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CONVERGENCE THEOREMS FOR A GENERALIZED EQUILIBRIUM PROBLEM AND TWO ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN HILBERT SPACES

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Abstract. In this paper, we introduce an iterative scheme for finding a common element of the set of solutions of a generalized equilibrium problem and the set of common fixed points of two asymptotically nonexpansive mappings in Hilbert spaces. Weak and strong convergence theorems are established for the iterative scheme.

1. Introduction

H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. Assume that a bifunction $F : C \times C \to R$ satisfies the following conditions:

(A1) $F(x, x) = 0, \ \forall\, x \in C$;
(A2) F is monotone, *i.e.,* $F(x, y) + F(y, x) \leq 0, \ \forall\, x, y \in C$;
(A3) $\lim_{t\downarrow 0} F(tz + (1-t)x, y) \leq F(x, y), \ \forall\, x, y, z \in C$;
(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

Let $A : C \to H$ be a nonlinear mapping. Then, we consider the following generalized equilibrium problem(GEP) which is to find $z \in C$ such that

$$\text{GEP: } F(z, y) + \langle Az, y - z\rangle \geq 0, \ \forall\, y \in C. \tag{1.1}$$

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In the case of $A \equiv 0$, this problem (1.1) reduces to the equilibrium problem(EP), which is to find $z \in C$ such that

$$\text{EP: } F(z, y) \geq 0, \ \ \forall\, y \in C. \tag{1.2}$$

In the case of $F \equiv 0$, this problem (1.1) reduces to the variational inequality problem(VIP), which is to find $z \in C$ such that

$$\text{VIP: } \langle Az, y - z\rangle \geq 0, \ \ \forall\, y \in C. \tag{1.3}$$

Denote the set of solutions of GEP by Ω, the set of solutions of EP by $EP(F)$ and the set of solutions of VIP by $VI(C, A)$. The problem (1.1) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, the Nash equilibrium problem in noncooperative games and others; see, for instance, [1,7]. Let $T : C \to C$ be a mapping. Recall that T is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, $\forall x, y \in C$, and T is asymptotically nonexpansive if there exists a sequence $\{t_n\} \subset [1, +\infty)$ with $\lim_{n\to\infty} t_n = 1$ such that

$$\|T^n x - T^n y\| \leq t_n \|x - y\|, \ \ \forall\, x, y \in C \text{ and } n \in \mathbb{N}.$$

The set of fixed points of T is denoted by $F(T)$. Many iterative methods for finding a common element of the set of solutions of the equilibrium problem(EP) or the variational inequality problem(VIP) and the set of fixed points of a nonexpansive mapping have been extensively investigated by many authors(see, *e.g.*, [2, 6, 8, 11, 13]). However iterative methods for finding a common element of the set of solutions of the generalized equilibrium problem(GEP) and the set of common fixed points of two asymptotically nonexpansive mappings are rarely studied.

Recently, Takahashi and Takahashi [10] introduced an iterative method for finding a common element of the set of solutions of the generalized equilibrium problem(GEP) and the set of fixed points of a nonexpansive mapping. More precisely, they proved the following theorem.

Theorem 1.1. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $F : C \times C \to R$ be a bifunction satisfying* (A1)-(A4). *Let A be an α-inverse-strongly monotone mapping of C into H and let S be a nonexpansive mapping of C into itself such that $F(S) \bigcap \Omega \neq \emptyset$. Let $u \in C$ and $x_1 \in C$ and let $\{z_n\} \subset C$ and $\{x_n\} \subset C$ be sequences generated by*

$$\begin{cases} F(z_n, y) + \langle Ax_n, y - z_n\rangle + \frac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \quad \forall\, y \in C, \\ \\ x_{n+1} = \beta_n x_n + (1 - \beta_n) S[\alpha_n u + (1 - \alpha_n) z_n], \quad \forall\, n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\} \subset [0, 1]$, $\{\beta_n\} \subset [0, 1]$ and $\{\lambda_n\} \subset [0, 2\alpha]$ satisfy

$$0 < c \leq \beta_n \leq d < 1, \quad 0 < a \leq \lambda_n \leq b < 2\alpha,$$
$$\lim_{n\to\infty} \alpha_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Then $\{x_n\}$ converges strongly to $z = P_{F(S)\bigcap\Omega}u$, where $P_{F(S)\bigcap\Omega}$ is the metric projection from C onto $F(S)\bigcap\Omega$.

In 1991, Schu [8] introduced the following modified Mann iteration process:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 1,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ which is bounded away from 0 and 1, i.e., $0 < a \leq \alpha_n \leq b < 1$ for all n and some constant a, b, to approximate some fixed point of the asymptotically nonexpansive self-mapping T in Hilbert spaces.

In 1994, Tan and Xu [15] studied the modified Ishikawa iteration process:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n\big((1-\beta_n)x_n + \beta_n T^n x_n\big), \quad n \geq 1,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$ such that $\{\alpha_n\}$ is bounded away from 0 and 1 and $\{\beta_n\}$ is bounded away from 1.

In 1998, Takahashi and Tamura [12] introduced the following iterative schemes known as Ishikawa iterative schemes for a pair of nonexpansive mappings T and S:

$$\begin{cases} x_1 = x \in C, \\ y_n = \beta_n T x_n + (1-\beta_n)x_n, \\ x_{n+1} = \alpha_n S y_n + (1-\alpha_n)x_n, \;\; n \geq 1, \end{cases}$$

where $\alpha_n, \beta_n \in [0,1]$. They proved strong and weak convergence of the sequence to a common fixed point of T and S.

Recently, Wang [16] used a similar iterative scheme to prove strong and weak convergence theorems for a pair of asymptotically nonexpansive mappings.

It is clear that the asymptotically nonexpansive mappings are important generalizations of nonexpansive mappings. For details, we refer the reader to [5].

Motivated and inspired by these facts, we introduce an iteration scheme for finding a common element of the set of solutions of the generalized equilibrium problem(GEP) and the set of common fixed points of two asymptotically nonexpansive mappings in Hilbert spaces. We obtain weak and strong convergence theorems.

2. Preliminaries

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x. We denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of positive integers and real numbers, respectively. For any $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall\, y \in C.$$

Such a P_C is called the metric projection of H onto C. It is known that P_C is nonexpansive and satisfies the following property:

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \quad \forall\, x \in H, y \in C. \tag{2.1}$$

Furthermore, for $x \in H$ and $u \in C$,

$$u = P_C x \Leftrightarrow \langle x - u, u - y\rangle \geq 0, \quad \forall\, y \in C. \tag{2.2}$$

Let S be a asymptotically nonexpansive mapping. We know that the set $F(S)$ of fixed points of S is closed and convex. Further, if C is bounded, closed and convex, then $F(S)$ is nonempty. A mapping $A : C \to H$ is called inverse-strongly monotone if there exists $\alpha > 0$ such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall\, x, y \in C.$$

Such a mapping A is also called α-inverse-strongly monotone. If A is an α-inverse-strongly monotone mapping of C to H, then it is obvious that A is $\frac{1}{\alpha}$-Lipschitz continuous. We also have that for all $x, y \in C$ and $\lambda > 0$,

$$\begin{aligned}
&\|(I - \lambda A)x - (I - \lambda A)y\|^2 \\
&= \|(x - y) - \lambda(Ax - Ay)\|^2 \\
&= \|x - y\|^2 - 2\lambda\langle x - y, Ax - Ay\rangle + \lambda^2\|Ax - Ay\|^2 \\
&\leq \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Ax - Ay\|^2.
\end{aligned} \tag{2.3}$$

So, if $\lambda \leq 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping of C into H.

A mapping $T : C \to C$ is said to be semi-compact, if for any sequence $\{x_n\}$ in C such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to $x^* \in C$.

Lemma 2.1. ([1,4]) *Let C be a nonempty closed convex subset of H and let F be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying* (A1)-(A4)*. Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall\, y \in C.$$

Further, if

$$T_r x = \left\{ z \in C : F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \ \ \forall \, y \in C \right\},$$

then the following hold:

(1) T_r *is single-valued;*
(2) T_r *is firmly nonexpansive, i.e.,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle, \ \ \forall \, x, y \in H;$$

(3) $F(T_r) = EP(F)$*;*
(4) $EP(F)$ *is closed and convex.*

Lemma 2.2. *There holds the identity in a Hilbert space* H*:*

$$\|\lambda x + (1-\lambda) y\|^2 = \lambda \|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2$$

for all $x, y \in H$ *and* $\lambda \in [0, 1]$.

Lemma 2.3. ([14]) *Let* $\{a_n\}$ *and* $\{t_n\}$ *be two sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq a_n + t_n, \quad \forall \, n \geq 1.$$

If $\sum_{n=1}^{\infty} t_n < \infty$, *then* $\lim_{n \to \infty} a_n$ *exists.*

Lemma 2.4. ([3]) *Let* X *be a uniformly convex Banach space,* C *be a nonempty closed convex subset of* X*, and let* $T : C \to X$ *be an asymptotically nonexpansive mapping with a sequence* $\{k_n\} \subset [1, +\infty)$ *and* $k_n \to 1$ *as* $n \to \infty$*. Then* $I - T$ *is demiclosed at zero, i.e., if* $x_n \rightharpoonup x$ *and* $x_n - Tx_n \to 0$*, then* $x \in F(T)$*, where* $F(T)$ *is the set of fixed points of* T*.*

3. Main Results

Theorem 3.1. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H *and let* $F : C \times C \to R$ *be a bifunction satisfying* (A1)-(A4)*. Let* A *be an* α*-inverse-strongly monotone mapping of* C *into* H *and let* $S, T : C \to C$ *be two asymptotically nonexpansive mappings with sequence* $\{s_n\} \subset [1, +\infty)$ *and* $\{t_n\} \subset [1, +\infty)$ *such that* $\sum_{n=1}^{\infty}(s_n - 1) < \infty$, $\sum_{n=1}^{\infty}(t_n - 1) < \infty$, $s_n \to 1$, $t_n \to 1$ *as* $n \to \infty$*, respectively and* $F = F(S) \bigcap F(T) \bigcap \Omega \neq \emptyset$*. From an arbitrary*

$x_1 \in C$, define the following sequence $\{x_n\}$:

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)T^n z_n, \\ z_n = \beta_n x_n + (1-\beta_n)S^n x_n, \\ x_{n+1} \in C \text{ such that} \\ F(x_{n+1}, y) + \langle Ay_n, y - x_{n+1}\rangle + \frac{1}{\lambda_n}\langle y - x_{n+1}, x_{n+1} - y_n\rangle \geq 0, \end{cases} \tag{3.1}$$

for all $y \in C$, $n \in \mathbb{N}$, where $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset (0,1)$ and $\{\lambda_n\} \subset [0, 2\alpha]$ satisfy:

(B1) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

(B2) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$,

(B3) $\lambda_n \in [a, b]$ *for some* $0 < a < b < 2\alpha$.

Then $\{x_n\}$ converges weakly to $z \in F$, where $z = \lim_{n\to\infty} P_F x_n$. Further, if one of T and S is completely continuous, then $\{x_n\}$ converges strongly to $z \in F$. Again, if one of T and S is semi-compact, then $\{x_n\}$ also converges strongly to $z \in F$.

Proof. Setting $t_n = 1+w_n$, $s_n = 1+v_n$. Since $\sum_{n=1}^{\infty}(s_n - 1) < \infty$, $\sum_{n=1}^{\infty}(t_n - 1) < \infty$, so, $\sum_{n=1}^{\infty} v_n < \infty$, $\sum_{n=1}^{\infty} w_n < \infty$. Note that x_{n+1} can be rewritten as

$$x_{n+1} = T_{\lambda_n}(y_n - \lambda_n A y_n)$$

for each $n \in \mathbb{N}$. Let $p \in F$. Since $p = T_{\lambda_n}(p - \lambda_n Ap)$, by Lemma 2.1 and (2.3), we have $\|x_{n+1} - p\| \leq \|y_n - p\|$. Using (3.1), we have

$$\begin{aligned} \|z_n - p\| &\leq \beta_n\|x_n - p\| + (1-\beta_n)(1+v_n)\|x_n - p\| \\ &= \|x_n - p\| + (1-\beta_n)v_n\|x_n - p\| \leq (1+v_n)\|x_n - p\| \end{aligned}$$

and so

$$\begin{aligned} \|x_{n+1} - p\| \leq \|y_n - p\| &\leq \alpha_n\|x_n - p\| + (1-\alpha_n)(1+w_n)\|z_n - p\| \\ &\leq \alpha_n\|x_n - p\| + (1-\alpha_n)(1+w_n)(1+v_n)\|x_n - p\| \\ &= \|x_n - p\| + (1-\alpha_n)[(1+w_n)(1+v_n) - 1]\|x_n - p\| \\ &\leq (1 + w_n + v_n + w_n v_n)\|x_n - p\| \\ &\leq \prod_{i=1}^{n}(1 + w_i + v_i + w_i v_i)\|x_1 - p\| \\ &\leq e^{\sum_{i=1}^{n}(w_i + v_i + w_i v_i)}\|x_1 - p\|. \end{aligned}$$

Since $\sum_{n=1}^{\infty}(w_n + v_n + w_n v_n) < \infty$, then $\{x_n\}$ is bounded. It implies that there exists a constant $M > 0$ such that $\|x_n - p\| \leq M$ for all $n \in \mathbb{N}$. So,

$$\|x_{n+1} - p\| \leq \|x_n - p\| + (w_n + v_n + w_n v_n)M.$$

It follows from Lemma 2.3 that $\lim_{n\to\infty} \|x_n - p\|$ exists. By (2.3) and Lemma 2.2, we have

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\leq \|y_n - p\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ay_n - Ap\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)t_n^2\|z_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - T^n z_n\|^2 \\
&\quad + \lambda_n(\lambda_n - 2\alpha)\|Ay_n - Ap\|^2 \\
&= \|x_n - p\|^2 + (1-\alpha_n)[t_n^2\|z_n - p\|^2 - \|x_n - p\|^2] \\
&\quad - \alpha_n(1-\alpha_n)\|x_n - T^n z_n\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ay_n - Ap\|^2 \\
&\leq \|x_n - p\|^2 + (1-\alpha_n)[t_n^2(\|x_n - p\|^2 + (1-\beta_n)(s_n^2 - 1)\|x_n - p\|^2 \\
&\quad - \beta_n(1-\beta_n)\|x_n - S^n x_n\|^2) - \|x_n - p\|^2] \\
&\quad - \alpha_n(1-\alpha_n)\|x_n - T^n z_n\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ay_n - Ap\|^2 \\
&= \|x_n - p\|^2 + (1-\alpha_n)[(t_n^2 - 1)\|x_n - p\|^2 \\
&\quad + t_n^2(1-\beta_n)(s_n^2 - 1)\|x_n - p\|^2] \\
&\quad - (1-\alpha_n)t_n^2\beta_n(1-\beta_n)\|x_n - S^n x_n\|^2 \\
&\quad - \alpha_n(1-\alpha_n)\|x_n - T^n z_n\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ay_n - Ap\|^2 \\
&\leq \|x_n - p\|^2 + [(t_n^2 - 1) + t_n^2(s_n^2 - 1)]M^2 \\
&\quad - (1-\alpha_n)\beta_n(1-\beta_n)\|x_n - S^n x_n\|^2 \\
&\quad - \alpha_n(1-\alpha_n)\|x_n - T^n z_n\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ay_n - Ap\|^2.
\end{aligned} \tag{3.2}$$

Hence,

$$\begin{aligned}
&(1-\alpha_n)\beta_n(1-\beta_n)\|x_n - S^n x_n\|^2 \\
&\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + [(t_n^2 - 1) + t_n^2(s_n^2 - 1)]M^2,
\end{aligned}$$

$$\begin{aligned}
&\alpha_n(1-\alpha_n)\|x_n - T^n z_n\|^2 \\
&\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + [(t_n^2 - 1) + t_n^2(s_n^2 - 1)]M^2
\end{aligned}$$

and

$$\begin{aligned}
&-\lambda_n(\lambda_n - 2\alpha)\|Ay_n - Ap\|^2 \\
&\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + [(t_n^2 - 1) + t_n^2(s_n^2 - 1)]M^2.
\end{aligned}$$

By $(B1)-(B3)$ and $\lim_{n\to\infty}\|x_n-p\|$ exists, $t_n\to 1$, $s_n\to 1$, we have

$$\lim_{n\to\infty}\|x_n-S^nx_n\|=\lim_{n\to\infty}\|x_n-T^nz_n\|=\lim_{n\to\infty}\|Ay_n-Ap\|=0. \tag{3.3}$$

Since

$$\|z_n-x_n\|=(1-\beta_n)\|S^nx_n-x_n\|\to 0,\ \text{as } n\to\infty,$$

we have

$$\begin{aligned}\|x_n-T^nx_n\| &\le \|x_n-T^nz_n\|+\|T^nz_n-T^nx_n\|\\ &\le \|x_n-T^nz_n\|+t_n\|z_n-x_n\|\\ &\to 0,\quad \text{as } n\to\infty.\end{aligned} \tag{3.4}$$

From (3.1) and (3.3), we have

$$\|y_n-x_n\|=(1-\alpha_n)\|T^nz_n-x_n\|\to 0,\ \text{as } n\to\infty. \tag{3.5}$$

Using Lemma 2.1 and (3.1), we have

$$\begin{aligned}&\|x_{n+1}-p\|^2\\ &=\|T_{\lambda_n}(y_n-\lambda_nAy_n)-T_{\lambda_n}(p-\lambda_nAp)\|^2\\ &\le\langle y_n-\lambda_nAy_n-(p-\lambda_nAp),x_{n+1}-p\rangle\\ &=\frac{1}{2}\big(\|y_n-\lambda_nAy_n-(p-\lambda_nAp)\|^2+\|x_{n+1}-p\|^2\\ &\quad-\|y_n-\lambda_nAy_n-(p-\lambda_nAp)-(x_{n+1}-p)\|^2\big)\\ &\le\frac{1}{2}\big(\|y_n-p\|^2+\|x_{n+1}-p\|^2-\|(y_n-x_{n+1})-\lambda_n(Ay_n-Ap)\|^2\big)\\ &=\frac{1}{2}\big(\|y_n-p\|^2+\|x_{n+1}-p\|^2-\|y_n-x_{n+1}\|^2\\ &\quad-\lambda_n^2\|Ay_n-Ap\|^2+2\lambda_n\langle y_n-x_{n+1},Ay_n-Ap\rangle\big).\end{aligned}$$

So, we have

$$\begin{aligned}\|x_{n+1}-p\|^2\le{}&\|y_n-p\|^2-\|y_n-x_{n+1}\|^2\\ &-\lambda_n^2\|Ay_n-Ap\|^2+2\lambda_n\langle y_n-x_{n+1},Ay_n-Ap\rangle.\end{aligned} \tag{3.6}$$

Then, from (3.2) and (3.6), we have

$$\begin{aligned}\|x_{n+1}-p\|^2&\le\|y_n-p\|^2-\|y_n-x_{n+1}\|^2-\lambda_n^2\|Ay_n-Ap\|^2\\ &\quad+2\lambda_n\langle y_n-x_{n+1},Ay_n-Ap\rangle\\ &\le\|x_n-p\|^2+[(t_n^2-1)+t_n^2(s_n^2-1)]M^2\\ &\quad-\|y_n-x_{n+1}\|^2+2\lambda_n\langle y_n-x_{n+1},Ay_n-Ap\rangle.\end{aligned}$$

So, we have

$$\begin{aligned}\|y_n-x_{n+1}\|^2\le{}&\|x_n-p\|^2-\|x_{n+1}-p\|^2+[(t_n^2-1)+t_n^2(s_n^2-1)]M^2\\ &+2\lambda_n\langle y_n-x_{n+1},Ay_n-Ap\rangle.\end{aligned}$$

Since $\lim_{n\to\infty} \|x_n - p\|$ exists, $t_n \to 1$, $s_n \to 1$, $\lim_{n\to\infty} \|Ay_n - Ap\| = 0$, we have

$$\lim_{n\to\infty} \|y_n - x_{n+1}\| = 0. \tag{3.7}$$

It follows from (3.5) and (3.7) that

$$\|x_n - x_{n+1}\| \le \|x_n - y_n\| + \|y_n - x_{n+1}\| \to 0, \quad \text{as } n \to \infty.$$

Hence,

$$\begin{aligned}\|Tx_n - x_n\| &\le \|Tx_n - T^{n+1}x_n\| + \|T^{n+1}x_n - T^{n+1}x_{n+1}\| \\ &\quad + \|T^{n+1}x_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| \\ &\le t_1\|x_n - T^n x_n\| + (t_\infty + 1)\|x_n - x_{n+1}\| \\ &\quad + \|T^{n+1}x_{n+1} - x_{n+1}\| \\ &\to 0, \quad \text{as } n \to \infty,\end{aligned} \tag{3.8}$$

where $t_\infty = \sup\{t_n : n \in \mathbb{N}\}$. Similarly, we have

$$\|Sx_n - x_n\| \to 0, \qquad \text{as } n \to \infty. \tag{3.9}$$

Noticing that $\{x_n\}$ is bounded, we obtain that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup w \in C$. By Lemma 2.4, we have $w \in F(T) \bigcap F(S)$. Let us show $w \in \Omega$. From (3.5) and (3.7), we have $y_{n_k} \rightharpoonup w$ and $x_{n_k+1} \rightharpoonup w$. Since $x_{n+1} = T_{\lambda_n}(y_n - \lambda_n Ay_n)$, for any $y \in C$ we have

$$F(x_{n+1}, y) + \langle y - x_{n+1}, Ay_n\rangle + \frac{1}{\lambda_n}\langle y - x_{n+1}, x_{n+1} - y_n\rangle \ge 0.$$

From (A2), we also have

$$\langle y - x_{n+1}, Ay_n\rangle + \frac{1}{\lambda_n}\langle y - x_{n+1}, x_{n+1} - y_n\rangle \ge F(y, x_{n+1}). \tag{3.10}$$

Put $z_t = ty + (1-t)w$ for all $t \in (0,1]$ and $y \in C$. Then, we have $z_t \in C$. So, from (3.10) we have

$$\begin{aligned}\langle z_t - x_{n+1}, Az_t\rangle &\ge \langle z_t - x_{n+1}, Az_t\rangle - \langle z_t - x_{n+1}, Ay_n\rangle \\ &\quad - \langle z_t - x_{n+1}, \frac{x_{n+1} - y_n}{\lambda_n}\rangle + F(z_t, x_{n+1}) \\ &= \langle z_t - x_{n+1}, Az_t - Ax_{n+1}\rangle + \langle z_t - x_{n+1}, Ax_{n+1} - Ay_n\rangle \\ &\quad - \langle z_t - x_{n+1}, \frac{x_{n+1} - y_n}{\lambda_n}\rangle + F(z_t, x_{n+1}).\end{aligned}$$

Since $\|x_{n+1} - y_n\| \to 0$, we have $\|Ax_{n+1} - Ay_n\| \to 0$. Further, from monotonicity of A, we have $\langle z_t - x_{n+1}, Az_t - Ax_{n+1}\rangle \ge 0$. So, replacing n by n_k, from (A4) we have

$$\langle z_t - w, Az_t\rangle \ge F(z_t, w), \ \text{ as } \ k \to \infty. \tag{3.11}$$

From (A1),(A4) and (3.11), we also have

$$\begin{aligned} 0 = F(z_t, z_t) &\leq tF(z_t, y) + (1-t)F(z_t, w) \\ &\leq tF(z_t, y) + (1-t)\langle z_t - w, Az_t\rangle \\ &= tF(z_t, y) + (1-t)t\langle y - w, Az_t\rangle \end{aligned}$$

and hence

$$0 \leq F(z_t, y) + (1-t)\langle y - w, Az_t\rangle.$$

Letting $t \to 0$, we have, for each $y \in C$,

$$0 \leq F(w, y) + \langle y - w, Aw\rangle.$$

This implies $w \in \Omega$. Therefore, $w \in F$. Define $u_n = P_F x_n$ for all $n \in \mathbb{N}$. Since $w \in F$, we have $\|u_n - x_n\| \leq \|w - x_n\|$. Then $\{u_n\}$ is bounded. From (3.2), we have

$$\|x_{n+1} - u_n\|^2 \leq \|x_n - u_n\|^2 + \theta_n \|x_n - u_n\|^2, \tag{3.12}$$

where $\theta_n = [(t_n^2 - 1) + t_n^2(s_n^2 - 1)]$. By $u_{n+1} = P_F x_{n+1}$ and $u_n = P_F x_n \in F$, we have

$$\|u_{n+1} - x_{n+1}\|^2 \leq \|u_n - x_{n+1}\|^2 \leq \|u_n - x_n\|^2 + \theta_n M^*,$$

where $M^* = \sup\{\|x_n - u_n\|^2 : n \in \mathbb{N}\}$. Since $\sum_{n=1}^{\infty} \theta_n < \infty$, it follows from Lemma 2.3 that $\lim_{n\to\infty} \|u_n - x_n\|$ exists. Again, using (3.12), for all $m \in \mathbb{N}$, we have

$$\|x_{n+m} - u_n\|^2 \leq \prod_{i=0}^{m-1} (1 + \theta_{n+i}) \|x_n - u_n\|^2.$$

From $u_{n+m} = P_F x_{n+m}$ and $u_n = P_F x_n \in F$, we have

$$\begin{aligned} \|u_n - u_{n+m}\|^2 &\leq \|u_n - x_{n+m}\|^2 - \|u_{n+m} - x_{n+m}\|^2 \\ &\leq \prod_{i=0}^{m-1} (1 + \theta_{n+i}) \|x_n - u_n\|^2 - \|u_{n+m} - x_{n+m}\|^2 \\ &\leq e^{\sum_{i=0}^{m-1} \theta_{n+i}} \|x_n - u_n\|^2 - \|u_{n+m} - x_{n+m}\|^2. \end{aligned}$$

Since $\sum_{n=1}^{\infty} \theta_n < \infty$ and $\lim_{n\to\infty} \|u_n - x_n\|$ exists, we obtain that $\{u_n\}$ is a Cauchy sequence. Since F is closed, we have that $\{u_n\}$ converges strongly to $z \in F$. On the other hand, noticing that $w \in F$ and $u_n = P_F x_n$, we have

$$\langle x_{n_k} - u_{n_k}, u_{n_k} - w\rangle \geq 0.$$

Letting $k \to \infty$, we have

$$\langle w - z, z - w\rangle \geq 0.$$

Hence, $w = z$. Therefore, $\{x_n\}$ converges weakly to $z \in F$, where $z = \lim_{n\to\infty} P_F x_n$.

If T or S is completely continuous, then we have $Tx_{n_k} \to z$ or $Sx_{n_k} \to z$, as $k \to \infty$. By (3.8) or (3.9), we have $x_n \to z$.

If one of T and S is semi-compact, then, by (3.8) or (3.9), there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to $q \in C$. It follows from (3.8), (3.9) and Lemma 2.4 that $q \in F$. Since $\lim_{n\to\infty} \|x_n - q\|$ exists, then $\{x_n\}$ converges strongly to q. Since $\{x_n\}$ converges weakly to $z \in F$, we have $q = z$, where $z = \lim_{n\to\infty} P_F x_n$. □

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $F : C \times C \to R$ be a bifunction satisfying* (A1)-(A4). *Let A be an α-inverse-strongly monotone mapping of C into H and let $S, T : C \to C$ be two nonexpansive mappings such that $F = F(S) \bigcap F(T) \bigcap \Omega \neq \emptyset$. If $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset (0,1)$ and $\{\lambda_n\} \subset [0, 2\alpha]$ satisfy $(B1) - (B3)$, then the sequence $\{x_n\}$ defined by:*

$$\begin{cases} x_1 \in C, \quad \text{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n) T z_n, \\ z_n = \beta_n x_n + (1-\beta_n) S x_n, \\ x_{n+1} \in C \text{ such that} \\ F(x_{n+1}, y) + \langle A y_n, y - x_{n+1} \rangle + \frac{1}{\lambda_n} \langle y - x_{n+1}, x_{n+1} - y_n \rangle \geq 0, \end{cases}$$

for all $y \in C$, $n \in \mathbb{N}$, converges weakly to $z \in F$, where $z = \lim_{n\to\infty} P_F x_n$. Further, if one of T and S is completely continuous, then $\{x_n\}$ converges strongly to $z \in F$. Again, if one of T and S is semi-compact, then $\{x_n\}$ also converges strongly to $z \in F$.

Proof. In Theorem 3.1, put $t_n = s_n = 1$ for all $n \in \mathbb{N}$. Then, we can obtain the desired result by Theorem 3.1. □

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