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LOCAL CONVERGENCE OF A MULTI-POINT-PARAMETER NEWTON-LIKE METHODS IN BANACH SPACE

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Abstract. In this paper, we present a local convergence analysis for the multi-point-parameter Newton-like-methods for solving nonlinear equations in a Banach space setting under weak conditions. Numerical examples validating our theoretical results are also provided in this study.

1. Introduction

In this study we are concerned with the problem of approximating a solution $x^\star$ of an equation

$$F(x) = 0, \tag{1.1}$$

where F is Fréchet differentiable operator defined on a non–empty, open and convex subset D of a Banach space $\mathcal{X}$ with values in a Banach space $\mathcal{Y}$.

A large number of problems in applied mathematics and engineering are solved by finding the solutions of certain equations. Except in special cases, the most commonly used solution methods are iterative. In fact, starting from one or several initial approximations a sequence is constructed that converges to a solution of the equation. The study about convergence matter of iterative procedures is usually based on two types: semi-local and local convergence

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analysis. The semi-local convergence matter is, based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls.

The famous Newton's method defined by

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \quad (n \geq 0,\ x_0 \in D)$$

converges quadratically to a solution of (1.1) [2, 4, 24, 25]. To attach a higher order, many methods have been developed [2, 4, 8], [11]-[23], [26, 27]. Among them, a classic iterative process with cubic convergence is Chebyshev's method (see [11]-[23]):

$$\begin{cases} x_0 \in D, \\ y_n = x_n - F'(x_n)^{-1}F(x_n), \\ x_{n+1} = y_n - \frac{1}{2}F'(x_n)^{-1}F''(x_n)(y_n - x_n)^2, \quad n \geq 0. \end{cases}$$

This one-point iterative process depends explicitly on the first and second derivatives of F (namely, $x_{n+1} = \psi(x_n, F(x_n), F'(x_n), F''(x_n))$). Ezquerro and Hernández introduced in [14]-[16] some modifications of Chebyshev's method that avoid the computation of the second derivative of F and reduce the number of evaluations of the first derivative of F. Actually, these authors have obtained a modification of the Chebyshev iterative process which only need to evaluate the first derivative of F, (namely, $x_{n+1} = \overline{\psi}(x_n, F'(x_n))$, but with third-order of convergence. In this paper we recall this method as the Chebyshev–Newton–type method (CNTM) and it is written as follows:

$$\begin{cases} x_0 \in D, \\ y_n = x_n - F'(x_n)^{-1}\ F(x_n), \\ z_n = x_n + a\ (y_n - x_n), \\ x_{n+1} = x_n - \dfrac{1}{a^2}\ F'(x_n)^{-1}\ ((a^2 + a - 1)\ F(x_n) + F(z_n)), \quad n \geq 0. \end{cases}$$

There is an interest in constructing families of iterative processes free of derivatives. To obtain a new family in [8] we considered an approximation of the first derivative of F from a divided difference of first order, that is, $F'(x_n) \approx [x_{n-1}, x_n, F]$, where, $[x, y; F]$ is a divided difference of order one for the operator F at the points x, $y \in D$. Then, we introduce the Chebyshev–Secant–type method (CSTM)

$$\begin{cases} x_{-1},\ x_0 \in D, \\ y_n = x_n - B_n^{-1}\ F(x_n), \quad B_n = [x_{n-1}, x_n; F], \\ z_n = x_n + a\ (y_n - x_n), \\ x_{n+1} = x_n - B_n^{-1}\ (b\ F(x_n) + c\ F(z_n)), \quad n \geq 0, \end{cases}$$

where a, b, c are non–negative parameters to be chosen so that sequence $\{x_n\}$ converges to $x^\star$. Note that (CSTM) is reduced to the secant method (SM) if $a = 0$, $b = c = 1/2$ and $y_n = x_{n+1}$.

We provided in [8] a semilocal convergence analysis for (CSTM) using recurrence sequences, and also illustrated its effectiveness through numerical examples. Bosarge and Falb [9], Dennis [13], Amat [1], Argyros [2]-[8] and others [11]-[22], have provided sufficient convergence conditions for the (SM) based on Lipschitz–type conditions on divided difference operator (see, also relevant works in [10, 13, 25]).

The usual conditions for the semilocal convergence of these methods are $(\mathcal{C})$:

$(\mathcal{C}_1)$ There exists $\Gamma_0 = F'(x_0)^{-1}$ and $\|\Gamma_0\| \leq \beta$;
$(\mathcal{C}_2)$ $\|\Gamma_0 F(x_0)\| \leq \eta$;
$(\mathcal{C}_3)$ $\|F''(x)\| \leq \beta_1$ for each $x \in D$;
$(\mathcal{C}_4)$ $\|F'''(x)\| \leq \beta_2$ for each $x \in D$;
$(\mathcal{C}_5)$ $\|F'''(x) - F'''(y)\| \leq \beta_3 \|x - y\|$ for each $x, y \in D$.

The local convergence conditions are similar but x_0 is x^* in $(\mathcal{C}_1)$ and $(\mathcal{C}_2)$.

In this paper, we continue the study of derivative free iterative processes. We introduce the Multi-point-parameter Newton-like method (MPPNLM) defined for each $n = 0, 1, 2, \cdots$ by

$$\begin{cases} x_0 \in D, \\ y_n = x_n - A_n^{-1}\, F(x_n), \quad A_n = A(x_n), \\ z_n = x_n + a\,(y_n - x_n), \\ x_{n+1} = x_n - A_n^{-1}\,(b\,F(x_n) + c\,F(z_n)), \quad n \geq 0, \end{cases}$$

where, a, b, c are real parameters and $A_n^{-1} \in L(Y, X)$. We assume the conditions $(\mathcal{A})$ to study the local convergence of (MPPNLM):

$(\mathcal{A}_1)$ $F : D \to Y$ is Fréchet-differentiable and there exists $x^* \in D$ such that $F(x^*) = 0$ and $F'(x^*)^{-1} \in L(Y, X)$. Moreover, $A(x) \in L(X, Y)$ and $A(x^*)^{-1} \in L(Y, X)$;
$(\mathcal{A}_2)$ $\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq K_0 \|x - x^*\|$ for each $x \in D$;
$(\mathcal{A}_3)$ $\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq K \|x - y\|$ for each $x, y \in D$;
$(\mathcal{A}_4)$ $\|F'(x^*)^{-1} F'(x)\| \leq N_1$ for each $x \in D$;
$(\mathcal{A}_5)$ $\|A(x^*)^{-1}(A(x) - A(x^*))\| \leq L\|x - x^*\| + l$ for each $x \in D$ and $l \in [0, 1)$;
$(\mathcal{A}_6)$ $\|A(x^*)^{-1}(A(x) - F'(x))\| \leq M\|x - x^*\| + \mu$ for each $x \in D$ and $\mu \in [0, 1)$;
$(\mathcal{A}_7)$ $\|A(x^*)^{-1} F'(x)\| \leq N_2$ for each $x \in D$;
$(\mathcal{A}_8)$ $|1 - a| + \frac{|a|\mu N}{1-l} < 1$; and
$(\mathcal{A}_9)$ $\frac{\mu N}{1-l} + \frac{(|1-b| + |c|(|1-a| + |a|\frac{\mu N}{1-l}))N_2}{1-\mu} < 1$.

Notice that we do not require hypotheses involving second or third Fréchet-derivatives. Hence, the applicability of (MPPNLM) is expanded this way.

The paper is organized as follows: Section 2 contains the local convergence of (MPPNLM) where the convergence ball as well as error estimates on the distances $\|x_n - x^*\|$, $\|y_n - x^*\|$ and $\|z_n - x^*\|$ are given. The numerical examples are presented in the concluding Section 3.

2. Local convergence

We present the local convergence of (MPPNLM) under the condition $(\mathcal{A})$. It is convenient for our local convergence analysis of (MPPNLM) to introduce some parameters and functions.

Define parameters R_0, R_1 and R_2 by

$$R_0 = \frac{l}{K_0}, \quad R_1 = \frac{1-l}{L} \quad \text{and} \quad R = \min\{R_1, R_2\}. \tag{2.1}$$

Define functions g_1 and G_1 on $[0, R)$ by

$$g_1(t) = \frac{Kt}{2(1-K_0 t)} + \frac{(Mt+\mu)N_1}{(1-(Lt+l))(1-K_0 t)} \tag{2.2}$$

and

$$G_1(t) = g_1(t) - 1. \tag{2.3}$$

We have by $(\mathcal{A}_9)$, (2.1)-(2.3) that

$$G_1(0) = \frac{\mu N}{1-l} - 1 < 0$$

and

$$G_1(t) \to \infty \text{ as } t \to R^-.$$

It follows by the intermediate value theorem that function G_1 has zeros in the interval $[0, R)$. Denote by r_1 the smallest such zero of function G_1. Define functions g_2 and G_2 on the interval $[0, R)$ by

$$g_2(t) = |1-a| + |a| g_1(t) \tag{2.4}$$

and

$$G_2(t) = g_2(t) - 1. \tag{2.5}$$

We have by $(\mathcal{A}_8)$, (2.1), (2.4) and (2.5) that

$$G_2(0) = g_2(0) - 1 = |1-a| + \frac{|a|\mu N}{1-l} - 1 < 0$$

and

$$G_2(t) \to \infty \text{ as } t \to R.$$

Then, function G_2 has zeros in the interval $(0, R)$. Denote by r_2 the smallest such zero of function G_2. Define functions g_3 and G_3 on the interval $[0, R)$ by

$$g_3(t) = g_1(t) + \frac{(|1-b| + |c|g_2(t))N_2}{1-(Lt+l)} \tag{2.6}$$

and

$$G_3(t) = g_3(t) - 1. \tag{2.7}$$

Then, we get by $(\mathcal{A}_9)$, (2.1) and (2.7) that

$$G_3(0) = g_3(0) - 1 < 0$$

and

$$G_3(t) \to \infty \text{ as } t \to R.$$

Hence G_3 has zeros in the interval $(0, R)$. Denote by r_3 the smallest such zero of function G_3. Set

$$r^* = \min\{r_1, r_2, r_3\} \tag{2.8}$$

and choose

$$r \in [0, r^*). \tag{2.9}$$

Then, we have that

$$g_1(t) < 1, \tag{2.10}$$

$$g_2(t) < 1, \tag{2.11}$$

and

$$g_3(t) < 1, \tag{2.12}$$

for each $t \in [0, r]$. Then, we can show the following local convergence result for (MPPNLM) under the $(\mathcal{A})$ conditions.

Theorem 2.1. *Suppose that the $(\mathcal{A})$ conditions and $\overline{U}(x^*, r) \subseteq D$, hold, where r is given by (2.9). Then, sequence $\{x_n\}$ generated by (MPPNLM) for some $x_0 \in U(x^*, r)$ is well defined, remains in $U(x^*, r)$ for each $n = 0, 1, 2, \cdots$ and converges to x^*. Moreover, the following estimates hold for each $n = 0, 1, 2, \cdots$,*

$$\|y_n - x^*\| \leq g_1(\|x_n - x^*\|)\|x_n - x^*\|, \tag{2.13}$$

$$\|z_n - x^*\| \leq g_2(\|x_n - x^*\|)\|x_n - x^*\|, \tag{2.14}$$

and

$$\|x_{n+1} - x^*\| \leq g_3(\|x_n - x^*\|\|x_n - x^*\|, \tag{2.15}$$

where, functions g_1, g_2, and g_3 are given by (2.2), (2.4) and (2.6), respectively.

Proof. We shall use induction to show that estimates (2.13)-(2.22) and that $y_n, z_n, x_{n+1} \in U(x^*, r)$ for each $n = 0, 1, 2, \cdots$. Using $(\mathcal{A}_1)$, $(\mathcal{A}_2)$ and the hypothesis $x_0 \in U(x^*, r)$ we have that

$$\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| \leq K_0\|x_0 - x^*\| < K_0 r < 1. \tag{2.16}$$

It follows from (2.16) and the Banach Lemma on invertible operators [2, 4, 24, 25] that $F'(x_0)^{-1} \in L(Y, X)$ and

$$\|F'(x_0)^{-1}F'(x^*)\| \leq \frac{1}{1 - K_0\|x_0 - x^*\|} < \frac{1}{1 - K_0 r}. \tag{2.17}$$

Similarly, using $(\mathcal{A}_5)$ we obtain that

$$\|A(x^*)^{-1}(A(x_0) - A(x^*))\| \leq L\|x_0 - x^*\| + l \leq Lr + l < 1 \tag{2.18}$$

so that $A(x_0)^{-1} \in L(Y, X)$ and

$$\|A(x_0)^{-1}A(x^*)\| \leq \frac{1}{1 - (L\|x_0 - x^*\| + l)} \leq \frac{1}{1 - (Lr + l)}. \tag{2.19}$$

Hence, we also have that y_0 is well defined. Then, using the first substep of (MPPNLM) for $n = 0$, (2.17), (2.19), $(\mathcal{A}_4)$, $(\mathcal{A}_6)$, $F(x^*) = 0$, (2.2) and (2.10) we get that

$$\begin{aligned}
&y_0 - x^* \\
&= x_0 - x^* - F'(x_0)^{-1}F(x_0) + (F'(x_0)^{-1} - A(x_0)^{-1})F(x_0) \\
&= -[F'(x_0)^{-1}F'(x^*)]\left[F'(x^*)^{-1}\int_0^1 (F'(x^* + \tau(x_0 - x^*)) - F'(x_0))(x_0 - x^*)d\tau\right] \\
&\quad + [A(x_0)^{-1}A(x^*)][A(x^*)^{-1}(A(x_0) - F'(x_0))][F'(x_0)^{-1}F'(x^*)] \\
&\quad \times \left[F'(x^*)^{-1}\int_0^1 (F'(x^* + \tau(x_0 - x^*)) - F'(x_0))(x_0 - x^*)d\tau\right]
\end{aligned} \tag{2.20}$$

so

$$\begin{aligned}
&\|y_0 - x^*\| \\
&\leq \|F'(x_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}\int_0^1 [F'(x^* + \theta(x_0 - x^*)) - F'(x_0)]d\theta\|\|x_0 - x^*\| \\
&\quad + \|A(x_0)^{-1}A(x^*)\|\|A(x^*)^{-1}(A(x_0) - F'(x_0))\|\|F'(x_0)^{-1}F'(x^*)\| \\
&\quad \times \|F'(x^*)^{-1}\int_0^1 (F'(x^* + \tau(x_0 - x^*)) - F'(x_0))d\tau\|\|x_0 - x^*\| \\
&\leq g_1(\|x_0 - x^*\|)\|x_0 - x^*\| \leq g_1(r)\|x_0 - x^*\| \\
&< \|x_0 - x^*\|,
\end{aligned} \tag{2.21}$$

which shows (2.13) for $n = 0$ and $y_0 \in U(x^*, r)$. Using the first substep of (MPPNLM) for $n = 0$, (2.4), (2.9), (2.11) and (2.21) we have that

$$\begin{aligned} z_0 - x^* &= x_0 - x^* + a((y_0 - x^*) + (x^* - x_0)) \\ &= (1-a)(x_0 - x^*) + a(y_0 - x^*), \end{aligned}$$

so,

$$\begin{aligned} \|z_0 - x^*\| &\leq |1-a|\|x_0 - x^*\| + a\|y_0 - x^*\| \\ &\leq |1-a|\|x_0 - x^*\| + |a| g_1(\|x_0 - x^*\|)\|x_0 - x^*\| \\ &= g_2(\|x_0 - x^*\|)\|x_0 - x^*\| \\ &\leq g_2(r)\|x_0 - x^*\| < \|x_0 - x^*\|, \end{aligned} \tag{2.22}$$

which shows (2.14) for $n = 0$, $z_0 \in U(x^*, r)$ and that x_1 is well defined. Moreover, using (2.6), (2.12), (2.19), (2.21), $(\mathcal{A}_7)$ and the third substep in (MPPNLM), for $n = 0$, we obtain in turn that

$$x_1 = y_0 + (1-b)A_0^{-1}F(x_0) - cA_0^{-1}F(z_0)$$

implies

$$\begin{aligned} & x_1 - x^* \\ & = y_0 - x^* \\ & \quad + (1-b)[A(x_0)^{-1}A(x^*)]\left[A(x^*)^{-1}\int_0^1 F'(x^* + \tau(x_0 - x^*))(x_0 - x^*)d\tau\right] \\ & \quad + c[A(x_0)^{-1}A(x^*)]\left[A(x^*)^{-1}\int_0^1 F'(x^* + \tau(z_0 - x^*))(z_0 - x^*)d\tau\right] \end{aligned}$$

so,

$$\begin{aligned} & \|x_1 - x^*\| \\ & = \|y_0 - x^*\| \\ & \quad + |1-b|\|A(x_0)^{-1}A(x^*)\|\|A(x^*)^{-1}\int_0^1 F'(x^* + \tau(x_0 - x^*))d\tau\|\|x_0 - x^*\| \\ & \quad + |c|\|A(x_0)^{-1}A(x^*)\|\|A(x^*)^{-1}\int_0^1 F'(x^* + \tau(z_0 - x^*))d\tau\|\|z_0 - x^*\| \\ & \leq \left[g_1(\|x_0 - x^*\|) + \frac{|1-b|N_2}{1-(L\|x_0 - x^*\| + l)} + \frac{|c|N_2 g_2(\|x_0 - x^*\|)}{1-(L\|x_0 - x^*\| + l)}\right]\|x_0 - x^*\| \\ & = g_3(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\|, \end{aligned} \tag{2.23}$$

which shows (2.22) for $n = 0$, $x_1 \in U(x^*, r)$. To complete the induction simply replace y_0, z_0, x_1 by y_k, z_k, x_{k+1} in the preceding estimates to obtain that

$$\|y_k - x^*\| \leq g_1(\|x_k - x^*\|)\|x_k - x^*\| \leq g_1(r)\|x_k - x^*\| \leq \|x_k - x^*\| < r,$$

$$\|z_k - x^*\| \leq g_2(\|x_k - x^*\|)\|x_k - x^*\| \leq g_2(r)\|x_k - x^*\| < \|x_k - x^*\| < r,$$

and

$$\|x_{k+1}-x^*\| \leq g_3(\|x_k-x^*\|)\|x_k-x^*\| \leq g_3(r)\|x_k-x^*\| < \|x_k-x^*\| < r,$$

which complete the induction for (2.13)-(2.22) and $y_k, z_k, x_{k+1} \in U(x^*, r)$. Finally, in particular from the estimate $\|x_{k+1}-x^*\| < \|x_k-x^*\|$, we deduce that that $\lim_{k\to\infty} x_k = x^*$. □

Remark 2.2. (a) Condition $(\mathcal{A}_2)$ can be dropped, since this condition follows from $(\mathcal{A}_3)$. Notice, however that

$$K_0 \leq K \tag{2.24}$$

holds in general and $\frac{K}{K_0}$ can be arbitrarily large [2]–[7].

(b) In view of condition $(\mathcal{A}_2)$ and the estimate

$$\begin{aligned}\|F'(x^*)^{-1}F'(x)\| &= \|F'(x^*)^{-1}[F'(x)-F'(x^*)]+I\| \\ &\leq 1+\|F'(x^*)^{-1}(F'(x)-F'(x^*))\| \\ &\leq 1+K_0\|x-x^*\|,\end{aligned}$$

condition $(\mathcal{A}_4)$ can be dropped and N_1 can be replaced by

$$N_1(r) = 1+K_0 r. \tag{2.25}$$

(c) It is worth noticing that if $A(x) = F'(x)$, $a = 0$ and $b = c = \frac{1}{2}$, we obtain Newton's method. Then, we get by (2.9) that

$$r = r_A = \frac{2}{2K_0+K}. \tag{2.26}$$

The convergence ball of radius r_A was given by us in [3] for Newton's method under conditions $(\mathcal{A}_1)$-$(\mathcal{A}_3)$. Estimate shows that the convergence ball of higher than two (MPPNLM) methods is smaller than the convergence ball of the quadratically convergent Newton's method. The convergence ball given by Rheinboldt [25] for Newton's method is $r_R = \frac{2}{3K} < r_A$ if $K_0 < K$ and $\frac{r_R}{r_A} \to \frac{1}{3}$ as $\frac{K_0}{K} \to 0$. Hence, we do not expect r to be larger than r_A no matter how we choose the parameters.

(d) The results can also be used to solve equations where the operator F' satisfies the autonomous differential equation [2, 4, 25]:

$$F'(x) = T(F(x)),$$

where T is a known continuous operator. Since $F'(x^*) = T(F(x^*)) = T(0)$, we can apply the results without actually knowing the solution x^*. Let as an example $F(x) = e^x - 1$. Then, we can choose $T(x) = x+1$ and $x^* = 0$.

(e) The local results can be used for projection methods such as Arnoldi's method, the generalized minimum residual method (GMREM), the generalized conjugate method(GCM) for combined Newton/finite projection methods and in connection to the mesh independence principle in order to develop the cheapest and most efficient mesh refinement strategy [2, 4, 25].

(f) Condition $(\mathcal{A}_8)$ and $(\mathcal{A}_9)$ are sufficient conditions used to show the existence of r_1, r_2 and r_3. These conditions can be replaced by the condition:
$(\mathcal{A}_{10})$ Functions G_1, G_2, G_3 have zeros in $(0, R)$.

3. Numerical examples

We present three numerical examples in this section for $A(x) = F'(x), a = c = 1$ and $b = 0$.

Example 3.1. Let $X = Y = \mathbb{R}^3$, $D = \overline{U}(0,1)$ and $x = (0,0,0)$. We define function F on D as

$$F(x,y,z) = \left(e^x - 1, \frac{e-1}{2}\, y^2 + y, z\right). \tag{3.1}$$

Then, the Fréchet derivative of F is given by

$$F'(x,y,z) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)\,y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.2}$$

Notice that we have:

$$\begin{aligned} &F(x^*) = 0, \quad F'(x^*) = F'(x^*)^{-1} = \operatorname{diag}\{1,1,1\}, \\ &K_0 = L = e-1, \quad N_1 = N_2 = K = e, \\ &M = l = \mu = 0. \end{aligned}$$

To ascertain the convergence-order of the method (MPPNLM), we use the concept of computational order of convergence (**COC**) [8]

$$\rho = \sup \frac{\ln\left(\frac{\|\mathbf{x}_{n+2}-\mathbf{x}_{n+1}\|}{\|\mathbf{x}_{n+1}-\mathbf{x}_{n}\|}\right)}{\ln\left(\frac{\|\mathbf{x}_{n+1}-\mathbf{x}_{n}\|}{\|\mathbf{x}_{n}-\mathbf{x}_{n-1}\|}\right)} \quad \text{for} \quad n \in \mathbb{N}_{>0}. \tag{3.3}$$

We solve the nonlinear system (3.1) by the (MPPNLM) for $\mathbf{x_0} = (0.1, 0.1, 0.1)^T$. Note that $x_0 \in U(x^*, r)$. Results of our computation are reported in the Table 1.

In the Table 1, we notice that $\rho = 2.87415 \approx 3$ and $r \approx 0.1482876006$. Thus our results are applicable for analysing convergence of the method (MPPNLM).

n	$\|x_n - x_{n-1}\|_2$	$\|F(\mathbf{x})\|_2$
0	$---$	0.181254010020148
1	0.172349059098655	0.001036567529705
2	0.001129080546855	0.000000001633261
3	0.000000001633894	0.000000000000000

Table 1. Solving (3.1) by the (MPPNLM) for $\mathbf{x_0} = (0.1, 0.1, 0.1)^T$.

Example 3.2. Let $X = Y = \mathbb{C}[0,1]$, the space of continuous functions defined on $[0,1]$ be equipped with the max norm and $D = \overline{U}(0,1)$. Define function F on D by

$$F(h)(x) = h(x) - 5\int_0^1 x\,\theta\,h(\theta)^3\,d\theta. \tag{3.4}$$

Then, the Fréchet derivative of F is given by

$$F'(h[u])(x) = u(x) - 15\int_0^1 x\,\theta\,h(\theta)^2\,u(\theta)\,d\theta \quad \text{for all } u \in D. \tag{3.5}$$

Some algebraic manipulations yield

$$M = l = \mu = 0, \qquad N_1 = N_2 = N_1(r) = N_2(r) = 1 + 7.5\,r,$$
$$L = K_0 = 7.5 \quad \text{and} \quad K = 15.$$

We obtain $r^* = 0.035726559$. Thus we must choose $r \in (0, r_1)$.

Example 3.3. Let $\mathcal{X} = \mathcal{Y} = \mathbb{R}^{m-1}$ for natural integer $n \geq 2$. $\mathcal{X}$ and $\mathcal{Y}$ are equipped with the max-norm $\|\mathbf{x}\| = \max_{1 \leq i \leq n-1} \|x_i\|$. The corresponding matrix norm is

$$\|A\| = \max_{1 \leq i \leq m-1} \sum_{j=1}^{j=m-1} |a_{ij}|$$

for $A = (a_{ij})_{1 \leq i,j \leq m-1}$. On the interval $[0,1]$, we consider the following two point boundary value problem

$$\begin{cases} v'' + v^2 = 0, \\ v(0) = v(1) = 0, \end{cases} \tag{3.6}$$

see [2, 4]. To discretize the above equation, we divide the interval $[0,1]$ into m equal parts with length of each part: $h = 1/m$ and coordinate of each point: $x_i = i\,h$ with $i = 0, 1, 2, \ldots, m$. A second-order finite difference discretization of equation (3.6) results in the following set of nonlinear equations

$$F(\mathbf{v}) := \begin{cases} v_{i-1} + h^2\,v_i^2 - 2v_i + v_{i+1} = 0 \ \text{ for } \ i = 1, 2, \ldots, (m-1), \\ \text{and from (3.6)}, \quad v_0 = v_m = 0, \end{cases} \tag{3.7}$$

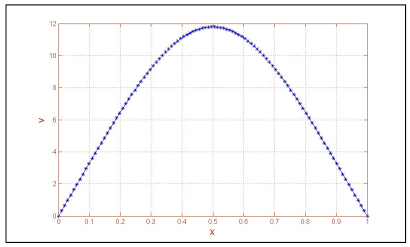

FIGURE 1. Solution of the boundary value problem (3.6).

where $\mathbf{v} = [v_1, v_2, \ldots, v_{(m-1)}]^{\mathrm{T}}$ For the above system-of-nonlinear-equations, we provide the Fréchet derivative

$$F'(\mathbf{v}) = \begin{bmatrix} \frac{2v_1}{m^2} - 2 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & \frac{2v_2}{m^2} - 2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \frac{2v_3}{m^2} - 2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & \frac{2v_{(m-1)}}{m^2} - 2 \end{bmatrix}. \tag{3.8}$$

Let $m = 101$, $x_0 = [5, 5, \ldots, 5]^{\mathrm{T}}$. To solve the linear systems (step 1 and step 2 in (MPPNLM), we employ MatLab routine "linsolve" which uses LU factorization with partial pivoting. Figure 1 plots our numerical solution.

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