Nonlinear Functional Analysis and Applications
Vol. 19, No. 3 (2014), pp. 391-399

http://nfaa.kyungnam.ac.kr/jour-nfaa.htm

ON THE RATE OF CONVERGENCE OF PICARD AND PICARD-MANN HYBRID ITERATIONS FOR CONTINUOUS FUNCTIONS ON AN ARBITRARY INTERVAL

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Abstract. In this paper, we compare the rate of convergence of Picard and Picard-Mann hybrid iterations under the same computational cost. A numerical example is provided which supports the theoretical result. Finally, we use the example provided by Chidume and Mutangadura [2] to show that the Picard-Mann hybrid iteration fails to converge for a Lipschitz pseudocontractive map with a unique fixed point.

1. Introduction

Let E be a closed interval on the real line and $f : E \to E$ be a continuous mapping. A point $p \in E$ is a fixed point of f if $f(p) = p$. We denote the set of fixed points of f by $F(f)$. It is known that if E is also bounded, then $F(f)$ is nonempty.

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[0]Received January 15, 2014. Revised June 5, 2014.
[0]2010 Mathematics Subject Classification: 47H05, 47H07, 47H10.
[0]Keywords: Picard iteration, Picard-Mann hybrid iteration, Lipschitz pseudocontractive mapping, convergence rate.
[0]Supported by National Natural Science Foundation of China (No. 11201476) and Fundamental Research Funds for the Central Universities (No. 3122013k004).
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Iterative methods are popular tools to approximate fixed points of nonlinear mappings. The Picard iteration [8] is defined by the sequence $\{u_n\}$:

$$u_{n+1} = f(u_n),$$

for all $n \geq 1$, where u_1 is an arbitrary initial value. Recently, Khan [5] and Sahu [9], individually, introduced the following iterative process which Khan referred it as Picard-Mann hybrid iteration (PMH):

$$\begin{cases} x_{n+1} = f(y_n), \\ y_n = (1-\alpha_n)x_n + \alpha_n f(x_n), \end{cases} \tag{1.1}$$

for all $n \geq 1$, where x_1 is an arbitrary initial value and $\{\alpha_n\}$ be a sequence in $[0, 1)$. Khan [5] proved that the Picard-Mann hybrid iteration converges faster than all of Picard, Mann and Ishikawa iterative processes in the sense of Berinde [1] for contractions.

Phuengrattana and Suantai [7] compared the convergence speed of Mann, Ishikawa and Noor iterations for continuous functions on an arbitrary interval. Recently, Dong *et al.*, [3] compared the rate of convergence of Mann, Ishikawa and Noor iterations from another point of view and come to a different conclusion.

The purpose of this paper is to compare the rate of convergence of Picard and Picard-Mann hybrid iterations under the same computational cost. We draw a different conclusion with Khan [5]. We also use an example to verify that the Picard-Mann hybrid iteration fails to converge for a Lipschitz pseudocontractive map with a unique fixed point.

2. Stability of the Wigner equation

In [3], the authors compared the Mann, Ishikawa and Noor iterations under the same computational cost and obtained different conclusions from [7].

Now, we give a definition and results about the rate of convergence of two iterations and compare Picard iteration with Picard-Mann hybrid iteration under the same computational cost. Also, we support the result with a numeric example.

Definition 2.1. Let E be a closed interval on the real line and $f: E \to E$ be a continuous function. Suppose that $\{x_n\}$ and $\{y_n\}$ are two iterations which converge to a fixed point p of f. Then $\{x_n\}$ is said to converge better than $\{y_n\}_{n=1}^{\infty}$ if

$$|x_n - p| \leq |y_n - p|, \tag{2.1}$$

for all $n \geq 1$.

For any sequence $\{x_n\}$ that converges to a point p, it is said that $\{x_n\}$ converges linearly to p, if there exists a constant $\mu \in (0,1)$ such that

$$\left|\frac{x_{n+1}-p}{x_n-p}\right| \le \mu, \tag{2.2}$$

for all $n \ge 1$, the number μ is called the rate of convergence.

To compare the rate of convergence of Picard and Picard-Mann hybrid iterations, we define a two-step Picard iteration (TSP):

$$\begin{cases} u_{n+1} = f(v_n), \\ v_n = f(u_n). \end{cases} \tag{2.3}$$

Remark 2.1. It should be noted that two-step Picard iteration isn't a new iteration and we introduce it just for comparing the rate of convergence of Picard and Picard-Mann hybrid iterations under the same computation cost.

Lemma 2.1. *Let E be a closed interval on the real line and $f : E \to E$ be a continuous and nondecreasing function. Let the Picard-Mann hybrid iteration $\{x_n\}$ and two-step Picard iteration $\{u_n\}$ be sequences defined by* (1.1) *and* (2.3)*, respectively, where $\{\alpha_n\}$ is a sequence in* $[0,1)$*. Then the following hold:*

(i) *if $f(x_1) < x_1$, then $f(x_n) \le x_n$ for all $n \ge 1$ and $\{x_n\}$ is nonincreasing;*
(ii) *if $f(x_1) > x_1$, then $f(x_n) \ge x_n$ for all $n \ge 1$ and $\{x_n\}$ is nondecreasing;*
(iii) *if $f(u_1) < u_1$, then $f(u_n) \le u_n$ for all $n \ge 1$ and $\{u_n\}$ is nonincreasing;*
(iv) *if $f(u_1) > u_1$, then $f(u_n) \ge u_n$ for all $n \ge 1$ and $\{x_n\}$ is nondecreasing.*

Proof. (i) Let $f(x_1) < x_1$. Then from the definition of $\{x_n\}$ we get that $f(x_1) < y_1 \le x_1$. Since f is nondecreasing, we have $f(y_1) = x_2 \le f(x_1) < y_1 \le x_1$. This implies $f(x_2) \le f(y_1)$. Thus

$$f(x_2) \le x_2.$$

Assume that $f(x_k) \le x_k$. So, we write $f(x_k) \le y_k \le x_k$. Since f is nondecreasing, we have $f(y_k) = x_{k+1} \le f(x_k) \le y_k \le x_k$. This implies that $f(x_{k+1}) \le f(y_k)$. Thus $f(x_{k+1}) \le x_{k+1}$. By mathematical induction, we obtain that $f(x_n) \le x_n$, for all $n \ge 1$. It follows that $x_{n+1} \le x_n$, for all $n \ge 1$. So, we get $\{x_n\}$ is a nonincreasing sequence.
(ii) In a similar way as in the proof (i), we get the desired conclusion.
(iii) Let $f(u_1) < u_1$. Then from the definition of $\{u_n\}$ we get that $f(u_1) = v_1 \le u_1$. Since f is nondecreasing, we have $f(v_1) = u_2 \le f(u_1) = v_1 \le u_1$. This implies $f(u_2) \le f(v_1)$. Thus

$$f(u_2) \le u_2.$$

Assume that $f(u_k) \leq u_k$. So, we write $f(u_k) = v_k \leq u_k$. Since f is nondecreasing, we have $f(v_k) = u_{k+1} \leq f(u_k) = v_k \leq u_k$. This implies that $f(u_{k+1}) \leq f(v_k)$. Thus $f(u_{k+1}) \leq u_{k+1}$. By mathematical induction, we obtain that $f(u_n) \leq u_n$, for all $n \geq 1$. It follows that $u_{n+1} \leq u_n$, for all $n \geq 1$. So, we get $\{u_n\}$ is a nonincreasing sequence.
(iv) In a similar way as in the proof (iii), we get the desired conclusion. □

Lemma 2.2. *Let E be a closed interval on the real line and $f : E \to E$ be a continuous and nondecreasing function. Let the Picard-Mann hybrid iteration $\{x_n\}$ and two-step Picard iteration $\{u_n\}$ be sequences defined by (1.1) and (2.3), respectively, where $\{\alpha_n\}$ are sequence in $[0, 1)$. Then the following are satisfied:*

(i) *if $p \in F(f)$ with $x_1 > p$, then $x_n \geq p$ for all $n \geq 1$;*
(ii) *if $p \in F(f)$ with $x_1 < p$, then $x_n \leq p$ for all $n \geq 1$;*
(iii) *if $p \in F(f)$ with $u_1 > p$, then $u_n \geq p$ for all $n \geq 1$;*
(iv) *if $p \in F(f)$ with $u_1 < p$, then $u_n \leq p$ for all $n \geq 1$.*

Proof. (i) Since $p \in f(f)$ with $x_1 > p$, and f is nondecreasing function we have $f(x_1) \geq f(p) = p$. Thus, from the definition of $\{x_n\}$, we get $y_1 > p$. It implies that $f(y_1) = x_2 \geq p$. Assume that $x_k \geq p$. So, we have $f(x_k) \geq p$. From the definition of $\{x_n\}$, we have $y_k \geq p$. Since f is nondecreasing, we get $f(y_k) = x_{k+1} \geq p$. By mathematical induction, we obtain that $x_n \geq p$, for all $n \geq 1$.
(ii) By using the same argument as in (i), we get the desired conclusion.
(iii) Since $p \in F(f)$ with $u_1 > p$, and f is nondecreasing function we have $f(u_1) \geq f(p) = p$. Thus, from the definition of $\{u_n\}$, we get $v_1 \geq p$. It implies that $f(v_1) = u_2 \geq p$. Assume that $u_k \geq p$. So, we have $f(u_k) \geq p$. From the definition of $\{u_n\}$, we have $v_k \geq p$. Since f is nondecreasing, we get $f(v_k) = u_{k+1} \geq p$. By mathematical induction, we obtain that $u_n \geq p$, for all $n \geq 1$.
(iv) By using the same argument as in (iii), we get the desired conclusion. □

Proposition 2.1. *Let E be a closed interval on the real line and $f : E \to E$ be a continuous and nondecreasing function such that $F(f)$ is nonempty and bounded with $x_1 > \sup\{p \in E : p = f(p)\}$. Let $\{\alpha_n\}$ be sequences in $[0, 1)$. If $f(x_1) > x_1$, then the sequence $\{x_n\}$ and $\{u_n\}$ defined by (1.1) and (2.3) don't converge to a fixed point of f.*

Proof. By Lemma 2.1 (ii) and (iv), $\{x_n\}$, $\{u_n\}$ are nondecreasing sequences. From hypothesis, since $x_1 > \sup\{p \in E : p = f(p)\}$, we have

$$f(x_n) \geq x_n \geq x_1 > \sup\{p \in E : p = f(p)\}$$

$$(f(u_n) \geq u_n \geq u_1 > \sup\{p \in E : p = f(p)\}).$$

It is clear that $\{x_n\}$ and $\{u_n\}$ don't converge to a fixed point of f. □

Proposition 2.2. *Let E be a closed interval on the real line and $f : E \to E$ be a continuous and nondecreasing function Such that $F(f)$ is nonempty and bounded with $x_1 < \inf\{p \in E : p = f(p)\}$. Let $\{\alpha_n\}$ be sequences in $[0, 1)$. If $f(x_1) < x_1$, then the sequence $\{x_n\}$ and $\{u_n\}$ defined by (1.1) and (2.3) don't converge to a fixed point of f.*

Proof. By Lemma 2.1 (i) and (iii), $\{x_n\}$, $\{u_n\}$ are nonincreasing sequences. From hypothesis, since $x_1 < \inf\{p \in E : p = f(p)\}$, we have

$$f(x_n) \leq x_n \leq x_1 < \inf\{p \in E : p = f(p)\}$$

$$(f(u_n) \leq u_n \leq u_1 < \inf\{p \in E : p = f(p)\}).$$

It is clear that $\{x_n\}$ and $\{u_n\}$ don't converge to a fixed point of f. □

Theorem 2.1. *Let E be a closed interval on the real line and $f : E \to E$ be a continuous and nondecreasing function such that $F(f)$ is nonempty and bounded. Let the sequence $\{x_n\}$ and $\{u_n\}$ defined by (1.1) and (2.3), respectively and $x_1 = u_1$. Let $\{\alpha_n\}$ be sequences in $[0, 1)$. If $\{x_n\}$ converges to $p \in F(f)$, then $\{u_n\}$ converges to $p \in F(f)$. Moreover, $\{u_n\}$ converges better than $\{x_n\}$.*

Proof. Let $U = \sup\{p \in E : p = f(p)\}$ and $L = \inf\{p \in E : p = f(p)\}$. Suppose that $\{x_n\}$, $\{u_n\}$ converges to $p \in F(f)$. We shall divide our proof into three cases:

Case 1. Let $U < x_1 = u_1$. By Proposition 2.1, we have $f(x_1) < x_1$ ($f(u_1) < u_1$). From Lemma 2.1 (i) and (iii), it follows $f(x_n) \leq x_n$ ($f(u_n) \leq u_n$) for all $n \geq 1$. Using (1) and (2), we obtain that $f(y_n) \leq y_n$ ($f(v_n) \leq v_n$) for all $n \geq 1$. It follows

$$\begin{aligned} v_1 - y_1 &= f(u_1) - (1 - \alpha_1)x_1 - \alpha_1 f(x_1) \\ &= f(x_1) - (1 - \alpha_1)x_1 - \alpha_1 f(x_1) \\ &= (1 - \alpha_1)f(x_1) - (1 - \alpha_1)x_1 \\ &= (1 - \alpha_1)(f(x_1) - x_1) \\ &< 0. \end{aligned}$$

Since f is nondecreasing function, we get $f(v_1) \leq f(y_1)$, thus $u_2 \leq x_2$. Now, assume that $u_k \leq x_k$. Since $f(u_k) \leq f(x_k)$, we have

$$\begin{aligned} v_k - y_k &= f(u_k) - (1-\alpha_k)x_k - \alpha_k f(x_k) \\ &= (1-\alpha_k)f(u_k) + \alpha_k f(u_k) - (1-\alpha_k)x_k - \alpha_k f(x_k) \\ &= (1-\alpha_k)(f(u_k) - x_k) + \alpha_k(f(u_k) - f(x_k)) \\ &\le (1-\alpha_k)(f(u_k) - f(x_k)) + \alpha_k(f(u_k) - f(x_k)) \\ &= f(u_k) - f(x_k) \\ &\le 0. \end{aligned}$$

Therefore, $v_k \le y_k$, and so $f(v_k) \le f(y_k)$. Thus, we get $u_{k+1} \le x_{k+1}$. By mathematical induction, we have $u_n \le x_n$ for all $n \ge 1$. From Lemma 2.2 (i) and (iii), and using $U < u_1$ and definition of $\{u_n\}$, from mathematical induction we can show that $U < u_k$. Since $p \le u_n \le x_n$, we get

$$|u_n - p| \le |x_n - p|, \quad \forall\, n \ge 1,$$

that is $\{u_n\}$ converges better than $\{x_n\}$.

Case 2. Let $x_1 = u_1 < L$. By Proposition 2.2, we get $f(x_1) > x_1$. As in Case 1, we can show that $u_n \ge x_n$ for all $n \ge 1$. Since $u_1 < L$, by using Lemma 2.2 (ii) and (iv) and definition of $\{u_n\}$, by mathematical induction. It is easy to see that $u_n < L$. This implies that

$$|u_n - p| \le |x_n - p|, \quad \forall\, n \ge 1,$$

that is $\{u_n\}$ converges better than $\{x_n\}$.

Case 3. Let $L \le x_1 = u_1 < U$, Assume that $f(x_1) \ne x_1$. If $f(x_1) < x_1$, then by Lemma 2.1 (i) and (iii), $\{x_n\}$, $\{u_n\}$ are nonincreasing sequences with limit p. So, it follow from Lemma 2.2 (i) and (iii) that $p \le u_n$ for all $n \ge 1$. As in Case 1, we have show that $u_n \le x_n$ for all $n \ge 1$. So, we have $p \le u_n \le x_n$. This implies that

$$|u_n - p| \le |x_n - p|, \quad \forall\, n \ge 1,$$

that is $\{u_n\}$ converges better than $\{x_n\}$. If $f(x_1) > x_1$, then by Lemma 2.1 (ii) and (iv), $\{x_n\}$, $\{u_n\}$ are nondecreasing sequences with limit p. So, it follow from Lemma 2.2 (ii) and (iv) that $p \ge u_n$ for all $n \ge 1$. As in Case 2, we have show that $u_n \ge x_n$ for all $n \ge 1$. So, we have $p \ge u_n \ge x_n$. This implies that

$$|u_n - p| \le |x_n - p|, \quad \forall\, n \ge 1,$$

that is $\{u_n\}$ converges better than $\{x_n\}$. □

Remark 2.2. From Theorem 2.1, we come to a conclusion that, under the same computational cost, Picard iteration is better than Picard-Mann hybrid iteration.

Next, we present a numerical example to compare the rate of convergence of Picard and Picard-Mann hybrid iterations.

Table 1. Comparison of rate of convergence of two-step Picard and Picard-Mann hybrid iterations

	TSP				PMH	
n	u_n	x_n	$\lvert f(u_n)-u_n\rvert$	$\frac{u_{n+1}-p}{u_n-p}$	$\lvert f(x_n)-x_n\rvert$	$\frac{x_{n+1}-p}{x_n-p}$
2	1.823457	2.154780	4.452439E-01	1.567365	5.824162E-01	1.773202
3	1.155927	1.501254	9.512107E-02	1.072000	2.882195E-01	1.201427
4	1.023152	1.199367	1.441974E-02	1.010052	1.207676E-01	1.068700
5	1.003282	1.075663	2.050156E-03	1.001412	4.674634E-02	1.024656
6	1.000462	1.028217	2.887510E-04	1.000199	1.756064E-02	1.008998
⋮	⋮	⋮	⋮	⋮	⋮	⋮
11	1.00000	1.000003	1.588351E-08	1.000000	1.253928E-04	1.000063
12	1.00000	1.000001	2.233619E-09	1.000000	4.678466E-05	1.000023
13	1.00000	1.000000	3.141025E-010	1.000000	1.746907E-05	1.000009
14	1.00000	1.000000	4.417089E-011	1.000000	6.526881E-06	1.000003

Example 2.1. Let $f:[0,4]\to[0,4]$ be defined by $f(x)=\frac{x^2+2\sqrt{x}+5}{8}$. Then it is clear that f is continuous and nondecreasing function with the fixed point $p=1$. In the following table, the comparison of the convergence for Picard and Picard-Mann hybrid iterations is given with the initial point $u_1=x_1=3.4$ and the sequences $\alpha_n=\frac{1}{n^2+1}$. From the table 1, we see that the under the same computational cost, Picard iteration converges better than the Picard-Mann hybrid iteration.

3. A result on the Picard-Mann hybrid iteration

Ishikawa [4] proved that, under some conditions, the Ishikawa sequence converges strongly to a fixed point of Lipschitz pseudocontractive mappings with nonempty fixed point sets. Chidume and Mutangadura [2] constructed an example of a Lipschitz pseudocontraction with a unique fixed point for which every nontrivial Mann sequence fails to converge. We now show Picard-Mann hybrid sequence also fails to converge.

Example 3.1. Let X be the real Hilbert space $\mathbb{R}^2$ under the usual Euclidean inner product. If $x=(a,b)\in X$ we define $x^{\perp}\in X$ to be $(b,-a)$. Trivially, we have $\langle x, x^\perp \rangle = 0$, $\|x^\perp\| = \|x\|$, $\langle x^\perp, y^\perp \rangle = \langle x, y \rangle$, $\|x^\perp - y^\perp\| = \|x - y\|$ and $\langle x^\perp, y \rangle + \langle x, y^\perp \rangle = 0$ for all $x, y \in X$. Take closed and bounded convex set K to be the closed unit ball in X and put $K_1 = \{x \in X : \|x\| \leq \frac{1}{2}\}$, $K_1 = \{x \in X : \frac{1}{2} \leq \|x\| \leq 1\}$. Define the map $T : K \longrightarrow K$ by

$$Tx = \begin{cases} x + x^\perp, & \text{if} \quad x \in K_1, \\ \frac{x}{\|x\|} - x + x^\perp, & \text{if} \quad x \in K_2. \end{cases}$$

The origin is the only fixed point of T.

Next, we prove that no Picard-Mann hybrid sequence for T is convergent for any nonzero starting point.

First, we show that no such Picard-Mann hybrid sequence converges to the fixed point. Let $x \in K$ be such that $x \neq 0$ and let $y = \lambda x + (1-\lambda)Tx$, $\lambda \in (0, 1)$. Then, in case $x \in K_1$, we have $\|y\|^2 = \|\lambda x + (1 - \lambda)Tx\|^2 = (1 + \lambda^2)\|x\|^2$, so $\|x\|^2 < \|y\|^2 < 2\|x\|^2$. If $x \in K_2$, then

$$\begin{aligned} \|y\|^2 &= \|\lambda x + (1 - \lambda)Tx\|^2 \\ &= \| \left(\frac{\lambda}{\|x\|} + 1 - 2\lambda \right) x + \lambda x \perp \|^2 \\ &= \left[\left(\frac{\lambda}{\|x\|} + 1 - 2\lambda \right)^2 + \lambda^2 \right] \|x\|^2 \\ &\geq \frac{1}{2}\|x\|^2. \end{aligned}$$

Furthermore if $y_n \in K_1$, we have $\|x_{n+1}\| = \|Ty_n\|^2 = 2\|y_n\|^2 \geq \|y_n\|^2$. If $y_n \in K_2$, we have $\|x_{n+1}\| = \|Ty_n\|^2 \geq \|y_n\|^2$. We therefore conclude that, in addition, any Picard-Mann hybrid iterate of any nonzero vector in K is itself nonzero. Thus any Picard-Mann hybrid sequence $\{x_n\}$, starting from a nonzero vector, must be infinite. For such a sequence to converge to the origin, x_n would have to lie in the neighborhood $K_0 = \{x \in X : \|x\| \leq \frac{\sqrt{2}}{4}\} \subset K_1$ of the origin and y_n lies in K_1 for all $n > N_0$, for some real N_0. This is not possible because, as already established for K_1, $\|x_n\| < \|y_n\| < \|x_{n+1}\|$ for all $n > N_0$.

Acknowledgments: The authors would like to express their thanks to Cuijie Zhang for helpful discussion.

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