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ANALYSIS ON THE LOCATION OF ZEROS OF POLYNOMIALS WITH ARBITRARY VARIABLES ASSOCIATED TO EXTREME COEFFICIENTS

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Abstract. In this paper, we obtain some interesting extensions and generalizations of well known Enestrom-Kakeya Theorem. We, in particular show that the bounds of the zeros of the polynomials are sharper for some sets of values of non negative real values of $\{\lambda, \mu, \tau, \rho\}$ associated to the constraints involving $\{\alpha_n, \beta_n, \alpha_0, \beta_0\}$ and compared the results obtained by earlier authors.

1. Introduction and statement of results

Many results on the location of zeros of polynomials are available in the literature. Among them the Enestrom and Kakeya Theorem [12] given below is well known in the theory of the location of the zeros of polynomials.

Theorem 1.1. *Let* $P(z)=\sum_0^n a_j z^j$ *be a polynomial of degree* n *such that*

$$a_n \geq a_{n-1} \geq a_{n-2} \geq \cdots \geq a_1 \geq a_0 > 0, \quad a_j \in \mathbb{R}.$$

Then $P(z)$ *has all its zeros in the disk* $|z| \leq 1$.

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In the literature [1] to [17], there exist some extensions and generalization of Enestrom-Kakeya Theorem. Shah and Liman [17] also considered polynomials with complex coefficients and proved the following result.

Theorem 1.2. *Let $P(z) = \sum_0^n a_j z^j$ be a polynomial of degree n with complex coefficients. If $Re(a_j) = \alpha_j$ and $Im(a_j)=\beta_j$, for $j = 0, 1, 2, \cdots, n$ such that for some $\lambda \geq 1$,*

$$\lambda\alpha_n \leq \alpha_{n-1} \leq \cdots \leq \alpha_{p+1} \leq \alpha_p\ , \ \ \alpha_p \geq \alpha_{p-1} \geq \cdots \geq \alpha_1 \geq \alpha_0,$$

$$\beta_n \geq \beta_{n-1} \geq \beta_{n-2} \geq \cdots \geq \beta_1 \geq \beta_0 > 0,$$

where $0 \leq p \leq n-1$, then all the zeros of P(z) lie in

$$\left| z + \frac{\alpha_n}{a_n}(\lambda - 1) \right| \leq \frac{1}{|a_n|}[2\alpha_p - \lambda\alpha_n - \alpha_0 + |\alpha_0| + \beta_n]. \tag{1.1}$$

Recently Gulzar [10] has proved that

Theorem 1.3. *Let $P(z)=\sum_0^n a_j z^j$ be a polynomial of degree n with complex coefficients. If $Re(a_j) = \alpha_j$ and $Im(a_j) = \beta_j$, for $j = 0, 1, 2, \cdots, n$ such that for some real numbers $0 < \lambda \leq 1$ and $0 < \tau \leq 1$,*

$$\lambda\alpha_n \leq \alpha_{n-1} \leq \cdots \leq \alpha_{p+1} \leq \alpha_p\ , \ \ \alpha_p \geq \alpha_{p-1} \geq \cdots \geq \alpha_1 \geq \tau\alpha_0,$$

$$\beta_n \geq \beta_{n-1} \geq \beta_{n-2} \geq \cdots \geq \beta_1 \geq \beta_0 > 0,$$

where $0 \leq p \leq n-1$, then all the zeros of P(z) lie in

$$\left| z - \frac{\alpha_n}{a_n}(1 - \lambda) \right| \leq \frac{1}{|a_n|}[2\alpha_p - \lambda\alpha_n + 2|\alpha_0| - \tau(\alpha_0 + |\alpha_0|) + \beta_n]. \tag{1.2}$$

While proving above theorem for $\tau = 1$, the author points out that the value of λ(associated to α_n) ≥ 1 is not correct and has no meaning in the statement and is never true for λ for extremely larger value such that $\lambda\alpha_n \geq \alpha_{n-1}$. However, we show in this paper that for the polynomial with restricted coefficients such as given in the theorem that the bound for $\lambda > 1$ sharpens the bound corresponding to $0 < \lambda \leq 1$. In fact, we prove in the illustration given below that for $\lambda > 1$, *i.e.*, when $\lambda = 2$ sharpens the bound by at least 15% than when $\lambda < 1$ corresponding to $\lambda = 0.3$. This suggests that for any type of restriction on the coefficient of polynomial of the above type, the value of λ is meaningful for $\lambda > 1$ as also for $0 < \lambda \leq 1$. Also, In this paper we present an interesting generalization and extension of Theorem 1.3 and discuss the above theorem subject to the following constraints:

Case 1: $\lambda \geq 1$ and $0 < \tau \leq 1$.
Case 2: $0 < \lambda \leq 1$ and $0 < \tau \leq 1$.

Case 3: $0 < \lambda \leq 1$ and $\tau \geq 1$.
Case 4: $\lambda \geq 1$ and $\tau \geq 1$.

Case 1: When $\lambda \geq 1$ and $0 < \tau \leq 1$,

Theorem 1.4. *Let $P(z) = \sum_0^n a_j z^j$ be a polynomial of degree n with complex coefficients. If $Re(a_j) = \alpha_j$ and $Im(a_j)=\beta_j$ for $j = 0, 1, 2, \cdots, n$ such that for some real numbers $\lambda, \mu \geq 1$, $0 < \tau, \rho \leq 1$, for*

$$\lambda\alpha_n \leq \alpha_{n-1} \leq \cdots \leq \alpha_{p+1} \leq \alpha_p \ , \ \ \alpha_p \geq \alpha_{p-1} \geq \cdots \geq \alpha_1 \geq \tau\alpha_0,$$

$$\mu\beta_n \geq \beta_{n-1} \geq \beta_{n-2} \geq \cdots \geq \beta_1 \geq \rho\beta_0,$$

where $0 \leq p \leq n-1$, then all the zeros of P(z) lie in

$$\left| z + \frac{(\lambda-1)\alpha_n + i(\mu-1)\beta_n}{|a_n|} \right| \leq \frac{1}{|a_n|}[2\alpha_p + (\mu\beta_n - \lambda\alpha_n) + 2(|\alpha_0| + |\beta_0|) - \tau(|\alpha_0| + \alpha_0) - \rho(\beta_0 + |\beta_0|)].$$

Proof. Consider the polynomial

$$\begin{aligned}
F(z) &= [1-z]P(z) \\
&= -a_n z^{n+1} + [a_n - a_{n-1}]z^n + \cdots + [a_1 - a_0]z + a_0 \\
&= -a_n z^{n+1} + [\alpha_n - \alpha_{n-1}]z^n + \cdots + [\alpha_{p+1} - \alpha_p]z^{p+1} \\
&\quad + [\alpha_p - \alpha_{p-1}]z^p + \cdots + [\alpha_1 - \alpha_0]z + \alpha_0 \\
&\quad + i([\beta_n - \beta_{n-1}]z^n + \cdots + [\beta_1 - \beta_0]z + \beta_0) \\
&= -a_n z^{n+1} + (\alpha_n - \lambda\alpha_n + \lambda\alpha_n - \alpha_{n-1})z^n + \cdots + (\alpha_{p+1} - \alpha_p)z^{p+1} \\
&\quad + (\alpha_p - \alpha_{p-1})z^p + \cdots + (\alpha_1 - \tau\alpha_0 + \tau\alpha_0 - \alpha_0)z + \alpha_0 \\
&\quad + i\{(\beta_n - \mu\beta_n + \mu\beta_n - \beta_{n-1})z^n + \cdots + (\beta_1 + \rho\beta_0 - \rho\beta_0 - \beta_0)z + \beta_0\}.
\end{aligned}$$

For $|z| > 1$, we have

$$\begin{aligned}
|F(z)| \geq |z|^n \Bigg\{ & |a_n z + (\lambda-1)\alpha_n + i(\mu-1)\beta_n| \\
& - \left(|\lambda\alpha_n - \alpha_{n-1}| + \frac{|\alpha_{n-1} - \alpha_{n-2}|}{|z|} + \cdots + \frac{|\alpha_{p+1} - \alpha_p|}{|z|^{n-p-1}} \right. \\
& \left. + \frac{|\alpha_p - \alpha_{p-1}|}{|z|^{n-p}} + \cdots + \frac{|\alpha_1 \tau\alpha_0|}{|z|^{n-1}} + \frac{|1-\tau||\alpha_0|}{|z|^{n-1}} + \frac{|\alpha_0|}{|z|^n} \right) \\
& - \left(|\mu\beta_n - \beta_{n-1}| + \frac{|\beta_{n-1} - \beta_{n-2}|}{|z|} + \cdots + \frac{|\beta_1 - \rho\beta_0|}{|z|^{n-1}} \right. \\
& \left. + \frac{|1-\rho||\beta_0|}{|z|^{n-1}} + \frac{|\beta_0|}{|z|^n} \right) \Bigg\}
\end{aligned}$$

$$
\begin{aligned}
&> |z|^n(|a_n z + (\lambda - 1)\alpha_n + i(\mu - 1)\beta_n| \\
&\quad - [-\lambda\alpha_n + \alpha_{n-1} - \alpha_{n-1} + \alpha_{n-2} + \cdots + \alpha_p - \alpha_{p+1} + \alpha_p - \alpha_{p-1} \\
&\quad + \cdots + \alpha_1 - \tau\alpha_0 + (1-\tau)|\alpha_0| + |\alpha_0| \\
&\quad + \mu\beta_n - \beta_{n-1} + \beta_{n-1} - \beta_{n-2} + \cdots + \beta_1 - \rho\beta_0 + (1-\rho)|\beta_0| + |\beta_0|]) \\
&= |z|^n[|a_n z + (\lambda - 1)\alpha_n + i(\mu - 1)\beta_n| \\
&\quad - [2\alpha_p - \lambda\alpha_n + 2|\alpha_0| - \tau(\alpha_0 + |\alpha_0|) + \mu\beta_n + 2|\beta_0| - \rho(\beta_0 + |\beta_0|)] \\
&> 0,
\end{aligned}
$$

if

$$
\begin{aligned}
\left|a_n z + (\lambda - 1)\alpha_n + i(\mu - 1)\beta_n\right| > [&2\alpha_p - \lambda\alpha_n + 2|\alpha_0| - \tau(\alpha_0 + |\alpha_0|) \\
&+ \mu\beta_n + 2|\beta_0| - \rho(\beta_0 + |\beta_0|)]
\end{aligned}
$$

or

$$
\begin{aligned}
\left|z + \frac{\alpha_n}{a_n}(\lambda - 1) + i\frac{\beta_n}{a_n}(\mu - 1)\right| \le \frac{1}{|a_n|}[&2\alpha_p - \lambda\alpha_n + 2|\alpha_0| - \tau(\alpha_0 + |\alpha_0|) \\
&+ \mu\beta_n + 2|\beta_0| - \rho(\beta_0 + |\beta_0|)].
\end{aligned}
$$

Therefore all the zeros of $F(z)$ whose modulus is greater than 1 lie in

$$
\begin{aligned}
\left|z + \frac{(\lambda - 1)\alpha_n + i(\mu - 1)\beta_n}{|a_n|}\right| \le \frac{1}{|a_n|}[&2\alpha_p - \lambda\alpha_n + 2|\alpha_0| - \tau(|\alpha_0| + \alpha_0) \\
&+ \mu\beta_n + 2|\beta_0| - \rho(\beta_0 + |\beta_0|)].
\end{aligned}
$$

But those zeros of $F(z)$ whose modulus is less than or equal to 1 already satisfy the inequality. Hence we conclude that all the zeros of $F(z)$ lie in the disk

$$
\begin{aligned}
\left|z + \frac{(\lambda - 1)\alpha_n + i(\mu - 1)\beta_n}{|a_n|}\right| \le \frac{1}{|a_n|}[&2\alpha_p - \lambda\alpha_n + 2|\alpha_0| - \tau(|\alpha_0| + \alpha_0) \\
&+ \mu\beta_n + 2|\beta_0| - \rho(\beta_0 + |\beta_0|)].
\end{aligned}
$$

Since every zero of $P(z)$ is also a zero of $F(z)$, it follows that all the zeros of $P(z)$ lie in the disk

$$
\begin{aligned}
\left|z + \frac{(\lambda - 1)\alpha_n + i(\mu - 1)\beta_n}{|a_n|}\right| \le \frac{1}{|a_n|}[&2\alpha_p + (\mu\beta_n - \lambda\alpha_n) + 2(|\alpha_0| + |\beta_0|) \\
&- \tau(|\alpha_0| + \alpha_0) - \rho(\beta_0 + |\beta_0|)].
\end{aligned}
$$

This completes the proof of the Theorem 1.4. □

Case 2: When $0 < \lambda \leq 1$ and $0 < \tau \leq 1$,

Theorem 1.5. *Let* $P(z) = \sum_0^n a_j z^j$ *be a polynomial of degree n with complex coefficients. If* $Re(a_j) = \alpha_j$ *and* $Im(a_j)=\beta_j$ *for* $j = 0, 1, 2, \cdots, n$ *such that for some real numbers* $0 < \lambda, \mu \leq 1$, $0 < \tau, \rho \leq 1$, *for*

$$\lambda\alpha_n \leq \alpha_{n-1} \leq \cdots \leq \alpha_{p+1} \leq \alpha_p \ , \ \ \alpha_p \geq \alpha_{p-1} \geq \cdots \geq \alpha_1 \geq \tau\alpha_0,$$

$$\mu\beta_n \geq \beta_{n-1} \geq \beta_{n-2} \geq \cdots \geq \beta_1 \geq \rho\beta_0,$$

where $0 \leq p \leq n-1$, *then all the zeros of P(z) lie in*

$$\left| z - \frac{(1-\lambda)\alpha_n + i(1-\mu)\beta_n}{|a_n|} \right| \leq \frac{1}{|a_n|}[2\alpha_p + (\mu\beta_n - \lambda\alpha_n) + 2(|\alpha_0| + |\beta_0|) - \tau(|\alpha_0| + \alpha_0) - \rho(\beta_0 + |\beta_0|)].$$

We omit the proof of the above since it holds on the parallel lines as given in Theorem 1.4. We now show that this result generalizes the results given by the previous authors.

Remark 1.6. If all the coefficients of $P(z)$ are positive under the conditions in Theorem 1.5, then all the zeros of $P(z)$ lie in

$$\left| z - \frac{(1-\lambda)\alpha_n + i(1-\mu)\beta_n}{|a_n|} \right| \leq \frac{1}{|a_n|}[2\alpha_p - \lambda\alpha_n + \mu\beta_n + 2\alpha_0(1-\tau) + 2\beta_0(1-\rho)]. \tag{1.3}$$

We remark here that in the above Theorem 1.5, if $\mu\beta_n \geq \beta_{n-1} \geq \beta_{n-2} \geq \cdots \geq \beta_1 \geq \rho\beta_0 > 0$ and $\mu = \rho = 1$, then the above result coincide with the result of Theorem 2 given in [10]. For $\mu = \lambda = 1$, $\rho = \tau = 1$, Theorem 1.5 reduces to Dewan and Bidkham [7] and if all the coefficients of $P(z)$ are real and $\lambda = \tau = 1$, then the above Theorem 1.5 reduces to Enestrom-Kakeya Theorem.

Corollary 1.7. *Let* $P(z) = \sum_0^n a_j z^j$ *be a polynomial of degree n with complex coefficients. If* $Re(a_j) = \alpha_j$ *and* $Im(a_j)=\beta_j$ *for* $j = 0, 1, 2, \cdots, n$ *such that for some real numbers* $\lambda, \mu \geq 1$, $0 < \tau, \rho \leq 1$, *such that*

$$\lambda\alpha_n \geq \alpha_{n-1} \geq \cdots \geq \alpha_1 \geq \tau\alpha_0,$$

$$\mu\beta_n \leq \beta_{n-1} \leq \beta_{n-2} \leq \cdots \leq \beta_{p+1} \leq \beta_p \ , \ \ \beta_p \geq \beta_{p-1} \geq \cdots \geq \beta_1 \geq \rho\beta_0,$$

where $0 \leq p \leq n-1$, *then all the zeros of P(z) lie in*

$$\left|z+\frac{(\lambda-1)\alpha_n+i(\mu-1)\beta_n}{|a_n|}\right| \leq \frac{1}{|a_n|}[2\beta_p+\lambda\alpha_n-\mu\beta_n+2|\alpha_0| - \tau(|\alpha_0|+\alpha_0)+2|\beta_0|-\rho(\beta_0+|\beta_0|)]. \tag{1.4}$$

Corollary 1.8. *Let $P(z)=\sum_0^n a_jz^j$ be a polynomial of degree n with complex coefficients. If $Re(a_j)=\alpha_j$ and $Im(a_j)=\beta_j$ for $j=0,1,2,\cdots,n$ such that for some real numbers $0<\lambda,\mu\leq 1$, $0<\tau,\rho\leq 1$, such that*

$$\lambda\alpha_n \geq \alpha_{n-1} \geq \cdots \geq \alpha_1 \geq \tau\alpha_0,$$

$$\mu\beta_n \leq \beta_{n-1} \leq \beta_{n-2} \leq \cdots \leq \beta_{p+1} \leq \beta_p\ ,\ \ \beta_p \geq \beta_{p-1} \geq \cdots \geq \beta_1 \geq \rho\beta_0,$$

where $0\leq p\leq n-1$, then all the zeros of $P(z)$ lie in

$$\left|z-\frac{(1-\lambda)\alpha_n+i(1-\mu)\beta_n}{|a_n|}\right| \leq \frac{1}{|a_n|}[2\beta_p+\lambda\alpha_n-\mu\beta_n+2|\alpha_0| - \tau(|\alpha_0|+\alpha_0)+2|\beta_0|-\rho(\beta_0+|\beta_0|)]. \tag{1.5}$$

Case 3: When $0<\lambda\leq 1$ and $\tau\geq 1$,

Theorem 1.9. *Let $P(z)=\sum_0^n a_jz^j$ be a polynomial of degree n with complex coefficients. If $Re(a_j)=\alpha_j$ and $Im(a_j)=\beta_j$ for $j=0,1,2,\cdots,n$ such that for some real numbers $0<\lambda\leq 1$, $\tau\geq 1$,*

$$\lambda\alpha_n \leq \alpha_{n-1} \leq \cdots \leq \alpha_{p+1} \leq \alpha_p\ ,\ \ \alpha_p \geq \alpha_{p-1} \geq \cdots \geq \alpha_1 \geq \tau\alpha_0,$$

$$\beta_n \geq \beta_{n-1} \geq \beta_{n-2} \geq \cdots \geq \beta_1 \geq \beta_0 > 0,$$

where $0\leq p\leq n-1$, then all the zeros of P(z) lie in

$$\left|z-\frac{\alpha_n}{a_n}(1-\lambda)\right| \leq \frac{1}{|a_n|}[2\alpha_p-\lambda\alpha_n-\tau\alpha_0+\tau|\alpha_0|+\beta_n]. \tag{1.6}$$

Case 4: When $\lambda\geq 1$ and $\tau\geq 1$,

Theorem 1.10. *Let $P(z)=\sum_0^n a_jz^j$ be a polynomial of degree n with complex coefficients. If $Re(a_j)=\alpha_j$ and $Im(a_j)=\beta_j$ for $j=0,1,2,\cdots,n$ such that for some real numbers $\lambda\geq 1$, $\tau\geq 1$,*

$$\lambda\alpha_n \leq \alpha_{n-1} \leq \cdots \leq \alpha_{p+1} \leq \alpha_p\ ,\ \ \alpha_p \geq \alpha_{p-1} \geq \cdots \geq \alpha_1 \geq \tau\alpha_0,$$

$$\beta_n \geq \beta_{n-1} \geq \beta_{n-2} \geq \cdots \geq \beta_1 \geq \beta_0 > 0,$$

where $0\leq p\leq n-1$, then all the zeros of $P(z)$ lie in

$$\left|z+\frac{\alpha_n}{a_n}(\lambda-1)\right| \leq \frac{1}{|a_n|}[2\alpha_p-\lambda\alpha_n-\tau\alpha_0+\tau|\alpha_0|+\beta_n]. \tag{1.7}$$

Theorem 1.11. *Let $P(z) = \sum_0^n a_j z^j$ be a polynomial of degree n with complex coefficients. If $Re(a_j) = \alpha_j$ and $Im(a_j)=\beta_j$ for $j = 0, 1, 2, \cdots, n$ such that for some real numbers $0 < \lambda, \mu \leq 1,\ 0 < \tau, \rho \leq 1$,*

$$\lambda\alpha_n \leq \alpha_{n-1} \leq \cdots \leq \alpha_{p+1} \leq \alpha_p\ ,\ \ \alpha_p \geq \alpha_{p-1} \geq \cdots \geq \alpha_1 \geq \tau\alpha_0,$$

$$\mu\beta_n \leq \beta_{n-1} \leq \cdots \leq \beta_{q+1} \leq \beta_q\ ,\ \ \beta_q \geq \beta_{q-1} \geq \cdots \geq \beta_1 \geq \rho\beta_0,$$

where $0 \leq p, q \leq n-1$, then all the zeros of $P(z)$ lie in

$$\begin{aligned}&\left|z - \frac{(1-\lambda)\alpha_n + i(1-\mu)\beta_n}{|a_n|}\right| \\ &\leq \frac{1}{|a_n|}[2(\alpha_p + \beta_q) - (\lambda\alpha_n + \mu\beta_n) + 2(|\alpha_0| + |\beta_0|) \\ &\quad - \tau(|\alpha_0| + \alpha_0) - \rho(\beta_0 + |\beta_0|)].\end{aligned} \tag{1.8}$$

Proof. Consider the polynomial

$$\begin{aligned}F(z) &= [1-z]P(z) \\ &= -a_n z^{n+1} + [a_n - a_{n-1}]z^n + \cdots + [a_1 - a_0]z + a_0 \\ &= -a_n z^{n+1} + [\alpha_n - \alpha_{n-1}]z^n + \cdots + [\alpha_{p+1} - \alpha_p]z^{p+1} \\ &\quad + [\alpha_p - \alpha_{p-1}]z^p + \cdots + [\alpha_1 - \alpha_0]z + \alpha_0 \\ &\quad + i([\beta_n - \beta_{n-1}]z^n + \cdots + [\beta_{q+1} - \beta_q]z^{q+1} \\ &\quad + [\beta_q - \beta_{q-1}]z^q + \cdots + [\beta_1 - \beta_0]z + \beta_0) \\ &= -a_n z^{n+1} + (\alpha_n - \lambda\alpha_n + \lambda\alpha_n - \alpha_{n-1})z^n + \cdots + (\alpha_{p+1} - \alpha_p)z^{p+1} \\ &\quad + (\alpha_p - \alpha_{p-1})z^p + \cdots + (\alpha_1 - \tau\alpha_0 + \tau\alpha_0 - \alpha_0)z + \alpha_0 \\ &\quad + i\{(\beta_n - \mu\beta_n + \mu\beta_n - \beta_{n-1})z^n + \cdots + (\beta_{q+1} - \beta_q)z^{q+1} \\ &\quad + (\beta_q - \beta_{q-1})z^q + \cdots + (\beta_1 + \rho\beta_0 - \rho\beta_0 - \beta_0)z + \beta_0\}.\end{aligned}$$

For $|z| > 1$, we have

$$\begin{aligned}|F(z)| \geq |z|^n\Bigg\{&|a_n z - (1-\lambda)\alpha_n - i(1-\mu)\beta_n| - \Bigg(|\lambda\alpha_n - \alpha_{n-1}| \\ &+ \frac{|\alpha_{n-1} - \alpha_{n-2}|}{|z|} + \cdots + \frac{|\alpha_{p+1} - \alpha_p|}{|z|^{n-p-1}} + \frac{|\alpha_p - \alpha_{p-1}|}{|z|^{n-p}} + \cdots \\ &+ \frac{|\alpha_1 - \tau\alpha_0|}{|z|^{n-1}} + \frac{|1-\tau||\alpha_0|}{|z|^{n-1}} + \frac{|\alpha_0|}{|z|^n}\Bigg) \\ &- \Bigg(|\mu\beta_n - \beta_{n-1}| + \frac{|\beta_{n-1} - \beta_{n-2}|}{|z|} + \cdots + \frac{|\beta_{q+1} - \beta_q|}{|z|^{n-q-1}} \\ &+ \frac{|\beta_q - \beta_{q-1}|}{|z|^{n-q}} + \cdots + \frac{|\beta_1 - \rho\beta_0|}{|z|^{n-1}} + \frac{|1-\rho||\beta_0|}{|z|^{n-1}} + \frac{|\beta_0|}{|z|^n}\Bigg)\Bigg\}\end{aligned}$$

$$\begin{aligned}
&> |z|^n\{|a_n z - (1-\lambda)\alpha_n - i(1-\mu)\beta_n| \\
&\quad - [-\lambda\alpha_n + \alpha_{n-1} - \alpha_{n-1} + \alpha_{n-2} + \cdots + \alpha_p \\
&\quad - \alpha_{p+1} + \alpha_p - \alpha_{p-1} + \cdots + \alpha_1 - \tau\alpha_0 + (1-\tau)|\alpha_0| + |\alpha_0|] \\
&\quad - [-\mu\beta_n + \beta_{n-1} - \beta_{n-1} + \beta_{n-2} + \cdots + \beta_q - \beta_{q+1} \\
&\quad + \beta_q - \beta_{q-1} + \cdots + \beta_1 - \rho\beta_0 + (1-\rho)|\beta_0| + |\beta_0|]\} \\
&= |z|^n\{|a_n z - (1-\lambda)\alpha_n - i(1-\mu)\beta_n| \\
&\quad - [2\alpha_p - \lambda\alpha_n - \mu\beta_n + 2\beta_q + 2|\alpha_0| - \tau(\alpha_0 + |\alpha_0|) + 2|\beta_0| \\
&\quad - \rho(\beta_0 + |\beta_0|)]\} \\
&> 0,
\end{aligned}$$

if

$$\begin{aligned}
&|a_n z - (1-\lambda)\alpha_n - i(1-\mu)\beta_n| \\
&\quad > [2\alpha_p - \lambda\alpha_n - \mu\beta_n + 2\beta_q + 2|\alpha_0| - \tau(\alpha_0 + |\alpha_0|) + 2|\beta_0| - \rho(\beta_0 + |\beta_0|)].
\end{aligned}$$

This shows that the zeros of $F(z)$ whose modulus is greater than 1 lie in the disk

$$\begin{aligned}
&\left| z - \frac{(1-\lambda)\alpha_n + i(1-\mu)\beta_n}{|a_n|} \right| \\
&\leq \frac{1}{|a_n|}[2(\alpha_p + \beta_q) - (\lambda\alpha_n + \mu\beta_n) + 2(|\alpha_0| + |\beta_0|) \\
&\quad - \tau(|\alpha_0| + \alpha_0) - \rho(\beta_0 + |\beta_0|)].
\end{aligned}$$

But those zeros of $F(z)$ whose modulus is less than or equal to 1 already satisfy the inequality. Hence we conclude that all the zeros of $F(z)$ lie in the disk

$$\begin{aligned}
&\left| z - \frac{(1-\lambda)\alpha_n + i(1-\mu)\beta_n}{|a_n|} \right| \\
&\leq \frac{1}{|a_n|}[2(\alpha_p + \beta_q) - (\lambda\alpha_n + \mu\beta_n) + 2(|\alpha_0| + |\beta_0|) \\
&\quad - \tau(|\alpha_0| + \alpha_0) - \rho(\beta_0 + |\beta_0|)].
\end{aligned}$$

Since every zero of $P(z)$ is also a zero of $F(z)$, it follows that all the zeros of $P(z)$ lie in the disk

$$\begin{aligned}
&\left| z - \frac{(1-\lambda)\alpha_n + i(1-\mu)\beta_n}{|a_n|} \right| \\
&\leq \frac{1}{|a_n|}[2(\alpha_p + \beta_q) - (\lambda\alpha_n + \mu\beta_n) + 2(|\alpha_0| + |\beta_0|) \\
&\quad - \tau(|\alpha_0| + \alpha_0) - \rho(\beta_0 + |\beta_0|)].
\end{aligned}$$

This completes the proof of the Theorem 1.11. □

2. Illustration

Case 1: When $\tau = 1$ and $\lambda \geq 1$ or $0 < \lambda \leq 1$,

N	$a_j = \alpha_j + i\beta_j$	Bounds by Shah & Liman	Bounds by Gulzar
2	(2,3),(5,2),(6,1) $\lambda\alpha_2 \leq \alpha_1 \leq \alpha_0$, $\beta_2 \geq \beta_1 \geq \beta_0 > 0$	When $\lambda = 2$, $\lvert z\rvert \leq 3.05$	When $\lambda = 0.3$, $\lvert z\rvert \leq 3.83$
2	(2,3),(5,2),(-6,1) $\lambda\alpha_2 \leq \alpha_1$, $\alpha_1 \geq \alpha_0$, $\beta_2 \geq \beta_1 \geq \beta_0 > 0$	When $\lambda = 2$, $\lvert z\rvert \leq 6.38$	When $\lambda = 0.3$, $\lvert z\rvert \leq 7.16$
3	(1,3),(4,2),(-9,2),(-10,1) $\lambda\alpha_3 \leq \alpha_2$, $\alpha_2 \geq \alpha_1 \geq \alpha_0$, $\beta_3 \geq \beta_2 \geq \beta_1 \geq \beta_0 > 0$	When $\lambda = 3$, $\lvert z\rvert \leq 9.49$	When $\lambda = 0.3$, $\lvert z\rvert \leq 9.87$

Remark 2.1. From the above table we find that corresponding to $\lambda = 0.3$ (*i.e.*, $0 < \lambda \leq 1$) we obtain the bound $|z| \leq 3.83$ which is greater than the bound $|z| \leq 3.05$ corresponding to $\lambda = 2$ (*i.e.*, $\lambda \geq 1$). This result clearly falsifies the statement given by Gulzar [10] that the values of λ can never exceed one. Analysising the above results in the above table we find that the bounds can be sharper corresponding to the values of λ greater than or equal to one or nonnegative and less than unity.

Case 2: When $0 < \lambda \leq 1$ and $\tau \geq 1$ or $0 < \tau \leq 1$, we find that some of the bounds may be sharper for $\tau \geq 1$ or $0 < \tau \leq 1$.

N	$a_j = \alpha_j + i\beta_j$	Bounds by Theorem 1.9.	Bounds by Gulzar	Bounds by Theorem 1.10.
2	$(1+3i)z^2$ $+(4+2i)z$ $+(3+2i) = 0$	When $\lambda = 1/2$, $\tau = 1.1$, $\lvert z\rvert \leq 3.48$	When $\lambda = 1/2$, $\tau = 1/3$, $\lvert z\rvert \leq 4.74$	When $\lambda = 2$, $\tau = 1.1$, $\lvert z\rvert \leq 3.16$

3. Conclusion

From the above discussion we find that for the polynomials with restricted coefficients, λ, μ are meaningful for all the values ≥ 1 or $0 < \lambda, \mu \leq 1$. The above results also hold true for λ and τ as given in Theorem 1.9 and Theorem 1.10. In Theorem 1.9 and Theorem 1.10, if we associate μ to β_n and ρ to β_0, then the bounds will be either less or more sharper corresponding to any set of values of μ and ρ greater or less than one.

References

[1] A. Aziz and B.A. Zargar, *Some extensions of Enestrom-Kakeya theorem*, Glasnik mathematicki, **31** (1996), 239–244.

[2] A. Aziz and Q.G. Mohammad, *On zeros of certain class of polynomials* & *related analytic function*, J. Math. Anal. Appl., **75** (1980), 495–502.

[3] A. Aziz and W.M. Shah, *On the zeros of polynomials and related analytic functions*, Glasnik Mat., **33**(53) (1998), 173–184.

[4] A. Aziz and W.M. Shah, *On the location of zeros of polynomials and related analytic functions*, Nonlinear Studies, **6**(1) (1999), 91–101.

[5] Y. Choo, *Some Results on the zeros of polynomials and related analytic functions*, Int. Journal of Math. Anal., **5**(35) (2011), 1741–1760.

[6] K.K. Dewan and N.K. Govil, *On the Enestrom Kakeya theorem*, J. Approx. Theory, **42**(1984), 239–246.

[7] K.K. Dewan and M. Bidkam, *On the Enestrom Kakeya theorem*, J. Math. Appl., **180** (1993), 29–36.

[8] N.K. Govil and Q.I. Rehman, *On the Enestrom Kakeya theorem*, Tahoku Math. J., **20** (1986), 126–136.

[9] N.K. Govil and G.N. McTune, *Some extensions of Enestrom Kakeya theorem*, International J. Applied Math., **11**(3) (2002), 245–253.

[10] M.H. Gulzar, *On the zeros of a polynomial with restricted coefficients*, Research Journal of Pure Algebra, **1**(9) (2011), 205–208.

[11] A. Joyal, G. Labelle and Q.I. Rehman, *On the location of zeros of polynomial*, Canad. Math. Bull., **10** (1967), 53–63.

[12] M. Marden, *Geometry of polynomials*, Math. Surveys. No. 3; Amer. Math. Soc., (R.I.:Providence.) (1966).

[13] B.L. Raina, H.B. Singh, K. Arunima and P.K. Raina, *Sharper Bounds for the zeros of Polynomials Using Enestrom Kakeya Theorem*, Int., Journal of Math. Anal., **V4** (2010), 861–872.

[14] B.L. Raina, S. Sripriya and P.K. Raina, *Sharper bounds for the zeros of complex polynomials*, Int. Journal of Mathematical Archieve, **3**(9) (2012), 3518–3524.

[15] B.L. Raina, S. Sripriya, S.K. Sahu and P.K. Raina, *Some generalization of Enestrom Kakeya theorem*, Research Journal of Pure Algebra, **2**(10) (2012), 305–311.

[16] N.A. Rather and S. Shakeel Ahmed, *A remark on the generalization of Enestrom Kakeya theorem*, Journal of Anal. & Comput., **3**(1) (2007), 33–41.

[17] W.M. Shah and A. Liman, *On Enestrom Kakeya theorem and related analytic functions*, Proc. Indian Acad. Sci., (Math. Sci.), **117**(3) (2007), 359–370.