheme alongside its corresponding convergence corollary.

Algorithm 4.4. Initialization: Select an initial point $z_0{\in}C$, parameters

$$0{<}\alpha{\leq}\lambda_n{\leq}\beta{<}\min\left\{\frac{1}{2c_1},\frac{1}{2c_2}\right\},\quad 0{<}\gamma{<}\beta_n{<}\sigma{<}1$$

and a sequence $\gamma_n^i{\subset}(0,\infty)$ satisfying

$$\liminf_{n\to\infty}\gamma_n^i{>}0.$$

Initialize the iteration counter $n{=}0$ and proceed to Step 1.

Step 1: Evaluate

$$t_n = J_{\gamma_n^N}^{\partial g_N} o\cdots o J_{\gamma_n^1}^{\partial g_1}(z_n).$$

Step 2: Evaluate

$$w_n = P_{K_M(t_n)}o\cdots oP_{K_1(t_n)}(t_n).$$

Step 3: Evaluate

$$x_n = \beta_n z_n \oplus (1-\beta_n)w_n.$$

Step 4: Solve the optimization task below and denote its minimizer by y_n, namely,

$$y_n = argmin_{y\in C}\left\{f(x_n,y)+\frac{1}{2\lambda_n}d^2(x_n,y)\right\}.$$

Step 5: Solve the optimization task below and denote its minimizer by z_{n+1}, namely,

$$z_{n+1} = argmin_{y\in C}\left\{f(y_n,y)+\frac{1}{2\lambda_n}d^2(x_n,y)\right\}.$$

Step 6: Update the counter $n := n+1$ and return to Step 1.

This framework directly yields the following consequence.

Corollary 4.5. *Provided that conditions $B1$–$B6$ are fulfilled, the iterative sequence $\{x_n\}$ produced by Algorithm 4.4 $\triangle$-converges to an element belonging to*

$$\Omega := \bigcap_{i=1}^{N} argmin_{y\in C} g_i(y) \bigcap S\left(f, \bigcap_{j=1}^{M} K_j\right).$$

5. NUMERICAL EXAMPLES

In this section, we present two computational experiments designed to validate the theoretical convergence results established previously, within the framework of an Hadamard space. The following example was solved using MATHEMATICA 11 and an 8-core computer.

Example 5.1. Consider the space $X = \mathbb{R}^2$ equipped with the metric

$$d_H(x,y) = \sqrt{(x_1 - y_1)^2 + (x_1^2 - x_2 - y_1^2 + y_2)^2},$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. As demonstrated in ([11], Example 5.2), the structure $(\mathbb{R}^2, d_H)$ constitutes an Hadamard space. The geodesic curve connecting x to y admits the explicit parametrization:

$$\gamma_{x,y}(t) = ((1-t)x_1 + ty_1, ((1-t)x_1 + ty_1)^2 - (1-t)(x_1^2 - x_2) - t(y_1^2 - y_2)).$$

Define two functionals $g_1 : \mathbb{R}^2 \to \mathbb{R}$ and $g_2 : \mathbb{R}^2 \to \mathbb{R}$ via

$$g_1(x_1,x_2) = 100\big((x_2+1) - (x_1+1)^2\big)^2 + x_1^2,$$
$$g_2(x_1,x_2) = 100x_1^2.$$

According to ([11], Example 5.2), g_1 is proper, convex, and lower semicontinuous with respect to the metric d_H, although it fails to be convex in the classical Euclidean sense. Consider the bifunction $f : X \times X \to \mathbb{R}$ given by

$$f(x,y) = d_H^2(y,0) - d_H^2(x,0).$$

It is straightforward to verify that f satisfies assumptions B_1–B_5 with constants $c_1 = c_2 = \dfrac{1}{4}$. Setting $N = 2$, we define $A_1 = \partial g_1$ and $A_2 = \partial g_2$. Let

$$C = \{x = (x_1,x_2) \in \mathbb{R}^2 : x_1, x_2 \geq 0\}, \quad M = 1$$

and consider the set-valued map $K(\cdot) : C \to 2^C$ specified by

$$K(x) = \Big\{x \in C : d(0,z) \leq 2 + \frac{1}{2}\sqrt{x_1^2 + (x_1^2 - x_2)^2}\Big\}, \qquad \forall x \in C.$$

One can verify that $K(\cdot) : C \to 2^C$ is a set-valued operator with nonempty, closed, and convex values that is both quasi-nonexpansive and demiclosed. Consequently, assumption $B6$ holds and the solution set S is nonempty.

Under this configuration, Algorithm 4.4 specializes to the following iterative scheme:

$$\begin{cases} v_n = argmin_{y \in C}\{g_1(y) + \frac{1}{2\gamma_n^1} d_H^2(y, z_n)\}, \\ t_n = argmin_{y \in C}\{g_2(y) + \frac{1}{2\gamma_n^2} d_H^2(y, v_n)\}, \\ w_n = P_{K(t_n)}(t_n), \\ x_n = \beta_n z_n \oplus (1 - \beta_n) w_n, \\ y_n = argmin_{y \in C}\{f(x_n, y) + \frac{1}{2\lambda_n} d^2(x_n, y)\}, \\ z_{n+1} = argmin_{y \in C}\{f(y_n, y) + \frac{1}{2\lambda_n} d^2(x_n, y)\}. \end{cases} \tag{5.1}$$

For the numerical implementation, we select the parameters $\beta_n = \frac{1}{2}$, $\lambda_n = \frac{1}{2} + \frac{1}{n+2}$, and $\gamma_n^1 = \gamma_n^2 = 2n$ for every $n \in \mathbb{N}$, with the initial iterate $z_1 = (0.3, 0.2)$. Now, using Algorithm (5.1), we have numerical results in Fig 1 and Fig 2. Finally, the number of iterations and CPU time are shown in Table 1.

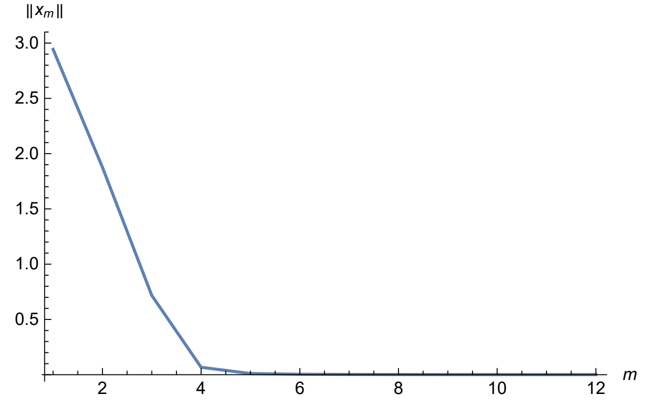

Figure 1. Plotting of $d_H(x_n, 0)$

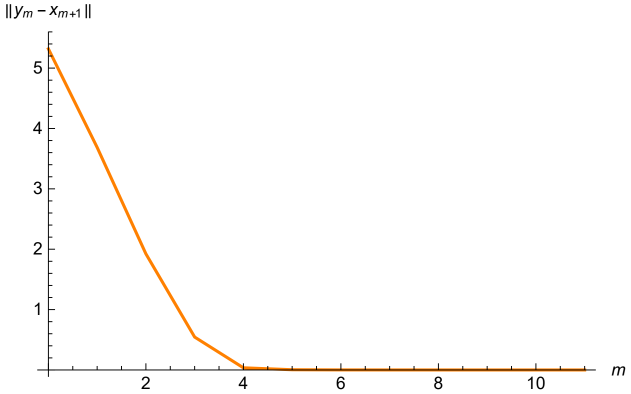

FIGURE 2. Plotting of $d_H(y_m, x_{m+1})$

Starting point: z_0	Average number of iterations	Average CPU time (Sec)
11	22	33
44	55	66

TABLE 1. The number of iterations and CPU time

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