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TWO-STEP ITERATIVE SCHEME FOR A PAIR OF SIMULTANEOUSLY GENERALIZED ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we study the notion of a pair of simultaneously generalized asymptotically quasi-nonexpansive mappings and prove a strong convergence theorem of two-step iterative scheme with errors for said mappings in the framework of Banach spaces. The result obtained in this paper is an extension and improvement of the corresponding result of [1]-[3], [5], [7, 8], [10]-[16] and [18]-[20].

1. Introduction

The concept of quasi-nonexpansive mapping was initiated by Tricomi in 1941 for real function. The concept of asymptotically nonexpansive mapping and the asymptotically nonexpansive type mapping were introduced by Goebel and Kirk [4] and Kirk [9], respectively, which are closely related to the theory of fixed points in Banach spaces. Shahzad and Zegeye [17] introduced the notion of generalized asymptotically quasi-nonexpansive mapping which is more general than both asymptotically nonexpansive and asymptotically nonexpansive type mappings and they established strong convergence theorem of an implicit iteration process for a finite family of mappings. Recently, Li et al. [10] introduced the notion of a pair of simultaneously asymptotically quasi-nonexpansive type mappings and established a general strong convergence

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theorem of the iteration scheme for a pair of said mappings in the setting of Banach spaces. Very recently, Imnang and Suantai [6] studied multi-step Noor iterations with errors for a finite family of generalized asymptotically quasi-nonexpansive mappings and established some strong convergence theorems in the setting of Banach spaces.

Inspired by [6, 10, 17] and many others, we study the notion of a pair of simultaneously generalized asymptotically quasi-nonexpansive mappings and prove a general strong convergence theorem of the iterative scheme with errors for a pair of simultaneously generalized asymptotically quasi-nonexpansive mappings in the framework of Banach spaces. Our results are extension and improvement of the corresponding results of [1]-[3], [5], [7, 8], [10]-[16] and [18]-[20].

2. Preliminaries

Throughout this paper, let E be a real Banach space, C be a nonempty subset of E, $S, T\colon C \to E$ a couple of mappings, $F(T)$ and $F(S)$ the set of fixed points of T and S respectively, that is, $F(T) = \{x \in C : Tx = x\}$ and $F(S) = \{y \in C : Sy = y\}$. Let m and n denote the nonnegative integers.

Definition 2.1. [3, 4, 12, 17] *Let $T\colon C \to E$ be a mapping,*

(1) T is said to be nonexpansive if

$$\|Tx - Ty\| \le \|x - y\| \tag{2.1}$$

for all $x, y \in C$;

(2) T is said to be quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$\|Tx - p\| \le \|x - p\| \tag{2.2}$$

for all $x \in C$ and $p \in F(T)$;

(3) T is said to be asymptotically nonexpansive if there exists a sequence $\{b_n\} \subset [0, \infty)$ with $b_n \to 0$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \le (1 + b_n)\|x - y\|, \tag{2.3}$$

for all $x, y \in C$ and $n \ge 0$;

(4) T is said to be asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{b_n\} \subset [0, \infty)$ with $b_n \to 0$ as $n \to \infty$ such that

$$\|T^n x - p\| \le (1 + b_n)\|x - p\|, \tag{2.4}$$

for all $x \in C$, $p \in F(T)$ and $n \ge 0$;

(5) generalized asymptotically quasi-nonexpansive [17] *if $F(T) \neq \emptyset$ and there exist two sequences of real numbers $\{b_n\}$ and $\{c_n\}$ with $\lim_{n\to\infty} b_n = 0 = \lim_{n\to\infty} c_n$ such that*

$$\|T^n x - p\| \le (1 + b_n)\|x - p\| + c_n, \tag{2.5}$$

for all $x \in C$, $p \in F(T)$ and $n \ge 1$.

Remark 2.2. *If in definition (5), $c_n = 0$ for all $n \ge 1$, then T becomes asymptotically quasi-nonexpansive, and hence the class of generalized asymptotically quasi-nonexpansive maps includes the class of asymptotically quasi-nonexpansive maps.*

Definition 2.3. *Let $S, T\colon C \to E$ be two mappings. (S,T) is said to be a pair of simutaneously generalized asymptotically quasi-nonexpansive mappings if $F(T) \neq \emptyset$, $F(S) \neq \emptyset$ and there exist two sequences of real numbers $\{b_n\}$ and $\{c_n\}$ with $\lim_{n\to\infty} b_n = 0 = \lim_{n\to\infty} c_n$ such that*

$$\|T^n x - p\| \le (1 + b_n)\|x - p\| + c_n, \tag{2.6}$$

for all $x \in C$, $p \in F(S)$ and $n \ge 1$, and

$$\|S^n x - p\| \le (1 + b_n)\|x - p\| + c_n, \tag{2.7}$$

for all $x \in C$, $p \in F(T)$ and $n \ge 1$.

For our main result, we need the following lemma.

Lemma 2.4. *(see* [18]*) Let $\{p_n\}$, $\{q_n\}$, $\{r_n\}$ be three sequences of nonnegative real numbers satisfying the following conditions:*

$$p_{n+1} \le (1 + q_n)p_n + r_n, \quad n \ge 0, \quad \sum_{n=0}^{\infty} q_n < \infty, \quad \sum_{n=0}^{\infty} r_n < \infty.$$

Then

(1) $\lim_{n\to\infty} p_n$ exists.

(2) In addition, if $\liminf_{n\to\infty} p_n = 0$, then $\lim_{n\to\infty} p_n = 0$.

3. Main Results

Theorem 3.1. *Let E be a real Banach space, C be a nonempty subset of E. (S,T) be a pair of simultaneously generalized asymptotically quasi-nonexpansive mappings on C with sequences $\{b_n\},\{c_n\} \subset [0,\infty)$ such that $\sum_{n=0}^{\infty} b_n < \infty$ and $\sum_{n=0}^{\infty} c_n < \infty$. Assume that there exist constants L_1, L_2, α_1 and $\alpha_2 > 0$ such that*

$$\|Tx - y^*\| \leq L_1 \|x - y^*\|^{\alpha_1}, \ \forall x \in C, \ \forall y^* \in F(S), \tag{3.1}$$

and

$$\|Sx - x^*\| \leq L_2 \|x - x^*\|^{\alpha_2}, \ \forall x \in C, \ \forall x^* \in F(T). \tag{3.2}$$

For any given $x_0 \in C$, the iteration scheme $\{x_n\}$ with errors is defined by

$$\begin{aligned} z_n &= (1-\beta_n)x_n + \beta_n S^n x_n + \beta_n v_n, \quad n \geq 0, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T^n z_n + \alpha_n u_n \quad n \geq 0, \end{aligned} \tag{3.3}$$

where $\{u_n\}$ and $\{v_n\}$ are bounded sequences in C and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ satisfying $\sum_{n=0}^{\infty} \alpha_n < \infty$. Suppose that $\{y_n\}$ is a sequence in C and define $\{\varepsilon_n\}$ by

$$\begin{aligned} w_n &= (1-\beta_n)y_n + \beta_n S^n y_n + \beta_n v_n, \quad n \geq 0, \\ \varepsilon_n &= \|y_{n+1} - (1-\alpha_n)y_n - \alpha_n T^n w_n - \alpha_n u_n\|, \ n \geq 0. \end{aligned} \tag{3.4}$$

If $F(S) \cap F(T) \neq \emptyset$, then we have the following:

(i) $\{x_n\}$ converges strongly to some common fixed point p^* of S and T if and only if

$$\liminf_{n\to\infty} d(x_n, F(S) \cap F(T)) = 0.$$

(ii) $\sum_{n=0}^{\infty} \varepsilon_n < \infty$ and $\liminf_{n\to\infty} d(y_n, F(S) \cap F(T)) = 0$ imply that $\{y_n\}$ converges strongly to some common fixed point p^* of S and T.

(iii) If $\{y_n\}$ converges strongly to some common fixed point p^* of S and T, then $\lim_{n\to\infty} \varepsilon_n = 0$.

To prove Theorem 3.1, we first give the following lemma.

Lemma 3.2. *Assume that all the assumptions in Theorem 3.1 hold and* $\sum_{n=0}^{\infty} \varepsilon_n < \infty$*. Then*

(i)

$$\|y_{n+1} - y^*\| \leq (1 + A_n)\|y_n - y^*\| + B_n + \varepsilon_n + \alpha_n K, \ \forall y^* \in F(S) \cap F(T),$$

where

$$A_n = b_n^2 + 2b_n \ \ \text{and} \ \ B_n = (2 + b_n)c_n$$

with $\sum_{n=0}^{\infty} A_n < \infty$ *and* $\sum_{n=0}^{\infty} B_n < \infty$ *since by assumptions* $\sum_{n=0}^{\infty} b_n < \infty$ *and* $\sum_{n=0}^{\infty} c_n < \infty$ *and*

$$K = \sup_{n \geq 0}\{(1 + b_n)\|v_n\| + \|u_n\|\} < \infty.$$

(ii)

$$\|y_m - y^*\| \leq K'\|y_n - y^*\| + K' \sum_{j=n}^{m-1} B_j + K' \sum_{j=n}^{m-1} \varepsilon_j + KK' \sum_{j=n}^{m-1} \alpha_j,$$

for all $y^* \in F(S) \cap F(T)$ *and* $m > n$*, where* $K' = e^{\sum_{j=n}^{\infty} A_j}$*.*

(iii)

$$\lim_{n \to \infty} d(y_n, F(S) \cap F(T)) \ \text{exists.}$$

Proof. Take any $y^* \in F(S) \cap F(T)$, it follows from (3.4) that

$$\begin{aligned} \|y_{n+1} - y^*\| &\leq \varepsilon_n + \|(1 - \alpha_n)(y_n - y^*) + \alpha_n(T^n w_n - y^*) + \alpha_n u_n\| \\ &\leq (1 - \alpha_n)\|y_n - y^*\| + \alpha_n \|T^n w_n - y^*\| \\ &\quad + \alpha_n \|u_n\| + \varepsilon_n \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \|w_n - y^*\| &= \|(1 - \beta_n)(y_n - y^*) + \beta_n(S^n y_n - y^*) + \beta_n v_n\| \\ &\leq (1 - \beta_n)\|y_n - y^*\| + \beta_n \|S^n y_n - y^*\| + \beta_n \|v_n\|. \end{aligned} \tag{3.6}$$

Since (S, T) is a pair of simultaneously generalized asymptotically quasi-nonexpansive mappings and since w_n is in C, from (3.5) and (3.6), we have

$$\begin{aligned} \|y_{n+1} - y^*\| &\leq (1-\alpha_n)\|y_n - y^*\| + \alpha_n[(1+b_n)\|w_n - y^*\| + c_n] \\ &\quad +\alpha_n\|u_n\| + \varepsilon_n \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \|w_n - y^*\| &\leq (1-\beta_n)\|y_n - y^*\| + \beta_n[(1+b_n)\|y_n - y^*\| + c_n] \\ &\quad +\beta_n\|v_n\| \\ &\leq (1+b_n)\|y_n - y^*\| + \beta_n c_n + \beta_n\|v_n\| \\ &\leq (1+b_n)\|y_n - y^*\| + c_n + \|v_n\|. \end{aligned} \tag{3.8}$$

Substituting (3.8) into (3.7), we have

$$\begin{aligned} \|y_{n+1} - y^*\| &\leq (1-\alpha_n)\|y_n - y^*\| + \alpha_n(1+b_n)[(1+b_n)\|y_n - y^*\| \\ &\quad +c_n + \|v_n\|] + \alpha_n c_n + \alpha_n\|u_n\| + \varepsilon_n \\ &\leq (1+b_n)^2\|y_n - y^*\| + \alpha_n c_n(1+b_n) + \alpha_n(1+b_n)\|v_n\| \\ &\quad +\alpha_n c_n + \alpha_n\|u_n\| + \varepsilon_n \\ &\leq (1+A_n)\|y_n - y^*\| + (2+b_n)c_n + \varepsilon_n \\ &\quad +\alpha_n[(1+b_n)\|v_n\| + \|u_n\|] \\ &\leq (1+A_n)\|y_n - y^*\| + (2+b_n)c_n + \varepsilon_n \\ &\quad +\alpha_n[(1+b_n)\|v_n\| + \|u_n\|] \\ &\leq (1+A_n)\|y_n - y^*\| + B_n + \varepsilon_n + \alpha_n K, \end{aligned} \tag{3.9}$$

where

$$A_n = b_n^2 + 2b_n \quad \text{and} \quad B_n = (2+b_n)c_n$$

with $\sum_{n=0}^{\infty} A_n < \infty$ and $\sum_{n=0}^{\infty} B_n < \infty$ since by assumption $\sum_{n=0}^{\infty} b_n < \infty$, $\sum_{n=0}^{\infty} c_n < \infty$ and

$$K = \sup_{n\geq 0}\{(1+b_n)\|v_n\| + \|u_n\|\} < \infty.$$

The conclusion (i) holds.

Note that when $x > 0$, $1 + x \leq e^x$. It follows from conclusion (i), we have, for any $m > n$,

$$
\begin{aligned}
\|y_m - y^*\| &\leq (1 + A_{m-1}) \|y_{m-1} - y^*\| + B_{m-1} + \varepsilon_{m-1} + \alpha_{m-1} K \\
&\leq e^{A_{m-1}} \|y_{m-1} - y^*\| + B_{m-1} + \varepsilon_{m-1} + \alpha_{m-1} K \\
&\leq e^{A_{m-1}} [e^{A_{m-2}} \|y_{m-2} - y^*\| + B_{m-2} + \varepsilon_{m-2} + \alpha_{m-2} K] \\
&\quad + B_{m-1} + \varepsilon_{m-1} + \alpha_{m-1} K \\
&\leq e^{\{A_{m-1} + A_{m-2}\}} \|y_{m-2} - y^*\| + e^{A_{m-1}} [B_{m-1} + B_{m-2}] \\
&\quad + e^{A_{m-1}} [\varepsilon_{m-1} + \varepsilon_{m-2}] + e^{A_{m-1}} K [\alpha_{m-1} + \alpha_{m-2}] \\
&\leq \cdots \\
&\leq \left(e^{\sum_{j=n}^{m-1} A_j} \right) \|y_n - y^*\| + \left(e^{\sum_{j=n}^{m-1} A_j} \right) \sum_{j=n}^{m-1} B_j \\
&\quad + \left(e^{\sum_{j=n}^{m-1} A_j} \right) \sum_{j=n}^{m-1} \varepsilon_j + K \left(e^{\sum_{j=n}^{m-1} A_j} \right) \sum_{j=n}^{m-1} \alpha_j \\
&= K' \|y_n - y^*\| + K' \sum_{j=n}^{m-1} B_j \\
&\quad + K' \sum_{j=n}^{m-1} \varepsilon_j + K K' \sum_{j=n}^{m-1} \alpha_j, \ \forall y^* \in F(S) \cap F(T). \qquad (3.10)
\end{aligned}
$$

where $K' = e^{\sum_{j=n}^{\infty} A_j}$.

This implies that the conclusion (ii) holds.

Again, it follows from conclusion (i) that

$$
\begin{aligned}
d(y_{n+1}, F(S) \cap F(T)) &\leq (1 + A_n) d(y_n, F(S) \cap F(T)) \\
&\quad + B_n + \varepsilon_n + \alpha_n K, \quad n \geq 1.
\end{aligned}
$$

Since $\sum_{n=0}^{\infty} A_n < \infty$, $\sum_{n=0}^{\infty} B_n < \infty$, $\sum_{n=0}^{\infty} \varepsilon_n < \infty$ and $K < \infty$, we have

$$
\sum_{n=0}^{\infty} \left(B_n + \varepsilon_n + \alpha_n K \right) < \infty.
$$

Thus, from Lemma 2.4, we know that $\lim_{n \to \infty} d(y_n, F(S) \cap F(T))$ exists. This implies that conclusion (iii) holds. This completes the proof of Lemma 3.2. □

Since the Lemma 3.2 holds for an arbitrary sequence $\{y_n\}$ in C, we have the following corollary as the proof of Lemma 3.2.

Corollary 3.3. *Assume that all the assumptions in Theorem 3.1 hold. Then there exists a constant $K > 0$ such that*

(i)

$$\|x_{n+1} - y^*\| \leq (1 + A_n)\|x_n - y^*\| + B_n + \alpha_n K, \ \forall y^* \in F(S) \cap F(T),$$

where

$$A_n = b_n^2 + 2b_n \ \ \textit{and} \ \ B_n = (2 + b_n)c_n$$

with $\sum_{n=0}^{\infty} A_n < \infty$ *and* $\sum_{n=0}^{\infty} B_n < \infty$ *since by assumptions* $\sum_{n=0}^{\infty} b_n < \infty$, $\sum_{n=0}^{\infty} c_n < \infty$ *and*

$$K = \sup_{n \geq 0}\{(1 + b_n)\|v_n\| + \|u_n\|\} < \infty.$$

(ii)

$$\|x_m - y^*\| \leq K'\|x_n - y^*\| + K' \sum_{j=n}^{m-1} B_j + KK' \sum_{j=n}^{m-1} \alpha_j,$$

for all $y^* \in F(S) \cap F(T)$ *and* $m > n$, *where* $K' = e^{\sum_{j=n}^{\infty} A_j}$.

(iii)

$$\lim_{n\to\infty} d(x_n, F(S) \cap F(T)) \ \ \textit{exists.}$$

Proof. **The Proof of Theorem 3.1**

The necessity of the conclusion (i) is obvious and the sufficiency follows from conclusion (ii) by setting $\varepsilon_n = 0$ for all $n \geq 0$ in (3.4) and considering (3.3). Now, we prove the conclusion (ii). It follows from Lemma 3.2(iii) that $\lim_{n\to\infty} d(y_n, F(S) \cap F(T))$ exists. Since

$$\liminf_{n\to\infty} d(y_n, F(S) \cap F(T)) = 0,$$

we have

$$\lim_{n\to\infty} d(y_n, F(S) \cap F(T)) = 0. \tag{3.11}$$

First, we have to prove that $\{y_n\}$ is a Cauchy sequence in E. In fact, it follows from (3.11), the assumptions $\sum_{n=0}^{\infty} \varepsilon_n < \infty$, $\sum_{n=0}^{\infty} B_n < \infty$ and $\sum_{n=0}^{\infty} \alpha_n < \infty$ that for any given $\varepsilon > 0$ there exists a positive integer n_1 such that

$$d(y_n, F(S) \cap F(T)) < \frac{\varepsilon}{8(K'+1)}, \ n \geq n_1 \tag{3.12}$$

$$\sum_{n=n_1}^{\infty} B_n < \frac{\varepsilon}{4K'} \tag{3.13}$$

$$\sum_{n=n_1}^{\infty} \varepsilon_n < \frac{\varepsilon}{4K'} \tag{3.14}$$

and

$$\sum_{n=n_1}^{\infty} \alpha_n < \frac{\varepsilon}{4KK'}. \tag{3.15}$$

By the definition of infimum, it follows from (3.12) that for any given $n \geq n_1$ there exists an $y^*(n) \in F(S) \cap F(T)$ such that

$$\|y_n - y^*(n)\| < \frac{\varepsilon}{4(K'+1)}. \tag{3.16}$$

On the other hand, for any $m, n \geq n_1$, without loss of generality $m > n_1$, it follows from Lemma 3.2(ii) that

$$\begin{aligned} \|y_m - y_n\| &\leq \|y_m - y^*(n)\| + \|y_n - y^*(n)\| \\ &\leq K' \|y_n - y^*(n)\| + K' \sum_{j=n}^{m-1} B_j + \sum_{j=n}^{m-1} \varepsilon_j \\ &\quad + KK' \sum_{j=n}^{m-1} \alpha_j + \|y_n - y^*(n)\| \\ &= (K'+1) \|y_n - y^*(n)\| + K' \sum_{j=n}^{m-1} B_j + \sum_{j=n}^{m-1} \varepsilon_j \\ &\quad + KK' \sum_{j=n}^{m-1} \alpha_j. \end{aligned} \tag{3.17}$$

Therefore from (3.13) - (3.17), for any $m > n \geq n_1$, we have

$$\begin{aligned} \|y_m - y_n\| &< (K'+1).\frac{\varepsilon}{4(K'+1)} + K'.\frac{\varepsilon}{4K'} \\ &\quad + K'.\frac{\varepsilon}{4K'} + KK'.\frac{\varepsilon}{4KK'} = \varepsilon. \end{aligned} \tag{3.18}$$

This shows that $\{y_n\}$ is a Cauchy sequence in E. Since E is complete, there exists an $y^* \in E$ such that $y_n \to y^*$ as $n \to \infty$.

Now, we prove that y^* is a fixed point of T. Since $y_n \to y^*$ and $d(y_n, F(S) \cap F(T)) \to 0$ as $n \to \infty$, for any given $\varepsilon > 0$, there exists a positive integer $n_2 \geq n_1$ such that

$$\|y_n - y^*\| < \varepsilon, \quad d(y_n, F(S) \cap F(T)) < \varepsilon, \tag{3.19}$$

for all $n \geq n_2$. The second inequality in (3.19) implies that there exists $z^* \in F(S) \cap F(T)$ such that

$$\|y_{n_2} - z^*\| < 2\varepsilon. \tag{3.20}$$

Since (S, T) is a pair of simultaneously generalized asymptotically quasi-nonexpansive mappings, thus from (3.19) and (3.20) and for any $n \geq n_2$, we have

$$\begin{aligned} \|T^n y^* - y^*\| &\leq \|T^n y^* - z^*\| + \|y^* - z^*\| \\ &\leq (2 + b_n)\,\|y^* - z^*\| + c_n \\ &\leq c_n + (2 + b_n)[\|y^* - y_{n_2}\| + \|z^* - y_{n_2}\|] \\ &< c_n + (2 + b_n)(\varepsilon + 2\varepsilon) = c_n + 3b_n\varepsilon + 6\varepsilon \\ &\quad = \varepsilon_1, \end{aligned} \tag{3.21}$$

where $\varepsilon_1 = c_n + 3b_n\varepsilon + 6\varepsilon$, since $c_n \to 0$, $b_n \to 0$ as $n \to \infty$ and $\varepsilon > 0$, it follows that $\varepsilon_1 > 0$. The inequality (3.21) implies that $T^n y^* \to y^*$ as $n \to \infty$. Again since

$$\begin{aligned} \|T^n y^* - Ty^*\| &\leq \|T^n y^* - z^*\| + \|Ty^* - z^*\| \\ &\leq (1 + b_n)\,\|y^* - z^*\| + c_n + \|Ty^* - z^*\|\,, \end{aligned} \tag{3.22}$$

for all $n \geq n_2$, by assumption (3.1) and using (3.19) and (3.20), we have

$$\begin{aligned} \|T^n y^* - Ty^*\| &\leq (1 + b_n)\,\|y^* - z^*\| + c_n + L_1\,\|y^* - z^*\|^{\alpha_1} \\ &\leq (1 + b_n)[\|y^* - y_{n_2}\| + \|z^* - y_{n_2}\|] + c_n \\ &\quad + L_1[\|y^* - y_{n_2}\| + \|z^* - y_{n_2}\|]^{\alpha_1} \\ &< 3(1 + b_n)\varepsilon + c_n + L_1(3\varepsilon)^{\alpha_1} = \varepsilon_1' \end{aligned} \tag{3.23}$$

where $\varepsilon_1' = 3(1 + b_n)\varepsilon + c_n + L_1(3\varepsilon)^{\alpha_1}$, since $c_n \to 0$, $b_n \to 0$ as $n \to \infty$, $\varepsilon > 0$ and $L_1 > 0$, it follows that $\varepsilon_1' > 0$, the inequality (3.23) shows that $T^n y^* \to Ty^*$ as $n \to \infty$. By the uniqueness of limit, we have $Ty^* = y^*$, that is, y^* is a fixed point of T.

Next, we prove that y^* is also a fixed point of S. Since $y_n \to y^*$ and $y^* \in F(T)$, $d(y_n, F(T)) \to 0$ (also follows from $d(y_n, F(S) \cap F(T)) \to 0$ and $d(y_n, F(T)) \leq d(y_n, F(S) \cap F(T)))$. Thus, for any given $\varepsilon > 0$, there exists a positive integer $n_3 \geq n_2 \geq n_1$ such that

$$\|y_n - y^*\| < \varepsilon, \quad d(y_n, F(T)) < \varepsilon, \tag{3.24}$$

for all $n \geq n_3$. The second inequality in (3.24) implies that there exists $z_1^* \in F(T)$ such that

$$\|y_{n_3} - z_1^*\| < 2\varepsilon. \tag{3.25}$$

Since (S, T) is a pair of simultaneously generalized asymptotically quasi-nonexpansive mappings. Thus, from (3.24) and (3.25), for any $n \geq n_3$, we have

$$\begin{aligned} \|S^n y^* - y^*\| &\leq \|S^n y^* - z_1^*\| + \|y^* - z_1^*\| \\ &\leq (2 + b_n)\|y^* - z_1^*\| + c_n \\ &\leq c_n + (2 + b_n)[\|y^* - y_{n_3}\| + \|z_1^* - y_{n_3}\| \\ &< c_n + (2 + b_n)(\varepsilon + 2\varepsilon) = c_n + 3b_n\varepsilon + 6\varepsilon \\ &\quad = \varepsilon_2, \end{aligned} \tag{3.26}$$

where $\varepsilon_2 = c_n + 3b_n\varepsilon + 6\varepsilon$, since $c_n \to 0$, $b_n \to 0$ as $n \to \infty$ and $\varepsilon > 0$, it follows that $\varepsilon_2 > 0$. The inequality (3.26) implies that $S^n y^* \to y^*$ as $n \to \infty$. Again since

$$\begin{aligned} \|S^n y^* - Sy^*\| &\leq \|S^n y^* - z_1^*\| + \|Sy^* - z_1^*\| \\ &\leq (1 + b_n)\|y^* - z_1^*\| + c_n + \|Sy^* - z_1^*\|, \end{aligned} \tag{3.27}$$

for all $n \geq n_3$, by assumption (3.2) and using (3.24) and (3.25), we have

$$\begin{aligned} \|S^n y^* - Sy^*\| &\leq (1 + b_n)\|y^* - z_1^*\| + c_n + L_2\|y^* - z_1^*\|^{\alpha_2} \\ &\leq (1 + b_n)[\|y^* - y_{n_3}\| + \|z_1^* - y_{n_3}\|] + c_n \\ &\quad + L_2[\|y^* - y_{n_3}\| + \|z_1^* - y_{n_3}\|]^{\alpha_2} \\ &< 3(1 + b_n)\varepsilon + c_n + L_2(3\varepsilon)^{\alpha_2} = \varepsilon_2' \end{aligned} \tag{3.28}$$

where $\varepsilon_2' = 3(1 + b_n)\varepsilon + c_n + L_2(3\varepsilon)^{\alpha_2}$, since $c_n \to 0$, $b_n \to 0$ as $n \to \infty$, $\varepsilon > 0$ and $L_2 > 0$, it follows that $\varepsilon_2' > 0$, the inequality (3.28) shows that $S^n y^* \to Sy^*$ as $n \to \infty$. By the uniqueness of limit, we have $Sy^* = y^*$, that is, y^* is also a fixed point of S. Hence y^* is a common fixed point of S and T. Thus, the conclusion (ii) holds.

Since w_n is in C, from (3.4) and (3.8), we have, for any given $\varepsilon > 0$,

$$\begin{aligned}
\varepsilon_n &\leq \|y_{n+1} - y^*\| + \|(1-\alpha_n)(y_n - y^*) + \alpha_n(T^n w_n - y^*) + \alpha_n u_n\| \\
&\leq \|y_{n+1} - y^*\| + (1-\alpha_n)\|y_n - y^*\| + \alpha_n \|T^n w_n - y^*\| \\
&\quad + \alpha_n \|u_n\| \\
&\leq \|y_{n+1} - y^*\| + (1-\alpha_n)\|y_n - y^*\| + \alpha_n\Big[(1+b_n)\|w_n - y^*\| + c_n\Big] \\
&\quad + \alpha_n \|u_n\| \\
&\leq \|y_{n+1} - y^*\| + (1-\alpha_n)\|y_n - y^*\| + \alpha_n(1+b_n)\Big[(1+b_n)\|y_n - y^*\| \\
&\quad + c_n + \|v_n\|\Big] + \alpha_n c_n + \alpha_n \|u_n\| \\
&\leq \|y_{n+1} - y^*\| + (1+b_n)^2\|y_n - y^*\| + c_n(2+b_n) + \alpha_n(1+b_n)\|v_n\| \\
&\quad + \alpha_n \|u_n\| \\
&= \|y_{n+1} - y^*\| + (1+b_n)^2\|y_n - y^*\| + c_n(2+b_n) \\
&\quad + \alpha_n\Big[(1+b_n)\|v_n\| + \|u_n\|\Big].
\end{aligned} \tag{3.29}$$

Since $y_n \to y^*$, $\sum_{n=0}^{\infty} b_n < \infty$, $\sum_{n=0}^{\infty} c_n < \infty$ and $\sum_{n=0}^{\infty} \alpha_n < \infty$, it follows that $\lim_{n\to\infty} \varepsilon_n = 0$. Thus the conclusion (iii) holds. This completes the proof of Theorem 3.1. □

Example 3.4. *Let E be the real line with the usual norm $|.|$ and $K = [0,1]$. Define S and $T\colon K \to K$ by*

$$Tx = sinx, \quad x \in [0,1] \quad and \quad Sx = x/3, \quad x \in [0,1],$$

for $x \in K$. Obviously $F(T) = \{0\}$, $F(S) = \{0\}$ and $F(S) \cap F(T) = \{0\}$, that is, 0 is a common fixed point of S and T. Now we check that S and T are generalized asymptotically quasi-nonexpansive mappings. In fact, if $x \in [0,1]$ and $p = 0 \in F(S) \cap F(T)$, then

$$|Tx - p| = |Tx - 0| = |sinx - 0| = |sinx| \leq |x| = |x - 0| = |x - p|,$$

that is

$$|Tx - p| \leq |x - p|,$$

This shows that T is quasi-nonexpansive. Similarly, we have

$$|Sx - p| = |Sx - 0| = |x/3 - 0| = |x/3| \leq |x| = |x - 0| = |x - p|,$$

that is

$$|Sx - p| \leq |x - p|,$$

This shows that S is quasi-nonexpansive. Hence S and T are asymptotically quasi-nonexpansive with constant sequence $\{k_n\} = \{1\}$ for each $n \geq 1$. Thus by remark 2.1, S and T are pair of simultaneously generalized asymptotically quasi-nonexpansive mappings.

Remark 3.5. *(1) Theorem 3.1 extends the corresponding result of Li et al.* [10] *to the case of more general class of asymptotically quasi-nonexpansive type mappings considered in this paper.*

(2) Theorem 3.1 also extends, improves and unifies the corresponding result in [1]-[3], [5], [7, 8], [11]-[16] *and* [18]-[20].

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