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L^P MEAN ESTIMATES FOR B-OPERATORS

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Abstract. If $P(z)$ be a polynomial of degree at most n which does not vanish in $|z| < 1$, then for $0 \leq p < \infty$ and $R > 1$, it is known that

$$\|B[P \circ \rho](z)\|_p \leq \frac{\|R^n \phi_n(\lambda_0, \lambda_1, \lambda_2) z + \lambda_0\|_p}{\|1 + z\|_p} \|P(z)\|_p ,$$

$B \in \mathcal{B}_n$, $\rho(z) = Rz$ and $\phi_n(\lambda_0, \lambda_1, \lambda_2)$ is defined by (1.13). The result is sharp as shown by $P(z) = az^n + b$, $|a| = |b| = 1$. In this paper, we present a compact generalization of above and other related results.

1. Introduction

Let $\mathscr{P}_n$ denote the space of all complex polynomials $P(z) = \sum_{j=0}^{n} a_j z^j$ of degree at most n. For $P \in \mathscr{P}_n$, define

$$\|P(z)\|_0 := \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log\left|P(e^{i\theta})\right| d\theta\right\},$$

$$\|P(z)\|_p := \left\{\frac{1}{2\pi}\int_0^{2\pi} \left|P(e^{i\theta})\right|^p\right\}^{1/p}, \quad 0 < p < \infty,$$

$$\|P(z)\|_\infty := \underset{|z|=1}{Max} |P(z)| ,$$

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and denote for any complex function $\rho : \mathbb{C} \to \mathbb{C}$, the composite function $P \circ \rho$ of P and ρ, defined by $(P \circ \rho)(z) := P(\rho(z)) \quad (z \in \mathbb{C})$.

A famous known result as Bernstein's inequality (for reference, see [13, p.531], [18, p.508] or [20] states that if $P \in \mathscr{P}_n$, then

$$\left\|P'(z)\right\|_\infty \leq n \left\|P(z)\right\|_\infty, \tag{1.1}$$

whereas concerning the maximum modulus of $P(z)$ on the circle $|z| = R > r \geq 1$, we have

$$\left\|P(Rz)\right\|_\infty \leq R^n \left\|P(z)\right\|_\infty, \quad R \geq 1, \tag{1.2}$$

(for reference, see [12, p.442] or [13, vol.I, p.137]).
Inequalities (1.1) and (1.2) can be obtained by letting $p \to \infty$ in the inequalities

$$\left\|P'(z)\right\|_p \leq n \left\|P(z)\right\|_p, \quad p \geq 1 \tag{1.3}$$

and

$$\left\|P(Rz)\right\|_p \leq R^n \left\|P(z)\right\|_p, \quad R > r \geq 1, \quad p > 0, \tag{1.4}$$

respectively. Inequality (1.3) was found by Zygmund [22] whereas inequality (1.4) is a simple consequence of a result of Hardy [9] (see also [16, Theorem 5.5]). Since inequality (1.3) was deduced from M. Riesz's interpolation formula [19] by means of Minkowski's inequality, it was not clear, whether the restriction on p was indeed essential. This question was open for a long time. Finally Arestov [2] proved that (1.3) remains true for $0 < p < 1$ as well.

If we restrict ourselves to the class of polynomials $P \in \mathscr{P}_n$ having no zero in $|z| < 1$, then inequalities (1.1) and (1.2) can be respectively replaced by

$$\left\|P'(z)\right\|_\infty \leq \frac{n}{2} \left\|P(z)\right\|_\infty \tag{1.5}$$

and

$$\left\|P(Rz)\right\|_\infty \leq \frac{R^n + 1}{2} \left\|P(z)\right\|_\infty, \quad R > r \geq 1. \tag{1.6}$$

Inequality (1.5) was conjectured by Erdös and later verified by Lax [10], whereas inequality (1.6) is due to Ankey and Ravilin [1].
Both the inequalities (1.5) and (1.6) can be obtain by letting $p \to \infty$ in the inequalities

$$\left\|P'(z)\right\|_p \leq n \frac{\left\|P(z)\right\|_p}{\left\|1+z\right\|_p}, \quad p \geq 0 \tag{1.7}$$

and

$$\left\|P(Rz)\right\|_p \leq \frac{\left\|R^n z + 1\right\|_p}{\left\|1+z\right\|_p} \left\|P(z)\right\|_p, \quad R > r \geq 1, \quad p > 0. \tag{1.8}$$

Inequality (1.7) is due to De-Bruijn [7] for $p \geq 1$. Rahman and Schmeisser [17] extended it for $0 \leq p < 1$ whereas the inequality (1.8) was proved by Boas and Rahman [6] for $p \geq 1$ and later it was extended for $0 \leq p < 1$ by Rahman and Schmeisser [17].

Q.I. Rahman [14] (see also Rahman and Schmeisser [18, p. 538]) introduced a class $\mathcal{B}_n$ of operators B that carries a polynomial $P \in \mathscr{P}_n$ into

$$B[P](z) := \lambda_0 P(z) + \lambda_1 \left(\frac{nz}{2}\right) \frac{P'(z)}{1!} + \lambda_2 \left(\frac{nz}{2}\right)^2 \frac{P''(z)}{2!}, \tag{1.9}$$

where λ_0, λ_1 and λ_2 are such that all the zeros of

$$U(z) := \lambda_0 + \lambda_1 C(n,1)z + \lambda_2 C(n,2)z^2, \tag{1.10}$$

where $C(n,r) = \dfrac{n!}{r!(n-r)!}$, $\quad 0 \leq r \leq n$, lie in half plane $|z| \leq |z - n/2|$.

As a generalization of inequality (1.1) and (1.5), Q.I. Rahman [14, inequality 5.2 and 5.3] proved that if $P \in \mathscr{P}_n$ and $B \in \mathcal{B}_n$, then

$$|B[P](z)| \leq |\phi_n(\lambda_0, \lambda_1, \lambda_2)| \|P(z)\|_\infty \quad \text{for} \quad |z| \geq 1 \tag{1.11}$$

and if $P \in \mathscr{P}_n$, $P(z) \neq 0$ in $|z| < 1$, then

$$|B[P](z)| \leq \frac{1}{2} \{|\phi_n(\lambda_0, \lambda_1, \lambda_2)| + |\lambda_0|\} \|P(z)\|_\infty \quad \text{for} \quad |z| \geq 1, \tag{1.12}$$

where

$$\phi_n(\lambda_0, \lambda_1, \lambda_2) = \lambda_0 + \lambda_1 \frac{n^2}{2} + \lambda_2 \frac{n^3(n-1)}{8}. \tag{1.13}$$

As a corresponding generalization of inequalities (1.2) and (1.4), Rahman and Schmeisser [18, p. 538] proved that if $P \in \mathscr{P}_n$, then

$$|B[P \circ \rho](z)| \leq R^n |\phi_n(\lambda_0, \lambda_1, \lambda_2)| \|P(z)\|_\infty \quad \text{for} \quad |z| = 1 \tag{1.14}$$

and if $P \in \mathscr{P}_n$, $P(z) \neq 0$ in $|z| < 1$, then as a special case of Corollary 14.5.6 in [18, p. 539], we have

$$|B[P \circ \rho](z)| \leq \frac{1}{2} \{R^n |\phi_n(\lambda_0, \lambda_1, \lambda_2)| + |\lambda_0|\} \|P(z)\|_\infty \quad \text{for} \quad |z| = 1, \tag{1.15}$$

where $\rho(z) := Rz$, $R \geq 1$ and $\phi_n(\lambda_0, \lambda_1, \lambda_2)$ is defined by (1.13).

Inequality (1.15) also follows by combining the inequalities (5.2) and (5.3) due to Rahman [14].

As an extension of inequality (1.14) to L_p-norm, recently Shah and Liman [21, Theorem 1] proved:

Theorem 1.1. *If $P \in \mathscr{P}_n$, then for every $R \geq 1$ and $p \geq 1$,*

$$\|B[P \circ \rho](z)\|_p \leq R^n |\phi_n(\lambda_1, \lambda_2, \lambda_3)| \|P(z)\|_p, \tag{1.16}$$

where $B \in \mathcal{B}_n$, $\rho(z) = Rz$ and $\phi_n(\lambda_0, \lambda_1, \lambda_2)$ is defined by (1.13).

While seeking the analogue of (1.15) in L_p norm, they [21, Theorem 2] have made an incomplete attempt by claiming to have proved the following result:

Theorem 1.2. *If $P \in \mathscr{P}_n$ and $P(z)$ does not vanish for $|z| \leq 1$, then for each $p \geq 1$, $R \geq 1$,*

$$\|B[P \circ \rho](z)\|_p \leq \frac{R^n|\phi_n(\lambda_1, \lambda_2, \lambda_3)| + |\lambda_0|}{\|1+z\|_p} \|P(z)\|_p , \tag{1.17}$$

where $B \in \mathcal{B}_n$, $\rho(z) = Rz$ and $\phi_n(\lambda_1, \lambda_2, \lambda_3)$ is defined by (1.13).

Unfortunately the proof of inequality (1.17) and other related results including the key lemma [21, Lemma 4] given by Shah and Liman is not correct as is pointed out by Rather and Shah [18] who in the same paper have given a correct proof of the inequality (1.17) and also extended it for $0 \leq p < 1$ as well. More precisely they proved:

Theorem 1.3. *If $P \in \mathscr{P}_n$ and $P(z)$ does not vanish for $|z| < 1$, then for $0 \leq p < \infty$ and $R > 1$,*

$$\|B[P \circ \rho](z)\|_p \leq \frac{\|R^n\phi_n(\lambda_0, \lambda_1, \lambda_2)z + \lambda_0\|_p}{\|1+z\|_p} \|P(z)\|_p , \tag{1.18}$$

$B \in \mathcal{B}_n$, $\rho(z) = Rz$ and $\phi_n(\lambda_0, \lambda_1, \lambda_2)$ is defined by (1.13). *The result is sharp as shown by $P(z) = az^n + b$, $|a| = |b| = 1$.*

2. Preliminaries

For the proofs of this theorem, we need the following lemmas. The first lemma follows from Corollary 18.3 of [11, p. 86].

Lemma 2.1. *If $P \in \mathscr{P}_n$ and $P(z)$ has all zeros in $|z| \leq 1$, then all the zeros of $B[P](z)$ also lie in $|z| \leq 1$.*

Lemma 2.2. *If $P \in \mathscr{P}_n$ and $P(z)$ have all its zeros in $|z| \leq 1$, then for every $R > r \geq 1$ and $|z| = 1$,*

$$|P(Rz)| \geq \left(\frac{R+1}{r+1}\right)^n |P(z)| .$$

Proof. Since all the zeros of $P(z)$ lie in $|z| \leq 1$, we write

$$P(z) = C\prod_{j=1}^{n}\left(z - r_je^{i\theta_j}\right),$$

where $r_j \leq 1$. Now for $0 \leq \theta < 2\pi$, $R > 1$, we have

$$\left|\frac{Re^{i\theta} - r_j e^{i\theta_j}}{e^{i\theta} - r_j e^{i\theta_j}}\right| = \left\{\frac{R^2 + r_j^2 - 2Rr_j\cos(\theta - \theta_j)}{1 + r_j^2 - 2r_j\cos(\theta - \theta_j)}\right\}^{1/2}$$

$$\geq \left\{\frac{R + r_j}{1 + r_j}\right\}$$

$$\geq \left\{\frac{R + 1}{r + 1}\right\}, \quad \text{for } j = 1, 2, \cdots, n.$$

Hence

$$\left|\frac{P(Re^{i\theta})}{P(e^{i\theta})}\right| = \prod_{j=1}^{n}\left|\frac{Re^{i\theta} - r_j e^{i\theta_j}}{e^{i\theta} - r_j e^{i\theta_j}}\right|$$

$$\geq \prod_{j=1}^{n}\left(\frac{R+1}{r+1}\right)$$

$$= \left(\frac{R+1}{r+1}\right)^n,$$

for $0 \leq \theta < 2\pi$. This implies for $|z| = 1$,

$$|P(Rz)| \geq \left(\frac{R+1}{r+1}\right)^n |P(z)|,$$

which completes the proof of Lemma 2.2. □

Lemma 2.3. *If $P \in \mathscr{P}_n$ and $P(z)$ has all its zeros in $|z| \leq 1$, then for every real or complex number α with $|\alpha| \leq 1$ and $|z| \geq 1$,*

$$|B[P \circ \rho](z) - \alpha B[P \circ \varrho](z)| \geq |R^n - \beta||\phi(\lambda_0, \lambda_1, \lambda_2)||z|^n m, \tag{2.1}$$

where $m = \underset{|z|=1}{Min}|P(z)|$, $\rho(z) = Rz$ and $\phi(\lambda_0, \lambda_1, \lambda_2)$ is given by (1.13).

Proof. By hypothesis, all the zeros of $P(z)$ lie in $|z| \leq 1$ and

$$m|z|^n \leq |P(z)| \quad \text{for } |z| = 1.$$

We first show that the polynomial $g(z) = P(z) - \beta m z^n$ has all its zeros in $|z| \leq 1$ for every real or complex number β with $|\beta| < 1$. This is obvious if $m = 0$, that is if $P(z)$ has a zero on $|z| = 1$. Henceforth, we assume $P(z)$ has all its zeros in $|z| < 1$, then $m > 0$ and it follows by Rouche's theorem that the polynomial $g(z)$ has all its zeros in $|z| < 1$ for every real or complex number β with $|\beta| < 1$. Applying Lemma 2.2 to the polynomial $g(z)$, we deduce

$$|g(Rz)| \geq \left(\frac{R+1}{r+1}\right)^n |g(z)| \quad \text{for } |z| = 1.$$

Since $R > r$, therefore $\frac{R+1}{r+1} > 1$, this gives

$$|g(Rz)| > |g(z)| \quad \text{for} \quad |z| = 1. \tag{2.2}$$

Since all the zeros of $G(Rz)$ lie in $|z| < 1/R < 1$, by Rouche's theorem again it follows from (2.2) that all the zeros of polynomial

$$H(z) = g(Rz) - \alpha g(z) = P(Rz) - \alpha P(z) - \beta(R^n - \alpha r^n)z^n m$$

lie in $|z| < 1$, for every α, β with $|\alpha| \leq 1$, $|\beta| < 1$. Applying Lemma 2.1 to $H(z)$ and noting that B is a linear operator, it follows that all the zeros of polynomial

$$\begin{aligned} B[H](z) &= B[g \circ \rho](z) - \alpha B[g](z) \\ &= \{B[P \circ \rho](z) - \alpha B[P \circ \varrho](z)\} - \beta(R^n - \alpha r^n) m B[z^n] \end{aligned} \tag{2.3}$$

lie in $|z| < 1$. This gives for $|z| \geq 1$,

$$|B[P \circ \rho](z) - \alpha B[P \circ \varrho](z)| \geq |R^n - \alpha r^n||\phi(\lambda_0, \lambda_1, \lambda_2)||z|^n m. \tag{2.4}$$

If (2.4) is not true, then there is point w with $|w| \geq 1$ such that

$$|B[P \circ \rho](w) - \alpha B[P \circ \varrho](w)| < |R^n - \alpha r^n||\phi(\lambda_0, \lambda_1, \lambda_2)||w|^n m. \tag{2.5}$$

We choose

$$\beta = \frac{B[P \circ \rho](w) - \alpha B[P \circ \varrho](w)}{(R^n - \alpha r^n)\phi(\lambda_0, \lambda_1, \lambda_2) w^n m},$$

then clearly $|\beta| < 1$ and with this choice of β, from (2.3), we get $B[H](w) = 0$ with $|w| \geq 1$. This is clearly a contradiction to the fact that all the zeros of $H(z)$ lie in $|z| < 1$. Thus for every real or complex α with $|\alpha| \leq 1$,

$$|B[P \circ \rho](z) - \alpha B[P \circ \varrho](z)| \geq |R^n - \alpha r^n||\phi(\lambda_0, \lambda_1, \lambda_2)||z|^n m$$

for $|z| \geq 1$ and $R > r \geq 1$. □

Lemma 2.4. *If $P \in \mathscr{P}_n$ and $P(z)$ has no zero in $|z| < 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$, $R > r \geq 1$ and $|z| \geq 1$,*

$$|B[P \circ \rho](z) - \alpha B[P \circ \varrho](z)| \leq |B[P^\star \circ \rho](z) - \alpha B[P^\star](z)|, \tag{2.6}$$

where $P^\star(z) = z^n\overline{P(1/\overline{z})}$ and $\rho(z) = Rz$.

Proof. Since the polynomial $P(z)$ has all its zeros in $|z| \geq 1$, therefore, for every real or complex number λ with $|\lambda| > 1$, the polynomial $f(z) = P(z) - \lambda P^\star(z)$, where $P^\star(z) = z^n\overline{P(1/\overline{z})}$, has all zeros in $|z| \leq 1$. Applying Lemma 2.2 to the polynomial $f(z)$, we obtain for every $R > 1$ and $0 \leq \theta < 2\pi$,

$$|f(Re^{i\theta})| \geq \left(\frac{R+1}{r+1}\right)^n |f(e^{i\theta})|. \tag{2.7}$$

Since $f(Re^{i\theta}) \neq 0$ for every $R > r \geq 1$, $0 \leq \theta < 2\pi$ and $R+1 > 2$, it follows from (2.7) that

$$|f(Re^{i\theta})| > \left(\frac{R+1}{r+1}\right)^n |f(Re^{i\theta})| \geq |f(e^{i\theta})|,$$

for every $R > r \geq 1$ and $0 \leq \theta < 2\pi$. This gives

$$|f(z)| < |f(Rz)| \quad \text{for} \quad |z| = 1, \quad R > 1.$$

Using Rouche's theorem and noting that all the zeros of $f(Rz)$ lie in $|z| \leq 1/R < 1$, we conclude that the polynomial

$$T(z) = f(Rz) - \alpha f(z) = \{P(Rz) - \alpha P(z)\} - \lambda \{P^\star(Rz) - \alpha P^\star(z)\}$$

has all its zeros in $|z| < 1$ for every real or complex α with $|\alpha| \geq 1$ and $R > 1$. Applying Lemma 2.1 to polynomial $T(z)$ and noting that B is a linear operator, it follows that all the zeros of polynomial

$$\begin{aligned} B[T](z) &= B[f \circ \rho](z) - \alpha B[f](z) \\ &= \{B[P \circ \rho](z) - \alpha B[P \circ \varrho](z)\} - \lambda \{B[P^\star \circ \rho](z) - \alpha B[P^\star](z)\} \end{aligned}$$

lie in $|z| < 1$ where $\rho(z) = Rz$. This implies

$$|B[P \circ \rho](z) - \alpha B[P \circ \varrho](z)| \leq |B[P^\star \circ \rho](z) - \alpha B[P^\star](z)| \tag{2.8}$$

for $|z| \geq 1$ and $R > r \geq 1$. If inequality (2.8) is not true, then there exits a point $z = z_0$ with $|z_0| \geq 1$ such that

$$|B[P \circ \rho](z_0) - \alpha B[P \circ \varrho](z_0)| > |B[P^\star \circ \rho](z_0) - \alpha B[P^\star](z_0)|. \tag{2.9}$$

But all the zeros of $P^\star(Rz)$ lie in $|z| < 1/R < 1$, therefore, it follows (as in case of $f(z)$) that all the zeros of $P^\star(Rz) - \alpha P^\star(z)$ lie in $|z| < 1$. Hence, by Lemma 2.1, we have

$$B[P^\star \circ \rho](z_0) - \alpha B[P^\star](z_0) \neq 0.$$

We take

$$\lambda = \frac{B[P \circ \rho](z_0) - \alpha B[P \circ \varrho](z_0)}{B[P^\star \circ \rho](z_0) - \alpha B[P^\star](z_0)},$$

then λ is well defined real or complex number with $|\lambda| > 1$ and with this choice of λ, we obtain $B[T](z_0) = 0$ where $|z_0| \geq 1$. This contradicts the fact that all the zeros of $B[T](z)$ lie in $|z| < 1$. Thus (2.8) holds true for $|\alpha| \leq 1$ and $R > r \geq 1$. □

Lemma 2.5. *If $P \in \mathscr{P}_n$ and $P(z)$ has no zero in $|z| < 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$, $R > r \geq 1$ and $|z| \geq 1$,*

$$\begin{aligned} &\left|B[P \circ \rho](z) - \alpha B[P \circ \varrho](z)\right| \\ &\leq |B[P^\star \circ \rho](z) - \alpha B[P^\star](z)| - (|R^n - \alpha r^n| - |1 - \alpha||\lambda_0|)m, \end{aligned} \tag{2.10}$$

where $P^\star(z) = z^n\overline{P(1/\overline{z})}$, $m = \underset{|z|=1}{Min}|P(z)|$ *and* $\rho(z) = Rz$.

Proof. By hypothesis $P(z)$ has all its zeros in $|z| \geq 1$ and

$$m \leq |P(z)| \quad \text{for} \quad |z| = 1. \tag{2.11}$$

We show $F(z) = P(z)+\lambda m$ does not vanish in $|z| < 1$ for every λ with $|\lambda| < 1$. This is obvious if $m = 0$ that is, if $P(z)$ has a zero on $|z| = 1$. So we assume all the zeros of $P(z)$ lie in $|z| > 1$, then $m > 0$ and by the maximum modulus principle, it follows from (2.11),

$$m < |P(z)| \quad \text{for} \quad |z| < 1. \tag{2.12}$$

Now if $F(z) = P(z) + \lambda m = 0$ for some z_0 with $|z_0| < 1$, then

$$P(z_0) + \lambda m = 0.$$

This implies

$$|P(z_0)| = |\lambda|m \leq m \quad \text{for} \quad |z_0| < 1, \tag{2.13}$$

which is clearly contradiction to (2.12). Thus the polynomial $F(z)$ does not vanish in $|z| < 1$ for every λ with $|\lambda| < 1$. Applying Lemma 2.4 to the polynomial $F(z)$, we get

$$|B[F \circ \rho](z) - \alpha B[F](z)| \leq |B[F^\star \circ \rho](z) - \alpha B[F^\star](z)$$

for $|z| = 1$ and $R > r \geq 1$. Replacing $F(z)$ by $P(z) + \lambda m$, we obtain

$$\begin{aligned} &|B[P \circ \rho](z) - \alpha B[P \circ \varrho](z) + \lambda(1-\alpha)\lambda_0 m| \\ &\leq |B[P^\star \circ \rho](z) - \alpha B[P^\star](z) + \bar{\lambda}(R^n - \alpha r^n)\phi(\lambda_0, \lambda_1, \lambda_2)z^n m|. \end{aligned} \tag{2.14}$$

Now choosing the argument of λ in the right hand side of (2.14) such that

$$\begin{aligned} &|B[P^\star \circ \rho](z) - \alpha B[P^\star](z) + \bar{\lambda}(R^n - \alpha r^n)\phi(\lambda_0, \lambda_1, \lambda_2)z^n m| \\ &= |B[P^\star \circ \rho](z) - \alpha B[P^\star](z)| - |\lambda||R^n - \alpha r^n||\phi(\lambda_0, \lambda_1, \lambda_2)|m \end{aligned}$$

for $|z| = 1$, which is possible by Lemma 2.3, we get

$$\begin{aligned} &|B[P^\star \circ \rho](z) - \alpha B[P^\star](z)| - |\lambda||1-\alpha||\lambda_0|m \\ &\leq |B[P^\star \circ \rho](z) - \alpha B[P^\star](z)| - |\lambda||R^n - \alpha r^n||\phi(\lambda_0, \lambda_1, \lambda_2)|m. \end{aligned}$$

Equivalently,

$$\begin{aligned} &|B[P \circ \rho](z) - \alpha B[P \circ \varrho](z)| \\ &\leq |B[P^\star \circ \rho](z) - \alpha B[P^\star](z)| - (|R^n - \alpha r^n| - |1-\alpha||\lambda_0|)m. \end{aligned}$$

This completes the proof of Lemma 2.5. □

Next we describe a result of Arestov [2]. For $\delta = (\delta_0, \delta_1, \cdots, \delta_n) \in \mathbb{C}^{n+1}$ and $P(z) = \sum_{j=0}^{n} a_j z^j \in \mathscr{P}_n$, we define

$$\Lambda_\delta P(z) = \sum_{j=0}^{n} \delta_j a_j z^j.$$

The operator Λ_δ is said to be admissible if it preserves one of the following properties:

(i) $P(z)$ has all its zeros in $\{z \in \mathbb{C} : |z| \leq 1\}$,
(ii) $P(z)$ has all its zeros in $\{z \in \mathbb{C} : |z| \geq 1\}$.

The result of Arestov [2] may now be stated as follows.

Lemma 2.6. ([2, Theorem 4]) *Let $\phi(x) = \psi(logx)$ where ψ is a convex non decreasing function on $\mathbb{R}$. Then for all $P \in \mathscr{P}_n$ and each admissible operator Λ_δ,*

$$\int_0^{2\pi} \phi(|\Lambda_\delta P(e^{i\theta})|)d\theta \leq \int_0^{2\pi} \phi(C(\delta, n)|P(e^{i\theta})|)d\theta,$$

where $C(\delta, n) = max(|\delta_0|, |\delta_n|)$.

In particular, Lemma 2.6 applies with $\phi : x \to x^p$ for every $p \in (0, \infty)$. Therefore, we have

$$\left\{\int_0^{2\pi} (|\Lambda_\delta P(e^{i\theta})|^p)d\theta\right\}^{1/p} \leq C(\delta, n)\left\{\int_0^{2\pi} |P(e^{i\theta})|^p d\theta\right\}^{1/p}. \tag{2.15}$$

We use (2.15) to prove the following interesting result.

Lemma 2.7. *If $P \in \mathscr{P}_n$ and $P(z)$ does not vanish in $|z| < 1$, then for every $p > 0$, $R > 1$ and for γ real, $0 \leq \gamma < 2\pi$,*

$$\begin{aligned}\int_0^{2\pi} \Big|&\Big\{B[P \circ \rho](e^{i\theta}) - \alpha B[P \circ \varrho](e^{i\theta})\Big\} e^{i\gamma} \\ &+ \Big\{B[P^\star \circ \rho]^\star(e^{i\theta}) - \bar{\alpha} B[P^\star]^\star(e^{i\theta})\Big\}\Big|^p d\theta \\ \leq \Big|&(R^n - \alpha)\phi(\lambda_0, \lambda_1, \lambda_2)e^{i\gamma} + (1 - \bar{\alpha})\bar{\lambda_0}\Big|^p \int_0^{2\pi} \Big|P(e^{i\theta})\Big|^p d\theta,\end{aligned} \tag{2.16}$$

where $B \in \mathcal{B}_n$, $\rho(z) := Rz$, $B[P^\star \circ \rho]^\star(z) := (B[P^\star \circ \rho](z))^\star$ and $\phi(\lambda_0, \lambda_1, \lambda_2)$ is defined by (1.13).

Proof. Since $P \in \mathscr{P}_n$ and $P^\star(z) = z^n\overline{P(1/\bar{z})}$, by Lemma 2.4, we have for $|z| \geq 1$,

$$|B[P \circ \rho](z) - \alpha B[P \circ \varrho](z)| \leq |B[P^\star \circ \rho](z) - \alpha B[P^\star](z)|. \tag{2.17}$$

Also, since $P^\star(Rz) - \alpha P^\star(z) = R^n z^n \overline{P(1/R\bar{z})} - \alpha z^n \overline{P(1/\bar{z})}$,

$$\begin{aligned}
&B[P^\star \circ \rho](z) - \alpha B[P^\star](z) \\
&= \lambda_0 \Big\{ R^n z^n \overline{P(1/R\bar{z})} - \alpha z^n \overline{P(1/\bar{z})} \Big\} \\
&\quad + \lambda_1 \left(\frac{nz}{2}\right) \Big\{ \Big(nR^n z^{n-1} \overline{P(1/R\bar{z})} - R^{n-1} z^{n-2} \overline{P'(1/R\bar{z})} \Big) \\
&\quad - \alpha \Big(nz^{n-1} \overline{P(1/\bar{z})} - z^{n-2} \overline{P'(1/\bar{z})} \Big) \Big\} \\
&\quad + \frac{\lambda_2}{2!} \left(\frac{nz}{2}\right)^2 \Big\{ \Big(n(n-1) R^n z^{n-2} \overline{P(1/R\bar{z})} \\
&\quad - 2(n-1) R^{n-1} z^{n-3} \overline{P'(1/R\bar{z})} + R^{n-2} z^{n-4} \overline{P''(1/R\bar{z})} \Big) \\
&\quad - \alpha \Big(n(n-1) z^{n-2} \overline{P(1/\bar{z})} - 2(n-1) z^{n-3} \overline{P'(1/\bar{z})} + r^{n-2} z^{n-4} \overline{P''(1/\bar{z})} \Big) \Big\}
\end{aligned}$$

and

$$\begin{aligned}
B[P^\star \circ \rho]^\star(z) - \bar{\alpha} B[P^\star]^\star(z) &= \Big(B[P^\star \circ \rho](z) - \alpha B[P^\star](z) \Big)^\star \\
= \Big(\bar{\lambda}_0 + \bar{\lambda}_1 \frac{n^2}{2} + \bar{\lambda}_2 \frac{n^3(n-1)}{8} \Big) &\Big\{ R^n P(z/R) - \bar{\alpha} P(z) \Big\} \\
- \Big(\bar{\lambda}_1 \frac{n}{2} + \bar{\lambda}_2 \frac{n^2(n-1)}{4} \Big) &\Big\{ R^{n-1} z P'(z/R) - \bar{\alpha} z P'(z) \Big\} \\
+ \bar{\lambda}_2 \frac{n^2}{8} &\Big\{ R^{n-2} z^2 P''(z/R) - \bar{\alpha} z^2 P''(z) \Big\}. \qquad (2.18)
\end{aligned}$$

Also,

$$|B[P^\star \circ \rho](z) - \alpha B[P^\star](z)| = |B[P^\star \circ \rho]^\star(z) - \bar{\alpha} B[P^\star]^\star(z)| \quad \text{for } |z| = 1.$$

Using this in (2.17), we get

$$|B[P \circ \rho](z) - \alpha B[P \circ \varrho](z)| \leq |B[P^\star \circ \rho]^\star(z) - \bar{\alpha} B[P^\star]^\star(z)| \quad \text{for } |z| = 1.$$

As in Lemma 2.4, the polynomial $P^\star \circ \rho(z) - \alpha P^\star(z)$ has all its zeros in $|z| < 1$ and by Lemma 2.1, $B[P^\star \circ \rho](z) - \alpha B[P^\star](z)$ also has all its zero in $|z| < 1$. Therefore, $B[P^\star \circ \rho]^\star(z) - \bar{\alpha} B[P^\star]^\star(z)$ has all its zeros in $|z| \geq 1$. Hence by the maximum modulus principle,

$$|B[P \circ \rho](z) - \alpha B[P \circ \varrho](z)| < |B[P^\star \circ \rho]^\star(z) - \bar{\alpha} B[P^\star]^\star(z)| \quad \text{for } |z| < 1. \quad (2.19)$$

A direct application of Rouche's theorem shows that with $P(z) = a_n z^n + \cdots + a_0$,

$$\begin{aligned}\Lambda_\delta P(z) =& \Big\{B[P\circ\rho](z) - \alpha B[P\circ\varrho](z)\Big\}e^{i\gamma} + B[P^\star\circ\rho]^\star(z) - \bar{\alpha}B[P^\star]^\star(z),\\ =& \left\{(R^n-\alpha)\left(\lambda_0 + \lambda_1\frac{n^2}{2} + \lambda_2\frac{n^3(n-1)}{8}\right)e^{i\gamma} + (1-\bar{\alpha})\bar{\lambda_0}\right\}a_n z^n\\ &+ \cdots + \left\{(R^n-\bar{\alpha})\left(\bar{\lambda_0} + \bar{\lambda_1}\frac{n^2}{2} + \bar{\lambda_2}\frac{n^3(n-1)}{8}\right) + e^{i\gamma}(1-\alpha)\lambda_0\right\}a_0,\end{aligned}$$

has all its zeros in $|z| \geq 1$, for every real γ, $0 \leq \gamma \leq 2\pi$. Therefore, Λ_δ is an admissible operator. Applying (2.15) of Lemma 2.6, the desired result follows immediately for each $p > 0$. □

We also need the following lemma [4].

Lemma 2.8. *If A, B, C are non-negative real numbers such that $B + C \leq A$, then for each real number γ,*

$$|(A-C)e^{i\gamma} + (B+C)| \leq |Ae^{i\gamma} + B|.$$

3. Main results

In this paper we establish L_p-mean extensions of the inequality (1.15) for $0 \leq p < \infty$ which in particular provides a generalization of inequality (1.18). In this direction, we present the following interesting compact generalization of Theorem 1.3 which yields L_p mean extension of the inequality (1.12) for $0 \leq p < \infty$.

Theorem 3.1. *If $P \in \mathscr{P}_n$ and $P(z)$ does not vanish for $|z| < 1$, then for $\alpha, \delta \in \mathbb{C}$ with $|\alpha| \leq 1, |\delta| \leq 1$, $0 \leq p < \infty$ and $R > r \geq 1$,*

$$\begin{aligned}&\left\|B[P\circ\rho](e^{i\theta}) - \alpha B[P\circ\varrho](e^{i\theta}) + \delta\Big\{\frac{(|R^n-\alpha r^n| - |1-\alpha||\lambda_0|)m}{2}\Big\}\right\|_p\\ &\leq \frac{\|(R^n-\alpha r^n)\phi_n(\lambda_0,\lambda_1,\lambda_2)z + (1-\alpha)\lambda_0\|_p}{\|1+z\|_p}\left\|P(z)\right\|_p, \qquad (3.1)\end{aligned}$$

where $m = Min_{|z|=1}|P(z)|$, $B \in \mathcal{B}_n$, $\rho(z) = Rz$ and $\phi_n(\lambda_0, \lambda_1, \lambda_2)$ is defined by (1.13)*. The result is best possible and equality in* (3.1) *holds for $P(z) = az^n + b$, $|a| = |b| = 1$.*

Proof. By hypothesis $P(z)$ does not vanish in $|z| < 1$, therefore by Lemma 2.5, we have

$$\begin{aligned}&|B[P\circ\rho](z)-\alpha B[P\circ\varrho](z)|\\&\le |B[P^{\star}\circ\rho](z)-\alpha B[P^{\star}](z)|-(|R^n-\alpha r^n|-|1-\alpha||\lambda_0|)m,\end{aligned}\tag{3.2}$$

for $|z| = 1$, $|\alpha| \le 1$ and $R > r \ge 1$ where $P^{\star}(z) = z^n\overline{P(1/\overline{z})}$.
Since $B[P^{\star}\circ\rho]^{\star}(z)-\bar{\alpha}B[P^{\star}]^{\star}(z)$ is the conjugate of $B[P^{\star}\circ\rho](z)-\alpha B[P^{\star}](z)$ and

$$|B[P^{\star}\circ\rho]^{\star}(z)-\bar{\alpha}B[P^{\star}]^{\star}(z)| = |B[P^{\star}\circ\rho](z)-\alpha B[P^{\star}](z)|.$$

Thus for $|z| = 1$, (3.2) can be written as

$$\begin{aligned}&|B[P\circ\rho](z)-\alpha B[P\circ\varrho](z)|+\frac{(|R^n-\alpha r^n|-|1-\alpha||\lambda_0|)m}{2}\\&\le |B[P^{\star}\circ\rho]^{\star}(z)-\bar{\alpha}B[P^{\star}]^{\star}(z)|-\frac{(|R^n-\alpha r^n|-|1-\alpha||\lambda_0|)m}{2}.\end{aligned}\tag{3.3}$$

Taking

$$A = |B[P^{\star}\circ\rho]^{\star}(z)-\bar{\alpha}B[P^{\star}]^{\star}(z)|\,,\ \ B = |B[P\circ\rho](z)-\alpha B[P\circ\varrho](z)|$$

and

$$C = \frac{(|R^n-\alpha r^n|-|1-\alpha||\lambda_0|)m}{2}$$

in Lemma 2.8 and noting by (3.3) that

$$B + C \le A - C \le A,$$

we get for every real γ,

$$\begin{aligned}&\Bigg|\left\{\left|B[P^{\star}\circ\rho]^{\star}(e^{i\theta})-\bar{\alpha}B[P^{\star}]^{\star}(e^{i\theta})\right|-\frac{(|R^n-\alpha r^n|-|1-\alpha||\lambda_0|)m}{2}\right\}e^{i\gamma}\\&\quad+\left\{\left|B[P\circ\rho](e^{i\theta})-\alpha B[P\circ\varrho](e^{i\theta})\right|+\frac{(|R^n-\alpha r^n|-|1-\alpha||\lambda_0|)m}{2}\right\}\Bigg|\\&\le\Big|\left|B[P^{\star}\circ]^{\star}(e^{i\theta})-\bar{\alpha}B[P^{\star}]^{\star}(e^{i\theta})\right|e^{i\gamma}+\left|B[P\circ\rho](e^{i\theta})-\alpha B[P\circ\varrho](e^{i\theta})\right|\Big|.\end{aligned}$$

This implies for each $p > 0$,

$$\int_0^{2\pi} \Big| \Big\{ \big| B[P^\star \circ \rho]^\star(e^{i\theta}) - \bar{\alpha} B[P^\star]^\star(e^{i\theta}) \big| - \frac{(|R^n - \alpha r^n| - |1-\alpha||\lambda_0|)m}{2} \Big\} e^{i\gamma}$$
$$+ \Big\{ \big| B[P \circ \rho](e^{i\theta}) - \alpha B[P \circ \varrho](e^{i\theta}) \big| + \frac{(|R^n - \alpha r^n| - |1-\alpha||\lambda_0|)m}{2} \Big\} \Big|^p d\theta$$
$$\leq \int_0^{2\pi} \Big| \big| B[P^\star \circ \rho]^\star(e^{i\theta}) - \bar{\alpha} B[P^\star]^\star(e^{i\theta}) \big| e^{i\gamma}$$
$$+ \big| B[P \circ \rho](e^{i\theta}) - \alpha B[P \circ \varrho](e^{i\theta}) \big| \Big|^p d\theta. \tag{3.4}$$

Integrating both sides of (3.4) with respect to γ from 0 to 2π, we get with the help of Lemma 2.7 for each $p > 0$,

$$\int_0^{2\pi}\int_0^{2\pi} \Big| \Big\{ \big| B[P^\star \circ \rho]^\star(e^{i\theta}) - \bar{\alpha} B[P^\star]^\star(e^{i\theta}) \big| - \frac{(|R^n - \alpha r^n| - |1-\alpha||\lambda_0|)m}{2} \Big\} e^{i\gamma}$$
$$+ \Big\{ \big| B[P \circ \rho](e^{i\theta}) - \alpha B[P \circ \varrho](e^{i\theta}) \big| + \frac{(|R^n - \alpha r^n| - |1-\alpha||\lambda_0|)m}{2} \Big\} \Big|^p d\theta d\gamma$$
$$\leq \int_0^{2\pi}\int_0^{2\pi} \Big| \big| B[P^\star \circ \rho]^\star(e^{i\theta}) - \bar{\alpha} B[P^\star]^\star(e^{i\theta}) \big| e^{i\gamma} + \big| B[P \circ \rho](e^{i\theta}) - \alpha B[P \circ \varrho](e^{i\theta}) \big| \Big|^p d\theta d\gamma.$$
$$\leq \int_0^{2\pi} \Bigg\{ \int_0^{2\pi} \Big| \big| B[P^\star \circ \rho]^\star(e^{i\theta}) - \bar{\alpha} B[P^\star]^\star(e^{i\theta}) \big| e^{i\gamma}$$
$$+ \big| B[P \circ \rho](e^{i\theta}) - \alpha B[P \circ \varrho](e^{i\theta}) \big| \Big|^p d\gamma \Bigg\} d\theta$$
$$\leq \int_0^{2\pi} \Bigg\{ \int_0^{2\pi} \Big| \big\{ B[P^\star \circ \rho]^\star(e^{i\theta}) - \bar{\alpha} B[P^\star]^\star(e^{i\theta}) \big\} e^{i\gamma}$$
$$+ \big\{ B[P \circ \rho](e^{i\theta}) - \alpha B[P \circ \varrho](e^{i\theta}) \big\} \Big|^p d\gamma \Bigg\} d\theta$$
$$\leq \int_0^{2\pi} \Bigg\{ \int_0^{2\pi} \Big| \big\{ B[P^\star \circ \rho]^\star(e^{i\theta}) - \bar{\alpha} B[P^\star]^\star(e^{i\theta}) \big\} e^{i\gamma}$$

$$+\left\{B[P\circ\rho](e^{i\theta})-\alpha B[P\circ\varrho](e^{i\theta})\right\}\Big|^p d\theta\Bigg\}d\gamma$$

$$\leq\int\limits_0^{2\pi}\left|(R^n-\alpha)\phi(\lambda_0,\lambda_1,\lambda_2)e^{i\gamma}+(1-\bar{\alpha})\bar{\lambda_0}\right|^p d\gamma\int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta. \qquad (3.5)$$

Now it can be easily verified that for every real number γ and $s\geq 1$,

$$\left|s+e^{i\alpha}\right|\geq\left|1+e^{i\alpha}\right|.$$

This implies for each $p>0$,

$$\int_0^{2\pi}\left|s+e^{i\gamma}\right|^p d\gamma\geq\int_0^{2\pi}\left|1+e^{i\gamma}\right|^p d\gamma. \qquad (3.6)$$

If $\left|B[P\circ\rho](e^{i\theta})-\alpha B[P\circ\varrho](e^{i\theta})\right|+\dfrac{(|R^n-\alpha r^n|-|1-\alpha||\lambda_0|)m}{2}\neq 0$, we take

$$s=\frac{\left|B[P^\star\circ\rho]^\star(e^{i\theta})-\bar{\alpha}B[P^\star]^\star(e^{i\theta})\right|-\dfrac{(|R^n-\alpha r^n|-|1-\alpha||\lambda_0|)m}{2}}{\left|B[P\circ\rho](e^{i\theta})-\alpha B[P\circ\varrho](e^{i\theta})\right|+\dfrac{(|R^n-\alpha r^n|-|1-\alpha||\lambda_0|)m}{2}},$$

then by (3.3), $s\geq 1$ and we get with the help of (3.6),

$$\int\limits_0^{2\pi}\Bigg|\left\{\left|B[P^\star\circ\rho]^\star(e^{i\theta})-\bar{\alpha}B[P^\star]^\star(e^{i\theta})\right|-\frac{(|R^n-\alpha r^n|-|1-\alpha||\lambda_0|)m}{2}\right\}e^{i\gamma}$$

$$+\left\{\left|B[P\circ\rho](e^{i\theta})-\alpha B[P\circ\varrho](e^{i\theta})\right|+\frac{(|R^n-\alpha r^n|-|1-\alpha||\lambda_0|)m}{2}\right\}\Bigg|^p d\gamma$$

$$=\left|\left|B[P\circ\rho](e^{i\theta})-\alpha B[P\circ\varrho](e^{i\theta})\right|+\frac{(|R^n-\alpha r^n|-|1-\alpha||\lambda_0|)m}{2}\right|^p$$

$$\times\int\limits_0^{2\pi}\left|e^{i\gamma}+\frac{\left|B[P^\star\circ\rho]^\star(e^{i\theta})-\bar{\alpha}B[P^\star]^\star(e^{i\theta})\right|-\dfrac{(|R^n-\alpha r^n|-|1-\alpha||\lambda_0|)m}{2}}{\left|B[P\circ\rho](e^{i\theta}-\alpha B[P\circ\varrho](e^{i\theta})\right|+\dfrac{(|R^n-\alpha r^n|-|1-\alpha||\lambda_0|)m}{2}}\right|^p d\gamma$$

$$=\left|\left|B[P\circ\rho](e^{i\theta})-\alpha B[P\circ\varrho](e^{i\theta})\right|+\frac{(|R^n-\alpha r^n|-|1-\alpha||\lambda_0|)m}{2}\right|^p$$

$$\times\int\limits_0^{2\pi}\left|e^{i\gamma}+\left|\frac{\left|B[P^\star\circ\rho]^\star(e^{i\theta})-\bar{\alpha}B[P^\star]^\star(e^{i\theta})\right|-\dfrac{(|R^n-\alpha r^n|-|1-\alpha||\lambda_0|)m}{2}}{\left|B[P\circ\rho](e^{i\theta})-\alpha B[P\circ\varrho](e^{i\theta})\right|+\dfrac{(|R^n-\alpha r^n|-|1-\alpha||\lambda_0|)m}{2}}\right|^p\right|d\gamma$$

$$\geq \left| \left|B[P\circ\rho](e^{i\theta}) - \alpha B[P\circ\varrho](e^{i\theta})\right| + \frac{(|R^n - \alpha r^n| - |1-\alpha||\lambda_0|)m}{2} \right|^p \times \int_0^{2\pi} |1+e^{i\gamma}|^p d\gamma. \tag{3.7}$$

For $\left|B[P\circ\rho](e^{i\theta}) - \alpha B[P\circ\varrho](e^{i\theta})\right| + \frac{(|R^n - \alpha r^n| - |1-\alpha||\lambda_0|)m}{2} = 0$, then (3.7) is trivially true. Using this in (3.5), we conclude for every real or complex number α with $|\alpha| \leq 1$, $R > r \geq 1$ and $p > 0$,

$$\int_0^{2\pi} \left| \left|B[P\circ\rho](e^{i\theta}) - \alpha B[P\circ\varrho](e^{i\theta})\right| + \frac{(|R^n - \alpha r^n| - |1-\alpha||\lambda_0|)m}{2} \right|^p d\theta \times \int_0^{2\pi} |1+e^{i\gamma}|^p d\gamma$$
$$\leq \int_0^{2\pi} \left|(R^n - \alpha)\phi(\lambda_0,\lambda_1,\lambda_2)e^{i\gamma} + (1-\bar{\alpha})\bar{\lambda_0}\right|^p d\gamma \int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta.$$

This gives for every real or complex number δ, α with $|\delta| \leq 1$, $|\alpha| \leq 1$, $R > r \geq 1$ and γ real

$$\int_0^{2\pi} \left| B[P\circ\rho](e^{i\theta}) - \alpha B[P\circ\varrho](e^{i\theta}) + \delta\Big\{\frac{(|R^n - \alpha r^n| - |1-\alpha||\lambda_0|)m}{2}\Big\} \right|^p d\theta \times \int_0^{2\pi} |1+e^{i\gamma}|^p d\gamma$$
$$\leq \int_0^{2\pi} \left|(R^n - \alpha)\phi(\lambda_0,\lambda_1,\lambda_2)e^{i\gamma} + (1-\bar{\alpha})\bar{\lambda_0}\right|^p d\gamma \int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta. \tag{3.8}$$

Since

$$\int_0^{2\pi} \left|(R^n - \alpha)\phi(\lambda_0,\lambda_1,\lambda_2)e^{i\gamma} + (1-\bar{\alpha})\bar{\lambda_0}\right|^p d\gamma \int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta$$
$$= \int_0^{2\pi} \left| |(R^n - \alpha)\phi(\lambda_0,\lambda_1,\lambda_2)|e^{i\gamma} + |(1-\bar{\alpha})\bar{\lambda_0}| \right|^p d\gamma \int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta$$

$$= \int_0^{2\pi} \left| |(R^n - \alpha)\phi(\lambda_0, \lambda_1, \lambda_2)| e^{i\gamma} + |(1-\alpha)\lambda_0| \right|^p d\gamma \int_0^{2\pi} \left| P(e^{i\theta}) \right|^p d\theta$$

$$= \int_0^{2\pi} \left| (R^n - \alpha)\phi(\lambda_0, \lambda_1, \lambda_2) e^{i\gamma} + (1-\alpha)\lambda_0 \right|^p d\gamma \int_0^{2\pi} \left| P(e^{i\theta}) \right|^p d\theta, \qquad (3.9)$$

the desired result follows immediately by combining (3.8) and (3.9). This completes the proof of Theorem 3.1 for $p > 0$. To establish this result for $p = 0$, we simply let $p \to 0+$. □

Setting $m = 0$ in (3.1), we get the following result.

Corollary 3.2. *If $P \in \mathscr{P}_n$ and $P(z)$ does not vanish for $|z| < 1$, then for $\alpha, \delta \in \mathbb{C}$ with $|\alpha| \le 1, |\delta| \le 1$, $0 \le p < \infty$ and $R > r \ge 1$,*

$$\left\| B[P \circ \rho](e^{i\theta}) - \alpha B[P \circ \varrho](e^{i\theta}) \right\|_p \le \frac{\|(R^n - \alpha r^n)\phi_n(\lambda_0, \lambda_1, \lambda_2) z + (1-\alpha)\lambda_0\|_p}{\|1+z\|_p} \|P(z)\|_p, \qquad (3.10)$$

$B \in \mathcal{B}_n$, $\rho(z) = Rz$ and $\phi_n(\lambda_0, \lambda_1, \lambda_2)$ is defined by (1.13). *The result is best possible and equality in* (3.1) *holds for $P(z) = az^n + b$, $|a| = |b| = 1$.*

Remark 3.3. If we take $\alpha = 0$ in (3.10), we obtain Theorem 1.3.

By using triangle inequality, the following result immediately follows from Theorem 3.1.

Corollary 3.4. *If $P \in \mathscr{P}_n$ and $P(z)$ does not vanish for $|z| < 1$, then for $\alpha, \delta \in \mathbb{C}$ with $|\alpha| \le 1$, $|\delta| \le 1$, $0 \le p < \infty$ and $R > r \ge 1$,*

$$\left\| B[P \circ \rho](e^{i\theta}) - \alpha B[P \circ \varrho](e^{i\theta}) + \delta \left\{ \frac{(|R^n - \alpha r^n| - |1-\alpha||\lambda_0|)m}{2} \right\} \right\|_p \le \frac{|(R^n - \alpha r^n)\phi_n(\lambda_0, \lambda_1, \lambda_2)| + |(1-\alpha)\lambda_0|}{\|1+z\|_p} \|P(z)\|_p, \qquad (3.11)$$

where $m = Min_{|z|=1}|P(z)|$, $B \in \mathcal{B}_n$, $\rho(z) = Rz$ and $\phi_n(\lambda_0, \lambda_1, \lambda_2)$ is defined by (1.13).

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