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NEW OSCILLATION CRITERIA FOR CERTAIN TYPE OF THIRD ORDER MIXED NEUTRAL DIFFERENTIAL EQUATIONS

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Abstract. The objective of this paper is to establish a new oscillation criteria for the third order mixed neutral differential equation of the form

$$(x(t) + b(t)x(t - \tau_1) + c(t)x(t + \tau_2))''' + q(t)x^{\alpha}(t - \sigma_1) + p(t)x^{\beta}(t + \sigma_2) = 0,$$

where $b(t), c(t), q(t)$ and $p(t)$ are positive continuous functions, α and β are ratios of odd positive integers, τ_1, τ_2, σ_1 and σ_2 are positive constants. We establish new sufficient conditions which ensure that all solutions are oscillatory. Examples are provided to illustrate the main results.

1. Introduction

In this paper, we are concerned with the following third order mixed type neutral differential equation of the form

$$(x(t)+b(t)x(t-\tau_1)+c(t)x(t+\tau_2))'''+q(t)x^{\alpha}(t-\sigma_1)+p(t)x^{\beta}(t+\sigma_2) = 0, \quad (1.1)$$

for $t \geq t_0$. Throughout this paper, we assume the following hypotheses hold.

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(H_1) $b(t), c(t), p(t), q(t) \in C([t_0), \infty, (0, \infty))$ and there exist b and c such that $b(t) \leq b$, $c(t) \leq c$ with $b + c < 1$;

(H_2) τ_1, τ_2, σ_1 and σ_2 are positive constants and α and β are ratios of odd positive integers.

Let $\theta = \max\{\tau_1, \sigma_1\}$. By a solution of equation (1.1), we mean a real continuous function $x(t)$ defined for all $t \geq t_0 - \theta$ and satisfying the equation (1.1) for all $t \geq t_0$. A solution of equation (1.1) is called oscillatory if it has large zeros on $[t_0, \infty)$, otherwise it is called nonoscillatory.

Recently there has been a great interest in studying the oscillatory and asymptotic behavior of differential equations.

In [3, 4], [6]-[16], the authors studied the oscillatory behavior of solutions of equation (1.1), when $c(t) \equiv 0$ and $p(t) \equiv 0$. In [8, 9, 15, 16], the authors discussed the oscillatory behavior of all solutions of equation (1.1) when $\alpha = \beta = 1$. Further in all the results the authors obtain sufficient conditions for the solutions of the third order differential equation is either oscillatory or tend to zero as $t \to \infty$. In [6] the authors obtain sufficient conditions which ensures that all solutions of equation (1.1) are oscillatory, when $b(t) \equiv -b$, $c(t) \equiv 0$ and $p(t) \equiv 0$. In [15], the authors obtain sufficient conditions for all solutions of equation(1.1) are either oscillatory or tend to zero as $t \to \infty$.

Motivated by the above observation, in this paper we obtain a stronger result in the sense that when all solutions of equation (1.1) are oscillatory. Therefore our result extend that of in [6] and improve the results given in [15].

In Section 2, we present new criteria for all solutions of equation (1.1) to be oscillatory. Examples are provided in Section 3 to illustrate the main results.

2. Oscillatory results

In this section, we present some new oscillation criteria for equation (1.1). For the sake of convenience, when we write a functional inequality without specifying its domain of validity, we assume that it holds for all large t.

We begin with the following lemmas which are crucial in the proof of the main results. For simplicity, we use the following notation, without further mention:

$$z(t) = x(t) + b(t)x(t - \tau_1) + c(t)x(t + \tau_2), \quad \sigma = \max(\sigma_1, \sigma_2),$$
$$Q(t) = \min\{q(t), q(t - \tau_1), q(t + \tau_2)\},$$
$$P(t) = \min\{p(t), p(t - \tau_1), p(t + \tau_2)\}.$$

Lemma 2.1. ([15]) *Assume* $A \geq 0, B \geq 0$. *If* $\delta \geq 1$, *then*

$$(A+B)^{\delta} \leq 2^{\delta-1}(A^{\delta}+B^{\delta}).$$

If $0 < \delta \leq 1$, *then* $(A+B)^{\delta} \leq A^{\delta}+B^{\delta}$.

Lemma 2.2. ([15]) *Let* $x(t)$ *be a positive solution of equation* (1.1). *Then there are only the following two cases for* $z(t)$ *for all sufficiently large* $t \geq t_1$.

(I) $z(t) > 0,\ \ z'(t) > 0,\ \ z''(t) > 0,\ \ (a(t)\ z''(t))' \leq 0$;
(II) $z(t) > 0,\ \ z'(t) < 0,\ \ z''(t) > 0,\ \ (a(t)\ z''(t))' \leq 0$.

Using the above lemmas, we present a new oscillation criteria for the oscillation of all solutions of equation (1.1).

Theorem 2.3. *Assume that* $0 < \alpha < 1 < \beta$ *and* $\sigma_i > \tau_i$ *for* $i = 1, 2$. *If*

$$\liminf_{t\to\infty} \int_{t-\frac{(\sigma-\tau_2)}{3}}^{t} (\sigma-\tau_2)^2 P^{\eta_1}(s) Q^{\eta_2}(s) ds \ > \ \left(\frac{9}{4e}\right) \eta_1^{\eta_1} \eta_2^{\eta_2} \left(4^{\beta-1}\right)^{\eta_1} \tag{2.1}$$

and

$$\int_{t_0}^{\infty} P^{\eta_1}(s) Q^{\eta_2}(s) ds = \infty, \tag{2.2}$$

where $\eta_1 = \dfrac{1-\alpha}{\beta-\alpha}, \eta_2 = \dfrac{\beta-1}{\beta-\alpha}$ *are satisfied, then every solution of equation* (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1). Without loss of generality, we assume that there exists a $t_1 \geq t_0$ such that $x(t) > 0,\ x(t-\sigma_1) > 0$ and $x(t-\tau_1) > 0$ for all $t \geq t_1$. From equation (1.1), we have

$$z'''(t) = -q(t)x^{\alpha}(t-\sigma_1) - p(t)x^{\beta}(t+\sigma_2) < 0 \quad \text{ for all } t \geq t_1.$$

Then as in Lemma 2.2, we have $z''(t) > 0$ for all $t \geq t_1$. Define a function $y(t)$ by

$$y(t) = z(t) + b^{\alpha} z(t-\tau_1) + c^{\alpha} z(t+\tau_2). \tag{2.3}$$

Since $z(t) > 0$ and $z''(t) > 0$, we have $y(t) > 0$, $y''(t) > 0$ and

$$\begin{aligned} &y'''(t) + Q(t)\left[x^{\alpha}(t-\sigma_1) + b^{\alpha}x^{\alpha}(t-\tau_1-\sigma_1) + c^{\alpha}x^{\alpha}(t+\tau_2-\sigma_1)\right] \\ &+ P(t)\left[x^{\beta}(t+\sigma_2) + b^{\alpha}x^{\beta}(t-\tau_1+\sigma_2) + c^{\alpha}x^{\beta}(t+\tau_2+\sigma_2)\right] \leq 0. \end{aligned}$$

Using Lemma 2.1 in the last inequality, we obtain

$$\begin{aligned} &y'''(t) + Q(t)\left[x(t-\sigma_1) + b\ x(t-\tau_1-\sigma_1) + c\ x(t+\tau_2-\sigma_1)\right]^{\alpha} \\ &+ P(t)\left[x^{\beta}(t+\sigma_2) + b^{\beta}x^{\beta}(t-\tau_1+\sigma_2) + c^{\beta}x^{\beta}(t+\tau_2+\sigma_2)\right] \leq 0. \end{aligned}$$

Now using again Lemma 2.1 in the last inequality, we have

$$y'''(t) + Q(t)z^{\alpha}(t-\sigma_1) + \frac{P(t)}{4^{\beta-1}}z^{\beta}(t+\sigma_2) \le 0, \quad t \ge t_1. \tag{2.4}$$

Next we consider two cases for $z'(t)$ as in Lemma 2.2 for $t \ge t_1$.

Case 1. Suppose $z'(t) > 0$. Then from (2.4), we have

$$y'''(t) + Q(t)z^{\alpha}(t-\sigma) + \frac{P(t)}{4^{\beta-1}}z^{\beta}(t-\sigma) \le 0, \quad t \ge t_1. \tag{2.5}$$

Define $u_1 = \eta_1^{-1}\frac{P(t)}{4^{\beta-1}}z^{\beta}(t-\sigma)$ and $u_2 = \eta_2^{-1}Q(t)z^{\alpha}(t-\sigma)$. Using arithmetic geometric mean inequality $\frac{u_1\eta_1 + u_2\eta_2}{\eta_1+\eta_2} \ge (u_1^{\eta_1}u_2^{\eta_2})^{\frac{1}{\eta_1+\eta_2}}$, in (2.5) we have

$$y'''(t) + \eta_1^{-\eta_1}\eta_2^{-\eta_2}\left(\frac{P(t)}{4^{\beta-1}}\right)^{\eta_1} Q^{\eta_2}(t)\; z(t-\sigma) \le 0, \quad t \ge t_1. \tag{2.6}$$

Further from the definition of $y(t)$, we have

$$y(t) \le (1 + b^{\alpha} + c^{\alpha})\; z(t+\tau_2-\sigma), \quad t \ge t_1. \tag{2.7}$$

Using the inequality (2.7) in (2.10), we obtain that $y(t)$ is a positive solution of

$$y'''(t) + \frac{\eta_1^{-\eta_1}\eta_2^{-\eta_2}}{(1 + b^{\alpha} + c^{\alpha})}\left(\frac{P(t)}{4^{\beta-1}}\right)^{\eta_1} Q^{\eta_2}(t)\; y(t-\tau_2+\sigma) \le 0, \quad t \ge t_1. \tag{2.8}$$

But by [6, Corollary 1], the condition (2.1) implies that inequality (2.8) is oscillatory, which is a contradiction.

Case 2. Suppose $z'(t) < 0$ for $t \ge t_1$. Then from (2.4) we have

$$y'''(t) + Q(t)z^{\alpha}(t+\sigma) + \frac{P(t)}{4^{\beta-1}}z^{\beta}(t+\sigma) \le 0, \quad t \ge t_1. \tag{2.9}$$

Again using arithmetic geometric mean inequality, we have from last inequality

$$y'''(t) + \eta_1^{-\eta_1}\eta_2^{-\eta_2}\left(\frac{P(t)}{4^{\beta-1}}\right)^{\eta_1} Q^{\eta_2}(t)\; z(t+\sigma) \le 0, \quad t \ge t_1. \tag{2.10}$$

Further the definition of $y(t)$, we have

$$y(t) \le (1 + b^{\alpha} + c^{\alpha})\; z(t-\tau_1), \quad t \ge t_1. \tag{2.11}$$

Using the inequality (2.11) in (2.10), we obtain that $y(t)$ is a positive solution of

$$y'''(t) + \frac{\eta_1^{-\eta_1}\eta_2^{-\eta_2}}{(1 + b^{\alpha} + c^{\alpha})}\left(\frac{P(t)}{4^{\beta-1}}\right)^{\eta_1} Q^{\eta_2}(t)\; y(t+\tau_1+\sigma) \le 0, \quad t \ge t_1. \tag{2.12}$$

But by [14, Corollary 3.5], the condition (2.2) implies that inequality (2.12) is oscillatory, which is a contradiction. This completes the proof. $\square$

Theorem 2.4. *Assume that $0 < \beta < 1 < \alpha$ and $\sigma_i > \tau_i$ for $i = 1, 2$. If*

$$\liminf_{t\to\infty} \int_{t-\frac{(\sigma-\tau_2)}{3}}^{t} (\sigma+\tau_2)^2 Q^{\eta_1}(s) P^{\eta_2}(s)ds > \left(\frac{9}{4e}\right) \eta_1^{\eta_1}\eta_2^{\eta_2}\left(4^{\alpha-1}\right)^{\eta_1}$$

and

$$\int_{t_0}^{\infty} P^{\eta_2}(s) Q^{\eta_1}(s)ds = \infty,$$

where $\eta_1 = \dfrac{1-\beta}{\alpha-\beta}$, $\eta_2 = \dfrac{\alpha-1}{\alpha-\beta}$ are satisfied, then every solution of equation (1.1) *is oscillatory.*

Proof. The proof is similar to that of Theorem 2.3 and hence the details are omitted. $\square$

3. Examples

Example 3.1. Consider the third order mixed type neutral differential equation

$$\left[x(t) + \frac{1}{4}x(t-\pi) + \frac{1}{3}x(t+\frac{\pi}{2})\right]''' + \frac{1}{12}x^{\frac{1}{3}}(t-2\pi) + \frac{1}{12}x^{\frac{5}{3}}(t+2\pi) = 0, \quad (3.1)$$

for $t \geq 2\pi$. Here $b(t) = \dfrac{1}{4}$, $c(t) = \dfrac{1}{3}$, $p(t) = q(t) = \dfrac{1}{12}$, $\tau_1 = \pi$, $\tau_2 = \pi/2$, $\sigma_1 = \sigma_2 = 2\pi$, $\alpha = \frac{1}{3}$, $\beta = \frac{5}{3}$. Further $\sigma = 2\pi$, $P(t) = Q(t) = \dfrac{1}{12}$, $\eta_1 = \eta_2 = \dfrac{1}{2}$. It is easy to see that all the conditions of Theorem 2.3 are satisfied and hence all solutions of (3.1) are oscillatory.

Example 3.2. Consider the third order mixed type neutral differential equation

$$\left[x(t) + \frac{1}{3}x(t-\frac{\pi}{2}) + \frac{1}{4}x(t+\pi)\right]''' + 2x^{\frac{5}{3}}(t-2\pi) + 3x^{\frac{1}{3}}(t+3\frac{\pi}{2}) = 0, \quad (3.2)$$

for $t \geq 2\pi$. Here $b(t) = \dfrac{1}{3}$, $c(t) = \dfrac{1}{4}$, $p(t) = 3$, $q(t) = 2$, $\tau_1 = \frac{\pi}{2}$, $\tau_2 = \pi$, $\sigma_1 = 2\pi$, $\sigma_2 = \frac{3\pi}{2}$, $\alpha = \frac{5}{3}$, $\beta = \frac{1}{3}$. Further $\sigma = 2\pi$, $P(t) = 3$, $Q(t) = 2$, $\eta_1 = \eta_2 = \dfrac{1}{2}$. It is easy to see that all the conditions of Theorem 2.4 are satisfied and hence all solutions of (3.2) are oscillatory.

We conclude this paper by the following remark.

Remark 3.3. In this paper we established sufficient conditions for the oscillation of all solutions of third order differential equations with mixed arguments and nonlinearities. Therefore our results are new and useful one.

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