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SOLVABILITY AND ASYMPTOTICALLY STABLE OF A MIXED FUNCTIONAL INTEGRAL EQUATION IN N VARIABLES

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Abstract. Using tools of functional analysis and a fixed point theorem of Krasnosel'skii type, this paper proves solvability and asymptotically stable of a mixed functional integral equation in N variables. Furthermore, the set of solutions is compact. In order to illustrate the results obtained here, an example is given.

1. Introduction

In this paper, we consider the mixed functional integral equation in N variables of the form

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$$u(x) = V\left(x, u(x), \int_{B_x} V_1\left(x, y, u(\sigma_1(y)), ..., u(\sigma_p(y))\right) dy\right) + \int_{\mathbb{R}_+^N} F\left(x, y, u(\chi_1(y)), ..., u(\chi_q(y))\right) dy, \quad (1.1)$$

where $x \in \mathbb{R}_+^N = \{(x_1, ..., x_N) \in \mathbb{R}^N : x_1 \geq 0, ..., x_N \geq 0\}$,

$$V : \mathbb{R}_+^N \times E^2 \to E, \quad V_1 : \Delta \times E^p \to E, \quad F : \mathbb{R}_+^{2N} \times E^q \to E,$$

$$\sigma_1, ..., \sigma_p, \chi_1, ..., \chi_q : \mathbb{R}_+^N \to \mathbb{R}_+^N \text{ are continuous},$$

$$\Delta = \{(x, y) \in \mathbb{R}_+^{2N} : y \in B_x\},\ B_x = [0, x_1] \times ... \times [0, x_N],$$

the functions $\sigma_1, ..., \sigma_p, \chi_1, ..., \chi_q : \mathbb{R}_+^N \to \mathbb{R}_+^N$ are continuous with $\sigma_1(x), ..., \sigma_p(x) \in B_x$, $\forall x \in \mathbb{R}_+^N$, E is a Banach space with norm $|\cdot|$.

It is well known that, nonlinear integral equations and nonlinear functional integral equations have been some topics of great interest in the field of nonlinear analysis for a long time. Since the pioneering work of Volterra up to our days, integral equations have attracted the interest of scientists not only because of their mathematical context but also because of their miscellaneous applications in various fields of science and technology, see [14]. The special cases of (1.1) occur in mechanics, population dynamics, engineering systems, the theory of "adiabatic tubular chemical reactors", etc. For the details of such problems, it can be found in, for example, Corduneanu [3] or Deimling [4]. It also can be found some applications of integral or integrodifferential equations to various problems occurring in contemporary research, such as the following integrodifferential equation is encountered in the mathematical description of coagulation process [3], under certain simplifying assumptions

$$f(t, x) = f_0(x) + \frac{1}{2}\int_0^t \int_0^x \phi(x-y, y) f(s, x-y) f(s, y) dy ds - \int_0^t \int_0^\infty f(s, x)\phi(x, y) f(s, y) dy ds.$$

In general, existence results of integral equations have been obtained via the fundamental methods in which the fixed point theorems are often applied, see [1]–[14] and the references given therein. Recently, using the technique of the measure of noncompactness and the Darbo fixed point theorem, Z. Liu *et al.*,, [6] have proved the existence and asymptotic stability of solutions for the equation

$$x(t) = f\left(t,\ x(t),\ \int_0^t u(t, s, x(a(s)), x(b(s)))\ ds\right),\ t \in \mathbb{R}_+.$$

In [2], using a fixed point theorem of Krasnosel'skii, Avramescu and Vladimirescu have proved the existence of asymptotically stable solutions to the following integral equation

$$u(t) = q(t) + \int_0^t K(t,s,u(s))ds + \int_0^\infty G(t,s,u(s))ds,\ t \in \mathbb{R}_+,$$

where the functions given with real values are supposed to be continuous satisfying suitable conditions. In case the Banach space E is arbitrary, recently in [10], [11], the existence of asymptotically stable solutions to the following integral equations

$$x(t) = q(t) + f(t,x(t)) + \int_0^t V(t,s,x(s))ds + \int_0^\infty G(t,s,x(s))ds,\ t \in \mathbb{R}_+$$

or

$$\begin{aligned} u(x,y) &= q(x,y) + f(x,y,u(x,y)) + \int_0^x \int_0^y V(x,y,s,t,u(s,t))\,dsdt \\ &\quad + \int_0^\infty \int_0^\infty F(x,y,s,t,u(s,t))\,dsdt, \quad (x,y) \in \mathbb{R}_+^2, \end{aligned}$$

also have been proved by using the fixed point theorem of Krasnosel'skii type as follows.

Theorem 1.1. ([9]) *Let $(X, |\cdot|_n)$ be a Fréchet space and let U, $C : X \to X$ be two operators. Assume that*

(i) *U is a $k-$contraction operator, $k \in [0,1)$ (depending on n), with respect to a family of seminorms $\|\cdot\|_n$ equivalent with the family $|\cdot|_n$;*
(ii) *C is completely continuous;*
(iii) $\lim\limits_{|x|_n \to \infty} \frac{|Cx|_n}{|x|_n} = 0,\ \forall n \in \mathbb{N}$.

Then $U + C$ has a fixed point.

In [8], Lungu and Rus established some results relative to existence, uniqueness, integral inequalities and data dependence for solutions of the following functional Volterra-Fredholm integral equation in two variables with deviating argument in a Banach space by Picard operators technique

$$u(x,y) = g(x,y,h(u)(x,y)) + \int_0^x \int_0^y K(x,y,s,t,u(s,t))\,dsdt,\ (x,y) \in \mathbb{R}_+^2.$$

In [12], based on the applications of the Banach fixed point theorem coupled with Bielecki type norm and the integral inequality with explicit estimates, B. G. Pachpatte studied some basic properties of solutions of the Fredholm type integral equation in two variables as follows

$$u(x,y) = f(x,y) + \int_0^a \int_0^b g\left(x,y,s,t,u(s,t),D_1u(s,t),D_2u(s,t)\right) dtds.$$

With the same methods, in [13], the existence, uniqueness and other properties of solutions of certain Volterra integral and integrodifferential equations in two variables were considered.

Applying the Banach fixed point theorem, in [5], El-Borai *et al.*, have proved the existence of a unique solution of a nonlinear integral equation of type Volterra-Hammerstein in n-dimensional of the form

$$\mu\phi(x,t) = f(x,t) + \lambda \int_0^t \int_\Omega F(t,\tau)K(x,y)\gamma\left(\tau,y,\phi(y,\tau)\right) dyd\tau,$$

where $x = (x_1,...,x_n)$, $y = (y_1,...,y_n)$; μ,λ are constants. After that, in [1], M. A. Abdou *et al.*, investigated the following mixed nonlinear integral equation of the second kind in $n-$dimensional

$$\begin{aligned}\mu\phi(x,t) &= \lambda \int_\Omega k(x,y)\gamma\left(t,y,\phi(y,t)\right) dy \\ &\quad +\lambda \int_0^t \int_\Omega G(t,\tau)k(x,y)\gamma\left(\tau,y,\phi(y,\tau)\right) dyd\tau \\ &\quad +\lambda \int_0^t F(t,\tau)\phi(x,\tau)d\tau + f(x,t),\end{aligned}$$

where $x = (x_1,...,x_n)$, $y = (y_1,...,y_n)$. Also using the Banach fixed point theorem, the existence of a unique solution of this equation was proved.

Motivated by the above mentioned works, because of mathematical context, we continue to show that Theorem 1.1 associated with tools of functional analysis can be applied in order to obtain the existence result and asymptotic stability of solutions of (1.1). This paper consists of five sections and the existence of solutions, the existence of asymptotically stable solutions for (1.1) will be presented in sections 2 and 3. On the other hand, the set of solutions is compact, see section 4. Finally, we give an illustrated example.

2. Preliminaries

Let $X = C(\mathbb{R}_+^N; E)$ be the space of all continuous functions on $\mathbb{R}_+^N$ to E which be equipped with the numerable family of seminorms

$$|u|_n = \sup_{x\in[0,n]^N} |u(x)|\,,\ n \geq 1.$$

Then X is complete with the metric

$$d(u,v) = \sum_{n=1}^{\infty} 2^{-n} \frac{|u-v|_n}{1+|u-v|_n}$$

and X is the Fréchet space.

Consider in X the other family of seminorms $\|\cdot\|_n$ defined by

$$\|u\|_n = |u|_{\gamma_n} + |u|_{h_n}, \quad n \geq 1,$$

where

$$\begin{aligned} |u|_{\gamma_n} &= \sup_{x\in[0,n]^N,\ |x|_1 \leq \gamma_n} |u(x)|, \\ |u|_{h_n} &= \sup_{x\in[0,n]^N,\ |x|_1 \geq \gamma_n} e^{-h_n(|x|_1-\gamma_n)} |u(x)|, \\ |x|_1 &= x_1 + ... + x_N, \end{aligned}$$

$\gamma_n \in (0,n)$ and $h_n > 0$ are arbitrary numbers. $\|\cdot\|_n$ and $|\cdot|_n$ are equivalent because

$$e^{-h_n(nN-\gamma_n)} |u|_n \leq \|u\|_n \leq 2|u|_n, \quad \forall u \in X, \quad \forall n \geq 1.$$

We have the following condition for the relative compactness of a subset of X. The proof of this condition is similar to that in Appendix of [9] via the Ascoli-Arzela's Theorem (see [7], p. 211).

Lemma 2.1. *Let $X = C(\mathbb{R}_+^N; E)$ be the Fréchet space defined as above and A be a subset of X. For each $n \in \mathbb{N}$, let $X_n = C([0,n]^N; E)$ be the Banach space of all continuous functions $u : [0,n]^N \to E$ with the norm*

$$|u|_n = \sup_{x\in[0,n]^N} |u(x)|$$

and $A_n = \{u|_{[0,n]^N} : u \in A\}$. The set A in X is relatively compact if and only if for each $n \in \mathbb{N}$, A_n is equicontinuous in X_n and for every $x \in [0,n]^N$, the set $A_n(x) = \{u(x) : u \in A_n\}$ is relatively compact in E.

Based on the notion of asymptotically stable solutions to the functional equation mentioned in [2] with citations and notes, we use the following definition and also note that it is stated on spaces of functions defined on $\mathbb{R}_+^N$ not necessarily bounded.

Definition 2.1. A function $\tilde{u}$ is said to be an *asymptotically stable solution* of (1.1) if for any solution u of (1.1), $\lim_{|x|_1 \to +\infty} |u(x) - \tilde{u}(x)| = 0$.

3. Main result

We make the following assumptions.

(A_1) There exist a constant $L \in [0,1)$ and a continuous function $\omega_0 : \mathbb{R}_+^N \to \mathbb{R}_+$ such that

$$|V(x;u,v) - V(x;\bar{u},\bar{v})| \leq L|u-\bar{u}| + \omega_0(x)|v-\bar{v}|,$$

for all $x \in \mathbb{R}_+^N$, $u, v, \bar{u}, \bar{v} \in E$.

(A_2) There exists a continuous function $\omega_1 : \Delta \to \mathbb{R}_+$ such that

$$|V_1(x,y;u_1,...,u_p) - V_1(x,y;\bar{u}_1,...,\bar{u}_p)| \leq \omega_1(x,y)\sum_{i=1}^{p}|u_i - \bar{u}_i|,$$

for all $(x,y;u_1,...,u_p)$, $(x,y;\bar{u}_1,...,\bar{u}_p) \in \Delta \times E^p$.

(A_3) F is completely continuous such that for all bounded subsets I_1, I_2 of $\mathbb{R}_+^N$ and for any bounded subset J of E^q, for all $\varepsilon > 0$, there exists $\delta > 0$, such that $\forall\, x, \bar{x} \in I_1$,

$$|x-\bar{x}|_1 < \delta \Longrightarrow |F(x,y;u_1,...,u_q) - F(\bar{x},y;u_1,...,u_q)| < \varepsilon,$$

for all $(y;u_1,...,u_q) \in I_2 \times J$.

(A_4) There exists a continuous function $\omega_2 : \mathbb{R}_+^{2N} \to \mathbb{R}_+$ such that for each bounded subset I of $\mathbb{R}_+^N$,

$$\int_{\mathbb{R}_+^N} \sup_{x \in I} \omega_2(x,y)dy < \infty$$

and

$$|F(x,y;u_1,...,u_q)| \leq \omega_2(x,y),$$

for all $(x,y;u_1,...,u_q) \in I \times \mathbb{R}_+^N \times E^q$.

(A_5) $\lim\limits_{\eta \to 0_+} \int_{B_x, |\sigma_i(y)|_1 \leq \eta} dy = 0, \quad \forall\, i = 1,...,p.$

Theorem 3.1. *Let* $(A_1) - (A_5)$ *hold. Then the equation* (1.1) *has a solution on* $\mathbb{R}_+^N$. *Furthermore, if*

$$\lim_{|x|_1 \to +\infty}\left[\bar{a}(x) + \bar{R}(x)\exp\left(\bar{R}(0)x_1x_2...x_N\right)\int_{B_x}\bar{a}(y)dy\right] = 0,$$

where

$$\begin{cases} \bar{a}(x) = a(x) + \sum_{i=1}^{p} a(\sigma_i(x)), \quad a(x) = \frac{1}{1-L}\int_{\mathbb{R}_+^N}\omega_2(x,y)dy, \\ \bar{R}(x) = R(x) + \sum_{i=1}^{p} R(\sigma_i(x)), \quad R(x) = \frac{1}{1-L}\omega_0(x)\omega_1(x,0), \\ \omega_0(x)\omega_1(x,y) \leq \omega_0(x)\omega_1(x,0) \leq \omega_0(0)\omega_1(0,0),\ \ \forall y \in B_x,\ \forall x \in \mathbb{R}_+^N, \end{cases}$$

then every solution u *of* (1.1) *is an asymptotically stable solution.*

Proof. First, we define

$$\Phi u(x) = V\left(x, u(x), \int_{B_x} V_1\left(x, y, u(\sigma_1(y)), ..., u(\sigma_p(y))\right) dy\right), \ (x,u) \in \mathbb{R}_+^N \times X$$

and choose $\|\cdot\|_n$ such that Φ is a L_n- contraction on Fréchet space $(X, \|\cdot\|_n)$ as below.

Let $n \in \mathbb{N}$ be fixed. Consider every $x \in [0,n]^N$. Assume $|x|_1 \leq \gamma_n$, with $\gamma_n \in (0,n)$ chosen later. It follows from (A_1), (A_2) and $\sigma_i(x) \in B_x$, $\forall x \in [0,n]^N$, that

$$\begin{aligned}
&|\Phi u(x) - \Phi v(x)| \\
&\leq L\,|u(x) - v(x)| + \omega_0(x) \sum_{i=1}^{p} \int_{B_x} \omega_1(x,y)\,|u(\sigma_i(y)) - v(\sigma_i(y))|\,dy \\
&\leq \left(L + p\bar{\omega}_n \frac{\gamma_n^N}{N^N}\right) |u - v|_{\gamma_n}, \quad \forall\, u, v \in X,
\end{aligned}$$

where

$$\bar{\omega}_n = \sup_{x\in[0,n]^N} \omega_0(x) \sup_{(x,y)\in\Delta_n} \omega_1(x,y), \quad \Delta_n = \{(x,y) \in [0,n]^{2N} : y \in B_x\}.$$

If $|x|_1 \geq \gamma_n$ then Φ has the following property

$$\begin{aligned}
&|\Phi u(x) - \Phi v(x)| \\
&\leq L\,|u(x) - v(x)| + \bar{\omega}_n \sum_{i=1}^{p} \int_{B_x,\ |\sigma_i(y)|_1 \leq \gamma_n} |u(\sigma_i(y)) - v(\sigma_i(y))|\,dy \\
&\quad + \bar{\omega}_n \sum_{i=1}^{p} \int_{B_x,\ |\sigma_i(y)|_1 \geq \gamma_n} |u(\sigma_i(y)) - v(\sigma_i(y))|\,dy,
\end{aligned}$$

leads to

$$\begin{aligned}
&|\Phi u(x) - \Phi v(x)|\, e^{-h_n(|x|_1 - \gamma_n)} \\
&\leq L\,|u - v|_{h_n} \\
&\quad + \bar{\omega}_n e^{-h_n(|x|_1 - \gamma_n)}\,|u - v|_{\gamma_n} \sum_{i=1}^{p} \int_{B_x,\ |\sigma_i(y)|_1 \leq \gamma_n} dy \\
&\quad + \bar{\omega}_n\,|u - v|_{h_n} \sum_{i=1}^{p} \int_{B_x,\ |\sigma_i(y)|_1 \geq \gamma_n} e^{h_n(|\sigma_i(y)|_1 - |x|_1)} dy \\
&\leq L\,|u - v|_{h_n} + \bar{\omega}_n e^{-h_n(|x|_1 - \gamma_n)}\,|u - v|_{\gamma_n}\,\varphi(\gamma_n) \\
&\quad + \bar{\omega}_n\,|u - v|_{h_n} \sum_{i=1}^{p} \frac{1}{h_n^N},
\end{aligned}$$

where $h_n > 0$ is also chosen later and $\varphi(\gamma_n) = \sum_{i=1}^{p} \int_{B_x,\ |\sigma_i(y)|_1 \leq \gamma_n} dy$, so

$$|\Phi u - \Phi v|_{h_n} \leq \left(L + p\bar{\omega}_n \frac{1}{h_n^N} \right) |u - v|_{h_n} + \bar{\omega}_n \varphi(\gamma_n) \, |u - v|_{\gamma_n} .$$

Consequently

$$\|\Phi u - \Phi v\|_n = |\Phi u - \Phi v|_{\gamma_n} + |\Phi u - \Phi v|_{h_n} \leq L_n \|u - v\|_n ,$$

with

$$L_n = \max \left\{ L + \bar{\omega}_n \left(p \frac{\gamma_n^N}{N^N} + \varphi(\gamma_n) \right),\ L + p\bar{\omega}_n \frac{1}{h_n^N} \right\}.$$

Φ becomes a L_n-contraction on $(X, \|\cdot\|_n)$ if we can choose $\gamma_n \in (0, n), h_n > 0$ such that $L_n < 1$. Clearly, we need choose $h_n > \sqrt[N]{\frac{p\bar{\omega}_n}{1-L}}$.

By (A_5), $\lim_{\gamma_n \to 0_+} \left(p \frac{\gamma_n^N}{N^N} + \varphi(\gamma_n) \right) = 0$, so γ_n can be chosen small enough such that $0 < p \frac{\gamma_n^N}{N^N} + \varphi(\gamma_n) < \frac{1-L}{\bar{\omega}_n}$. Thus, Φ has one fixed point ξ.

Next, with the transformation $u = v + \xi$, Eq (1.1) is written in the form

$$v(x) = Uv(x) + Cv(x),\ \ x \in \mathbb{R}_+^N, \tag{3.1}$$

where

$$\begin{cases} Uv(x) = -\xi(x) + V\bigg(x, v(x) + \xi(x), \\ \qquad\qquad \int_{B_x} V_1 \left(x, y, (v + \xi)\left(\sigma_1(y)\right), ..., (v + \xi)\left(\sigma_p(y)\right)\right) dy \bigg), \\ Cv(x) = \int_{\mathbb{R}_+^N} F\left(x, y; (v + \xi)\left(\chi_1(y)\right), ..., (v + \xi)\left(\chi_q(y)\right)\ \right) dy, \ \ x \in \mathbb{R}_+^N. \end{cases}$$

It is similar to Φ, we can show that U is a L_n-contraction, with respect to $\|\cdot\|_n$.

Now we prove that C is completely continuous on X. By $(A_3) - (A_5)$, using the dominated convergence theorem and Lemma 2.1, this proof as follows.

(i) For any $v_0 \in X$, let $\{v_m\}$ be a sequence in X such that $\lim_{m\to\infty} v_m = v_0$. Let $n \in \mathbb{N}$ be fixed. For any given $\varepsilon > 0$, by $\int_{\mathbb{R}_+^N} \sup_{x \in [0,n]^N} \omega_2(x, y) dy < \infty$, there exists $T_n \in \mathbb{N}$ (T_n is big enough) such that

$$\int_{\mathbb{R}_+^N \setminus \bar{B}_n} \omega_2(x, y) dy \leq \int_{\mathbb{R}_+^N \setminus \bar{B}_n} \sup_{x \in [0,n]^N} \omega_2(x, y) dy < \frac{\varepsilon}{4},\ \ \forall x \in [0, n]^N. \tag{3.2}$$

where $\bar{B}_n = \{y \in \mathbb{R}_+^N : y_1^2 + y_2^2 + ... + y_N^2 \leq T_n^2\}$.

Put $K_1 = \{(v_m + \xi)\left(\chi_1(y)\right) : y \in \bar{B}_n,\ m \in \mathbb{Z}_+\}$, then K_1 is compact in E. The same holds true for $K_2, ..., K_q$. For $\varepsilon > 0$ be given as above, by F is continuous on the compact set $[0, n]^N \times \bar{B}_n \times K_1 \times ... \times K_q$, there exists

$\delta > 0$ such that for every $(u_1, ..., u_q)$, $(\bar{u}_1, ..., \bar{u}_q) \in K_1 \times ... \times K_q$, $|u_i - v_i| < \delta$, $i = 1, ..., q$,

$$|F(x, y; u_1, ..., u_q) - F(x, y; \bar{u}_1, ..., \bar{u}_q)| < \frac{\varepsilon}{2mes(\bar{B}_n)},$$

for all $(x, y) \in [0, n]^N \times \bar{B}_n$. With $i = 1, ..., q$, by

$$\lim_{m\to\infty} \sup_{y\in\bar{B}_n} |(v_m + \xi)(\chi_i(y)) - (v_0 + \xi)(\chi_i(y))| = 0,$$

there exists m_0 such that for $m > m_0$,

$$|(v_m + \xi)(\chi_i(s)) - (v_0 + \xi)(\chi_i(s))| < \delta,$$

for all $y \in \bar{B}_n$, $i = 1, ..., q$. This implies that for all $x \in [0, n]^N$, for all $m > m_0$,

$$\begin{aligned} |Cv_m(x) - Cv_0(x)| \;\leq\; & \int_{\bar{B}_n} |F\left(x, y; (v_m + \xi)(\chi_1(y)), ..., (v_m + \xi)\left(\chi_q(y)\right)\right) \\ & - F\left(x, y; (v_0 + \xi)(\chi_1(y)), ..., (v_0 + \xi)\left(\chi_q(y)\right)\right)| \, dy \\ & + 2\int_{\mathbb{R}_+^N \setminus \bar{B}_n} \omega_2(x, y) dy \\ < \; & mes(\bar{B}_n) \times \frac{\varepsilon}{2mes(\bar{B}_n)} + 2\frac{\varepsilon}{4} = \varepsilon, \end{aligned}$$

so $|Cv_m - Cv_0|_n < \varepsilon$, for all $m > m_0$, and the continuity of C is proved.
(ii) It remains to show that C maps a bounded set Ω of X into relatively compact set. Let $n \in \mathbb{N}$ be fixed. Consider any $\varepsilon > 0$ given. Then, there exists $T_n \in \mathbb{N}$ such that (3.2) is valid.

(a) For any $v \in \Omega$, for all x, $\bar{x} \in [0, n]^N$,

$$\begin{aligned} & |Cv(x) - Cv(\bar{x})| \\ \leq\; & \int_{\bar{B}_n} |F\left(x, y; (v + \xi)(\chi_1(y)), ..., (v + \xi)\left(\chi_q(y)\right)\right) \\ & - F\left(\bar{x}, y; (v + \xi)(\chi_1(y)), ..., (v + \xi)\left(\chi_q(y)\right)\right)| \, dy \\ & + \int_{\mathbb{R}_+^N \setminus \bar{B}_n} \left(\omega_2(x, y) + \omega_2(\bar{x}, y)\right) dy. \end{aligned} \tag{3.3}$$

According to (3.2), (3.3) and the hypothesis (A_4), $(C\Omega)_n$ is equicontinuous on X_n.

(b) For every $x \in [0, n]^N$, consider the set $(C\Omega)_n(x) = \{Cv|_{[0,n]^N}(x) : v \in \Omega\}$ and let $\{Cv_k|_{[0,n]^N}(x)\}_k$, $v_k \in \Omega$, be a sequence in $(C\Omega)_n(x)$. We need show that there exists a convergent subsequence of $\{Cv_k|_{[0,n]^N}(x)\}_k$.

Put $S_i = \{(y + \xi)(\chi_i(y)) : y \in \Omega,\ y \in \bar{B}_n\}$, $i = 1, ..., q$. Then $S_1, ..., S_q$ are bounded in E and consequently the set $F([0, n]^N \times \bar{B}_n \times S_1 \times ... \times S_q)$ is relatively compact in E, since F is completely continuous. The sequence

$\{F(x,y;(v_k+\xi)(\chi_1(y)),...,(v_k+\xi)(\chi_q(y)))\}_k$ belongs to $F([0,n]^N\times\bar{B}_n\times S_1\times ...\times S_q)$, so there exists a subsequence

$$\left\{F\left(x,y;(v_{k_j}+\xi)(\chi_1(y)),...,(v_{k_j}+\xi)(\chi_q(y))\ \right)\right\}_j$$

and $\Psi(x,y)\in E$, such that

$$\left|F\left(x,y;(v_{k_j}+\xi)(\chi_1(y)),...,(v_{k_j}+\xi)(\chi_q(y))\ \right)-\Psi(x,y)\right|\to 0 \tag{3.4}$$

as $j\to\infty$. On the other hand, by (A_4),

$$\left|F\left(x,y;(v_{k_j}+\xi)(\chi_1(y)),...,(v_{k_j}+\xi)(\chi_q(y))\ \right)\right|\le\omega_2(x,y),$$

for all $(x,y)\in[0,n]^N\times\bar{B}_n$. Hence

$$\left|F\left(x,y;(v_{k_j}+\xi)(\chi_1(y)),...,(v_{k_j}+\xi)(\chi_q(y))\ \right)-\Psi(x,y)\right|\le 2\omega_2(x,y), \tag{3.5}$$

for all $(x,y)\in[0,n]^N\times\bar{B}_n,\ \omega_2(x,\cdot)\in L^1\left(\bar{B}_n\right)$. Using the dominated convergence theorem, (3.4) and (3.5) yield

$$\int_{\bar{B}_n}\left|F\left(x,y;(v_{k_j}+\xi)(\chi_1(y)),...,(v_{k_j}+\xi)(\chi_q(y))\ \right)-\Psi(x,y)\right|dy\to 0$$

as $j\to\infty$. It means that, for given $\varepsilon>0$, there exists j_0 such that for $j>j_0$,

$$\int_{\bar{B}_n}\left|F\left(x,y;(v_{k_j}+\xi)(\chi_1(y)),...,(v_{k_j}+\xi)(\chi_q(y))\ \right)-\Psi(x,y)\right|dy<\frac{\varepsilon}{2}.$$

Consequently, for $j>j_0$,

$$\begin{aligned}&\left|Cv_{k_j}(x)-\int_{\bar{B}_n}\Psi(x,y)dy\right|\\ \le\ &\int_{\bar{B}_n}\left|F\left(x,y;(v_{k_j}+\xi)(\chi_1(y)),...,(v_{k_j}+\xi)(\chi_q(y))\ \right)-\Psi(x,y)\right|dy\\ &+\int_{\mathbb{R}_+^N\setminus\bar{B}_n}\left|F\left(x,y;(v_{k_j}+\xi)(\chi_1(y)),...,(v_{k_j}+\xi)(\chi_q(y))\ \right)\right|dy\\ \le\ &\frac{\varepsilon}{2}+\int_{\mathbb{R}_+^N\setminus\bar{B}_n}\omega_2(x,y)dy<\frac{\varepsilon}{2}+\frac{\varepsilon}{4}<\varepsilon,\end{aligned}$$

$\left\{Cv_{k_j}(x)\right\}_j$ is a convergent subsequence of $\{Cv_k(x)\}_k$, then $(C\Omega)_n(x)$ is relatively compact in E. In view of Lemma 2.1, $C(\Omega)$ is relatively compact in X. Therefore, C is completely continuous.

On the other hand, it follows from (A_4) that

$$|Cv(x)|\le\int_{\mathbb{R}_+^N}\sup_{x\in[0,n]^N}\omega_2(x,y)dy<\infty,\ \ \forall\, x\in[0,n]^N.$$

The result is $\lim\limits_{|v|_n\to\infty}\frac{|Cv|_n}{|v|_n}=0$. Applying Theorem 1.1, $U+C$ has a fixed point v in X. Hence, (1.1) has a solution $u=v+\xi$ on $\mathbb{R}_+^N$.

Finally, we show that every solution u of (1.1) is an asymptotically stable solution. Note that for all $x \in \mathbb{R}_+^N$,

$$\xi(x) = V\left(x, \xi(x), \int_{B_x} V_1\left(x, y, \xi(\sigma_1(y)), ..., \xi(\sigma_p(y))\right) dy\right).$$

So, with $v = u - \xi$, we obtain

$$|v(x)| \leq L\,|v(x)| + \omega_0(x) \sum_{i=1}^{p} \int_{B_x} \omega_1(x,y)|v(\sigma_i(y))|dy + \int_{\mathbb{R}_+^N} \omega_2(x,y)dy.$$

It follows that

$$|v(x)| \leq \sum_{i=1}^{p} \int_{B_x} r(x,y)|v(\sigma_i(y))|dy + a(x),$$

where

$$\begin{aligned} a(x) &= \frac{1}{1-L} \int_{\mathbb{R}_+^N} \omega_2(x,y)dy, \\ r(x,y) &= \frac{1}{1-L}\omega_0(x)\omega_1(x,y) \leq r(x,0) \leq r(0,0). \end{aligned}$$

It implies the next property of $|v(x)|$, the proof will be presented in Remark 3.2 below,

$$|v(x)| \leq \bar{a}(x) + \bar{R}(x) \exp\left(\bar{R}(0)x_1x_2...x_N\right) \int_{B_x} \bar{a}(y)dy. \tag{3.6}$$

Obviously, if the following condition holds

$$\lim_{|x|_1 \to +\infty} \left[\bar{a}(x) + \bar{R}(x) \exp\left(\bar{R}(0)x_1x_2...x_N\right) \int_{B_x} \bar{a}(y)dy\right] = 0, \tag{3.7}$$

then

$$\lim_{|x|_1 \to +\infty} |v(x)| = \lim_{|x|_1 \to +\infty} |u(x) - \xi(x)| = 0.$$

So, for any solution $\tilde{u}$ of (1.1), $\lim\limits_{|x|_1 \to +\infty} |u(x) - \tilde{u}(x)| = 0$. Theorem 3.1 is proved. □

Remark 3.1. Assumption (A_5) is reasonable. Can choose the following two examples.

Example 3.1. Consider $\sigma_i(x) = x$, then σ_i satisfies (A_5). Indeed,

$$\int_{y\in[0,n]^N,\ |y|_1\le\eta} dy \le \int_{\mathbb{R}_+^N,\ |y|_1\le\eta} dy$$
$$= \underset{\mathbb{R}_+^N,\ y_1+...+y_N\le\eta}{\int ... \int} dy_1...dy_N = \frac{\eta^N}{N!} \to 0,$$

as $\eta \to 0_+$.

Example 3.2. In the case of $\sigma_i(y) = by$, $0 < b < 1$, (A_5) also holds. Indeed, we have

$$\int_{y\in[0,n]^N,\ |by|_1\le\eta} dy = \int_{y\in[0,n]^N,\ |y|_1\le\frac{\eta}{b}} dy \le \frac{\left(\frac{\eta}{b}\right)^N}{N!} = \frac{\eta^N}{N!b^N} \to 0,$$

as $\eta \to 0_+$. So is the condition (3.7). We give an example in which ω_0, ω_1, ω_2, σ_i satisfying (3.7).

Example 3.3.

$$\begin{cases} \omega_1(x,y) = \frac{\sqrt{(1-L)\alpha_1}}{\sqrt{1+\beta_1\exp\left(\gamma_1|x|_1^N\right)+\beta_2|y|_1^{\lambda_1}}},\ \ \omega_0(x) = \frac{\sqrt{(1-L)\alpha_1}}{\sqrt{1+\beta_1\exp\left(\gamma_1|x|_1^N\right)}}, \\ \omega_2(x,y) = \frac{\exp\left(-\gamma_2|x|_1\right)}{1+|y|_2^{\lambda_2}},\ \ |y|_2 = \sqrt{y_1^2+...+y_N^2};\ \ \sigma_i(x) = \bar{\sigma}_i x,\ \ 1\le i\le p, \end{cases}$$

where $\alpha_1,\ \beta_1,\ \beta_2,\ \gamma_1,\ \gamma_2,\ \lambda_1,\ \lambda_2,\ \bar{\sigma}_i$ $(1 \le i \le p)$ are positive constants with $\lambda_1 > N$, $\lambda_2 > N$, $0 < \bar{\sigma}_i \le 1$, $\sigma_{\min} = \min\limits_{1\le i\le p} \bar{\sigma}_i$, $\ \gamma_1 > \frac{(p+1)\alpha_1}{N^N(1+\beta_1)\sigma_{\min}}$.

Calculating the functions $r(x,y)$, $R(x)$, $a(x)$:

$$\begin{cases} r(x,y) = \frac{1}{1-L}\omega_0(x)\omega_1(x,y) = \frac{\alpha_1}{\sqrt{1+\beta_1\exp\left(\gamma_1|x|_1^N\right)+\beta_2|y|_1^{N+1}}\sqrt{1+\beta_1\exp\left(\gamma_1|x|_1^N\right)}}, \\ r(x,y) \le r(x,0) = \frac{\alpha_1}{1+\beta_1\exp\left(\gamma_1|x|_1^N\right)} \\ \qquad\qquad\qquad \equiv R(x) \le r(0,0) = \frac{\alpha_1}{1+\beta_1}, \\ a(x) = \frac{1}{1-L}\int_{\mathbb{R}_+^N}\omega_2(x,y)dy = \frac{\exp\left(-\gamma_2|x|_1\right)}{1-L}\int_{\mathbb{R}_+^N}\frac{dy}{1+|y|_2^{\lambda_2}} \\ \qquad = \frac{\exp\left(-\gamma_2|x|_1\right)}{1-L}\omega_N\int_0^\infty\frac{r^{N-1}dy}{1+r^{\lambda_2}} \\ \qquad \equiv \alpha_2\exp\left(-\gamma_2\left|x\right|_1\right), \\ \alpha_2 = \frac{\omega_N}{1-L}\int_0^\infty\frac{r^{N-1}dy}{1+r^{\lambda_2}}, \text{ where } \omega_N \text{ is the area of unit sphere in } \mathbb{R}^N. \end{cases}$$

And calculating the functions $\bar{a}(x)$, $\int_{B_x} \bar{a}(y)dy$, $\bar{R}(x)$:

$$\begin{aligned} \bar{a}(x) &= a(x) + \sum\nolimits_{i=1}^{p} a(\sigma_i(x)) \\ &= \alpha_2 \left[\exp\left(-\gamma_2 \left|x\right|_1\right) + \sum_{i=1}^{p} \exp\left(-\gamma_2 \left|\bar{\sigma}_i x\right|_1\right) \right] \\ &\leq (p+1)\alpha_2 \exp\left(-\sigma_{\min}\gamma_2 \left|x\right|_1\right) \to 0, \text{ as } \left|x\right|_1 \to +\infty, \end{aligned}$$

$$\begin{aligned} \int_{B_x} \bar{a}(y)dy &\leq \frac{(p+1)\alpha_2}{(\sigma_{\min}\gamma_2)^N}(1 - e^{-\sigma_{\min}\gamma_2 x_1})...(1 - e^{-\sigma_{\min}\gamma_2 x_N}) \\ &\leq \frac{(p+1)\alpha_2}{(\sigma_{\min}\gamma_2)^N} \text{ for all } x \in \mathbb{R}_+^N, \end{aligned}$$

$$\begin{aligned} \bar{R}(x) &= R(x) + \sum_{i=1}^{p} R(\sigma_i(x)) \\ &= \frac{\alpha_1}{1 + \beta_1 \exp\left(\gamma_1 \left|x\right|_1^N\right)} + \sum_{i=1}^{p} \frac{\alpha_1}{1 + \beta_1 \exp\left(\gamma_1 \left|\bar{\sigma}_i x\right|_1^N\right)} \\ &\leq \frac{(p+1)\alpha_1}{1 + \beta_1 \exp\left(\sigma_{\min}\gamma_1 \left|x\right|_1^N\right)} \leq \frac{(p+1)\alpha_1}{\beta_1} \exp\left(-\sigma_{\min}\gamma_1 \left|x\right|_1^N\right), \\ \bar{R}(0) &= \frac{(p+1)\alpha_1}{1+\beta_1}. \end{aligned}$$

Since $\gamma_1 > \frac{(p+1)\alpha_1}{N^N(1+\beta_1)\sigma_{\min}}$, it follows that

$$\begin{aligned} &\bar{R}(x) \exp\left(\bar{R}(0) x_1 x_2 ... x_N\right) \\ &\leq \frac{(p+1)\alpha_1}{\beta_1} \exp\left(-\sigma_{\min}\gamma_1 \left|x\right|_1^N\right) \exp\left(\frac{(p+1)\alpha_1}{1+\beta_1} \frac{\left|x\right|_1^N}{N^N}\right) \\ &= \frac{(p+1)\alpha_1}{\beta_1} \exp\left(-\left[\sigma_{\min}\gamma_1 - \frac{(p+1)\alpha_1}{N^N(1+\beta_1)}\right] \left|x\right|_1^N\right) \to 0, \end{aligned}$$

as $\left|x\right|_1 \to +\infty$. Then (3.7) holds.

Remark 3.2. The inequality (3.6) is true. Indeed, put

$$\begin{aligned} w(x) &= |v(x)|, \quad Aw(x) = \int_{B_x} r(x,y)w(y)dy, \\ R(x) &= r(x,0), \;\; \bar{R}(x) = R(x) + \sum_{i=1}^{p} R(\sigma_i(x)), \;\; x \in \mathbb{R}_+^N. \end{aligned}$$

(i) Assume that $p = 1$, $\sigma_1(y) = y$, then

$$Aw(x) = \int_{B_x} r(x,y)w(y)dy \le R(x)\int_{B_x} w(y)dy, \quad \forall\, w \in C(\mathbb{R}_+^N;\mathbb{R}_+).$$

It implies that

$$\begin{aligned} w(x) &\le a(x) + Aw(x) \le a(x) + A\left(a + Aw\right)(x) \\ &= a(x) + Aa(x) + A^2 w(x) \le ... \\ &\le a(x) + \sum_{k=0}^{n-1} A^{k+1}a(x) + A^{n+1}w(x). \end{aligned}$$

By induction, the result is

$$A^{k+1}w(x) \le R(x)\frac{\left(R(0)x_1x_2...x_N\right)^k}{\left(k!\right)^N}\int_{B_x} w(y)dy.$$

So

$$\begin{aligned} w(x) \quad \le \quad & a(x) + R(x)\sum\nolimits_{k=0}^{n-1}\frac{\left(R(0)x_1x_2...x_N\right)^k}{\left(k!\right)^N}\int_{B_x} a(y)dy \\ & +R(x)\frac{\left(R(0)x_1x_2...x_N\right)^n}{\left(n!\right)^N}\int_{B_x} w(y)dy. \end{aligned} \tag{3.8}$$

For $X_0 > 0$ is given, it leads to

$$\left|\frac{\left(R(0)x_1x_2...x_N\right)^k}{\left(k!\right)^N}\right| \le \frac{\left(R(0)X_0^N\right)^k}{\left(k!\right)^N}, \quad \forall\, x \in [0,X_0]^N, \;\; \forall\, k \in \mathbb{N}.$$

The positive series $\sum_{k=0}^{\infty}\frac{\left(R(0)X_0^N\right)^k}{\left(k!\right)^N}$ converges (via a standard of D'Alembert) and then $\sum_{k=0}^{\infty}\frac{\left(R(0)x_1x_2...x_N\right)^k}{\left(k!\right)^N}$ converges uniformly on $[0,X_0]^N$ (via a standard of Weierstrass). By the continuity of the function $x \longmapsto \frac{\left(R(0)x_1x_2...x_N\right)^k}{\left(k!\right)^N}$ on $[0,X_0]^N$, the sum of the series $\sum_{k=0}^{\infty}\frac{\left(R(0)x_1x_2...x_N\right)^k}{\left(k!\right)^N}$ is continuous on $[0,X_0]^N$. On the other hand, $X_0 > 0$ is arbitrary, so the sum of this series is continuous on ${}^N_+$.

Note that $\frac{\left(R(0)x_1x_2...x_N\right)^n}{\left(n!\right)^N} \to 0$ as $n \to \infty$, for all $x \in \mathbb{R}_+^N$, it implies from (3.8) that

$$w(x) \le a(x) + R(x)\sum_{k=0}^{\infty}\frac{\left(R(0)x_1x_2...x_N\right)^k}{\left(k!\right)^N}\int_{B_x} a(y)dy, \quad \forall\, x \in \mathbb{R}_+^N.$$

(ii) Let $p \geq 2$, note that $B_{\sigma_i(x)} \subset B_x$, for all $x \in \mathbb{R}_+^N$, so we have

$$w(\sigma_i(x)) \leq a(\sigma_i(x)) + R(\sigma_i(x)) \sum_{j=1}^{p} \int_{B_x} w(\sigma_j(y))dy.$$

Hence

$$\sum_{i=1}^{p} w(\sigma_i(x)) \leq \sum_{i=1}^{p} a(\sigma_i(x)) + \sum_{i=1}^{p} R(\sigma_i(x)) \sum_{j=1}^{p} \int_{B_x} w(\sigma_j(y))dy.$$

Put $\bar{w}(x) = w(x) + \sum_{i=1}^{p} w(\sigma_i(x))$, consequently

$$\bar{w}(x) \leq \bar{a}(x) + \bar{R}(x) \int_{B_x} \bar{w}(y)dy.$$

By

$$0 \leq \frac{\left(\bar{R}(0)x_1x_2...x_N\right)^k}{(k!)^N} \leq \frac{\left(\bar{R}(0)x_1x_2...x_N\right)^k}{k!}, \quad \forall x \in \mathbb{R}_+^N.$$

Consequently,

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{\left(\bar{R}(0)x_1x_2...x_N\right)^k}{(k!)^N} &\leq \sum_{k=0}^{\infty} \frac{\left(\bar{R}(0)x_1x_2...x_N\right)^k}{k!} \\ &= \exp\left(\bar{R}(0)x_1x_2...x_N\right), \quad \forall x \in \mathbb{R}_+^N. \end{aligned}$$

Therefore

$$w(x) \leq \bar{w}(x) \leq \bar{a}(x) + \bar{R}(x) \exp\left(\bar{R}(0)x_1x_2...x_N\right) \int_{B_x} \bar{a}(y)dy,$$

for all $x \in \mathbb{R}_+^N$. Then (3.6) holds.

4. Compactness of the set of solutions

Theorem 4.1. *Let $(A_1) - (A_5)$ hold. Then the set of solutions of the problem (1.1) is nonempty and compact.*

Proof. Put

$$\begin{aligned} \Phi u(x) &= V\left(x, u(x), \int_{B_x} V_1\left(x, y, u(\sigma_1(y)), ..., u(\sigma_p(y))\right) dy\right), \qquad (4.1) \\ \bar{C}u(x) &= \int_{\mathbb{R}_+^N} F\left(x, y, u(\chi_1(y)), ..., u(\chi_q(y))\right) dy, \quad (x, u) \in \mathbb{R}_+^N \times X. \end{aligned}$$

It is similar to C, we can show that $\bar{C} : X \to X$ is completely continuous such that $\lim_{|u|_n \to \infty} \frac{|\bar{C}u|_n}{|u|_n} = 0$, $\forall n \in \mathbb{N}$. Then $\Phi + \bar{C}$ has a fixed point, it implies that $Q = \{u \in X : u = (I - \Phi)^{-1}\bar{C}u\} \neq \phi$. We shall show that Q is compact.

First, Q bounded in X. Indeed, by Assumption (A_4), for all $n \in \mathbb{N}$, for all $(x,u) \in [0,n]^N \times X$, we get

$$\begin{aligned} \left|\bar{C}u(x)\right| &\leq \int_{\mathbb{R}_+^N} \left|F\left(x,y,u(\chi_1(y)),....,u(\chi_q(y))\right)\right| dy \\ &\leq \int_{\mathbb{R}_+^N} \omega_2(x,y)dy \leq \int_{\mathbb{R}_+^N} \sup_{x\in[0,n]^N} \omega_2(x,y)dy \\ &\equiv D_n < \infty. \end{aligned} \tag{4.2}$$

Hence

$$\left|\bar{C}u\right|_n \leq D_n. \tag{4.3}$$

Then, for all $u \in Q$, we have

$$\begin{aligned} \|u\|_n &= \left\|\Phi u + \bar{C}u\right\|_n \leq \|\Phi u - \Phi 0\|_n + \|\Phi 0\|_n + \left\|\bar{C}u\right\|_n \\ &\leq L_n \|u\|_n + \|\Phi 0\|_n + \left\|\bar{C}u\right\|_n . \end{aligned}$$

Thus

$$\begin{aligned} \|u\|_n &\leq \frac{\|\Phi 0\|_n + \left\|\bar{C}u\right\|_n}{1-L_n} \leq \frac{\|\Phi 0\|_n + 2\left|\bar{C}u\right|_n}{1-L_n} \\ &\leq \frac{\|\Phi 0\|_n + 2D_n}{1-L_n}, \quad \forall\, u \in Q. \end{aligned} \tag{4.4}$$

Next, from the compactness of the operator $(I-\Phi)^{-1}\bar{C} : X \to X$, it follows from (4.4) that $Q = (I-\Phi)^{-1}\bar{C}(Q)$ is relatively compact. It remains to prove that Q is closed. Let $\{u_m\}_m \subset Q$ be a sequence and $u_m \to u_0$ in X.

For all $n \in \mathbb{N}$, by the continuity of the operators Φ, $\bar{C} : X \to X$, we have

$$\begin{aligned} \left|\Phi u_0 + \bar{C}u_0 - u_0\right|_n &\leq \left|u_m - u_0 - \Phi u_m + \Phi u_0 - \bar{C}u_m + \bar{C}u_0\right|_n \\ &\leq \left|u_m - u_0\right|_n + \left|\Phi u_m - \Phi u_0\right|_n + \left|\bar{C}u_m - \bar{C}u_0\right|_n \\ &\to 0. \end{aligned}$$

So

$$u_0 = \Phi u_0 + \bar{C}u_0,$$

which implies that $u_0 \in Q$. Therefore, Q is closed. The proof of Theorem 4.1 is complete. □

5. An example

Let us illustrate the results obtained by means of an example.

Let $E = C([0,1];\mathbb{R})$ be the Banach space of all continuous functions $v : [0,1] \to \mathbb{R}$ with the norm

$$\|v\| = \sup_{0\leq t\leq 1} |v(t)|, \quad v \in E.$$

Then, for all $u \in X = C(\mathbb{R}_+^2; E)$, for any $x \in \mathbb{R}_+^2$, $u(x)$ is an element of E and we denote

$$u(x)(t) = u(x,t), \quad 0 \le t \le 1.$$

Consider (1.1) in form

$$\begin{aligned} u(x) &= V\left(x, u(x), \int_{B_x} V_1\left(x, y, u(\sigma_1(y)), ..., u(\sigma_p(y))\right) dy\right) \quad (5.1)\\ &\quad + \int_{\mathbb{R}_+^2} F\left(x, y, u(\chi_1(y)), ..., u(\chi_q(y))\right) dy, \quad x \in \mathbb{R}_+^2, \end{aligned}$$

where $\sigma_i(x) = \bar{\sigma}_i x$, $0 < \bar{\sigma}_i \le 1$, $i = 1, ..., p$; $\chi_i(x) = \bar{\chi}_i x$, $0 < \bar{\chi}_i \le 1$, $i = 1, ..., q$; $B_x = [0, x_1] \times [0, x_2]$. Giving the continuous functions V, V_1, F as follows.
(i) Function $V : \mathbb{R}_+^2 \times E^2 \to E$,

$$V(x,u,v)(t) = 2(1-k_1)u_*(x,t) + k_1 \left|u(t)\right| + e^{-\gamma |x|_1^2} \left|v(t)\right|,$$

$0 \le t \le 1$, $(x,u,v) \in \mathbb{R}_+^2 \times E^2$ with $u_*(x,t) = \frac{1}{t + e^{|x|_1}}$ and γ, k_1 are given constants such that $0 < k_1 < 1$, $\gamma > \frac{(1+p)\pi}{2(1-k_1)\theta} > 0$, $\theta = \min_{1 \le i \le p} \bar{\sigma}_i^2$.
(ii) Function $V_1 : \Delta \times E^p \to E$,

$$V_1\left(x, y; u_1, ..., u_p\right)(t) = e^{-2|y|_1} u_*(x,t) \sum_{i=1}^{p} \sin\left(\pi \frac{u_i(t)}{u_*(\sigma_i(y), t)}\right),$$

$0 \le t \le 1$, $(x, y; u_1, ..., u_p) \in \Delta \times E^p$, $\Delta = \{(x,y) \in \mathbb{R}_+^2 \times \mathbb{R}_+^2 : y \in B_x\}$.
(iii) Function $F : \mathbb{R}_+^2 \times \mathbb{R}_+^2 \times E^q \to E$,

$$\begin{aligned} &F\left(x, y; u_1, ..., u_q\right)(t) \\ &= \frac{4}{q}\left(k_1 - 1\right) e^{-2|y|_1} u_*(x,t) \sum_{i=1}^{q} \sin\left(\frac{\pi}{2} \int_0^1 \frac{u_i(s)}{u_*(\chi_i(y), s)} ds\right), \end{aligned}$$

$0 \le t \le 1$, $(x, y; u_1, ..., u_q) \in \mathbb{R}_+^2 \times \mathbb{R}_+^2 \times E^q$.

We can prove that $(A_1) - (A_5)$ hold. It is easy to see that (A_5) holds, see Remark 3.1.

(a) Assumption (A_1) holds, by for all (x,u,v), $(x, \bar{u}, \bar{v}) \in \mathbb{R}_+^2 \times E^2$, $\forall t \in [0,1]$,

$$\left\| V(x,u,v) - V(x,\bar{u},\bar{v}) \right\| \le k_1 \left\| u - \bar{u} \right\| + \omega_0(x) \left\| v - \bar{v} \right\|$$

with $\omega_0(x) = e^{-\gamma |x|_1^2}$.

(b) Assumption (A_2) holds, for all $(x, y; u_1, ..., u_p)$, $(x, y; \bar{u}_1, ..., \bar{u}_p) \in \Delta \times E^p$, $\Delta = \{(x, y) \in \mathbb{R}_+^2 \times \mathbb{R}_+^2 : y \in B_x\}$, $\forall t \in [0, 1]$,

$$\begin{aligned}
& |V_1(x, y; u_1, ..., u_p)(t) - V_1(x, y; \bar{u}_1, ..., \bar{u}_p)(t)| \\
\leq \;& e^{-2|y|_1} u_*(x, t) \sum_{i=1}^{p} \frac{\pi}{u_*(\sigma_i(y), t)} |u_i(t) - \bar{u}_i(t)| \\
\leq \;& \pi e^{-2|y|_1} \frac{1}{t + e^{|x|_1}} \sum_{i=1}^{p} (t + e^{|\sigma_i(y)|_1}) \|u_i - \bar{u}_i\| \\
= \;& \pi e^{-|y|_1} \frac{1}{t + e^{|x|_1}} \sum_{i=1}^{p} (t e^{-|y|_1} + e^{-|y|_1 + |\sigma_i(y)|_1}) \|u_i - \bar{u}_i\| \\
\leq \;& 2\pi e^{-|x|_1 - |y|_1} \sum_{i=1}^{p} \|u_i - \bar{u}_i\| \\
= \;& \omega_1(x, y) \sum\nolimits_{i=1}^{p} \|u_i - \bar{u}_i\|,
\end{aligned}$$

in which

$$\omega_1(x, y) = 2\pi e^{-|x|_1 - |y|_1}.$$

(c) Assumption (A_3) is also fulfilled.

First, we can show $F : \mathbb{R}_+^2 \times \mathbb{R}_+^2 \times E^q \to E$ is continuous.

Next, we show $F : \mathbb{R}_+^2 \times \mathbb{R}_+^2 \times E^q \to E$ is compact. Let B is bounded in $\mathbb{R}_+^2 \times \mathbb{R}_+^2 \times E^q$, we deduce from

$$\begin{aligned}
\|F(x, y; u_1, ..., u_q)\| &\leq \omega_2(x, y) = 4(1 - k_1) e^{-|x|_1 - 2|y|_1} \\
&\leq 4(1 - k_1) \equiv M, \quad \forall (x, y; u_1, ..., u_q) \in B,
\end{aligned}$$

that $F(B)$ is uniformly bounded in E. For all $t_1, t_2 \in [0, 1]$, $(x, y; u_1, ..., u_q) \in B$,

$$\begin{aligned}
& F(x, y; u_1, ..., u_q)(t_1) - F(x, y; u_1, ..., u_q)(t_2) \\
= \;& \frac{4}{q}(k_1 - 1) e^{-2|y|_1} \frac{t_2 - t_1}{\left(t_1 + e^{|x|_1}\right)\left(t_2 + e^{|x|_1}\right)} \sum_{i=1}^{q} \sin\left(\frac{\pi}{2} \int_0^1 \frac{u_i(s)}{u_*(\chi_i(y), s)} ds\right),
\end{aligned}$$

so

$$\begin{aligned}
& |F(x, y; u_1, ..., u_q)(t_1) - F(x, y; u_1, ..., u_q)(t_2)| \\
\leq \;& 4(1 - k_1) e^{-2|y|_1} \frac{|t_2 - t_1|}{\left(t_1 + e^{|x|_1}\right)\left(t_2 + e^{|x|_1}\right)} \\
\leq \;& 4(1 - k_1) |t_2 - t_1|,
\end{aligned}$$

it implies that $F(B)$ is equicontinuous.

Finally, for all bounded subsets I_1, I_2 of $\mathbb{R}_+^2$ and for any bounded subset J of E^q, for all $\varepsilon > 0$, there exists $\delta > 0$, such that

$$\forall\, x, \bar{x} \in I_1, \ \ |x - \bar{x}|_1 < \delta \implies \|F(x, y; u_1, ..., u_q) - F(\bar{x}, y; u_1, ..., u_q)\| < \varepsilon,$$

for all $(y; u_1, ..., u_q) \in I_2 \times J$.

We get the above property, since

$$\|F(x, y; u_1, ..., u_q) - F(\bar{x}, y; u_1, ..., u_q)\| \leq 4(1 - k_1)\,|x - \bar{x}|_1\,,$$

for all $x, \bar{x} \in I_1$, $(y; u_1, ..., u_q) \in I_2 \times J$. Indeed,

$$\begin{aligned}
&F(x, y; u_1, ..., u_q)(t) - F(\bar{x}, y; u_1, ..., u_q)(t) \\
= \ & \frac{4}{q}\,(k_1 - 1)\,e^{-2|y|_1}[u_*(x, t) - u_*(\bar{x}, t)] \sum_{i=1}^{q} \sin\left(\frac{\pi}{2}\int_0^1 \frac{u_i(s)}{u_*(\chi_i(y), s)} ds\right) \\
= \ & \frac{4}{q}\,(k_1 - 1)\,e^{-2|y|_1} \frac{e^{|\bar{x}|_1} - e^{|x|_1}}{(t + e^{|x|_1})(t + e^{|\bar{x}|_1})} \sum_{i=1}^{q} \sin\left(\frac{\pi}{2}\int_0^1 \frac{u_i(s)}{u_*(\chi_i(y), s)} ds\right),
\end{aligned}$$

so

$$\begin{aligned}
& |F(x, y; u_1, ..., u_q)(t) - F(\bar{x}, y; u_1, ..., u_q)(t)| \\
\leq \ & 4(1 - k_1)e^{-2|y|_1} \frac{\left|e^{|\bar{x}|_1} - e^{|x|_1}\right|}{(t + e^{|x|_1})(t + e^{|\bar{x}|_1})} \\
\leq \ & 4(1 - k_1)e^{-2|y|_1}\,\left||\bar{x}|_1 - |x|_1\right| \\
\leq \ & 4(1 - k_1)\,|\bar{x} - x|_1\,.
\end{aligned}$$

(d) Assumption (A_4) is also clearly, by the fact that, for all bounded subset $I \subset \mathbb{R}_+^2$, $\forall (x, y; u_1, ..., u_q) \in I \times \mathbb{R}_+^2 \times E^q$, $\forall t \in [0, 1]$,

$$\begin{aligned}
|F\,(x, y; u_1, ..., u_q)\,(t)| \ &\leq \ 4\,(1 - k_1)\,e^{-2|y|_1} u_*(x, t) \leq \frac{4(1 - k_1)e^{-2|y|_1}}{t + e^{|x|_1}} \\
&\leq \ 4(1 - k_1)e^{-|x|_1 - 2|y|_1} = \omega_2(x, y), \\
\int_{\mathbb{R}_+^2} \sup_{x \in I} \omega_2(x, y) dy \ &\leq \ 4(1 - k_1) \int_{\mathbb{R}_+^2} e^{-2|y|_1} dy = 1 - k_1 < \infty,
\end{aligned}$$

since $\int_{\mathbb{R}_+^2} e^{-2|y|_1} dy = \frac{1}{4}$. On the other hand, the condition (3.7) is true. Indeed,

$$\begin{aligned}
\omega_0(x) &= e^{-\gamma|x|_1^2}, \\
\omega_1(x, y) &= 2\pi e^{-|x|_1 - |y|_1}, \\
\omega_2(x, y) &= 4(1 - k_1)e^{-|x|_1 - 2|y|_1}.
\end{aligned}$$

(i) $\bar{a}(x) \to 0$ as $|x|_1 \to +\infty$:

$$\begin{aligned} a(x) &= \frac{1}{1-k_1}\int_{\mathbb{R}^2_+}\omega_2(x,y)dy \\ &= \frac{1}{1-k_1}4(1-k_1)e^{-|x|_1}\int_{\mathbb{R}^2_+}e^{-2|y|_1}dy = e^{-|x|_1}, \quad \forall x\in\mathbb{R}^2_+; \\ \bar{a}(x) &= a(x)+\sum_{i=1}^p a(\sigma_i(x)) = e^{-|x|_1}+\sum_{i=1}^p e^{-\bar{\sigma}_i|x|_1} \ \to \ 0. \end{aligned}$$

(ii) $\bar{R}(x)\exp\left(\bar{R}(0)x_1x_2\right)\int_{B_x}\bar{a}(y)dy \ \to \ 0$ as $|x|_1\to+\infty$:
(ii_1) $\int_{B_x}\bar{a}(y)dy$ is bounded:

$$\begin{aligned} \int_{B_x}\bar{a}(y)dy &= \int_{B_x}e^{-|y|_1}dy+\sum_{i=1}^p\int_{B_x}e^{-\bar{\sigma}_i|y|_1}dy \\ &= (1-e^{-x_1})(1-e^{-x_2})+\sum_{i=1}^p\frac{1}{\bar{\sigma}_i^2}(1-e^{-\bar{\sigma}_ix_1})(1-e^{-\bar{\sigma}_ix_2}) \\ &\le 1+\sum_{i=1}^p\frac{1}{\bar{\sigma}_i^2}, \end{aligned}$$

(ii_2) $\bar{R}(x)\exp\left(\bar{R}(0)x_1x_2\right)\to 0$ as $|x|_1\to+\infty$:

$$\begin{aligned} R(x) &= \frac{1}{1-L}\omega_0(x)\omega_1(x,0) = \frac{2\pi}{1-k_1}e^{-\gamma|x|_1^2-|x|_1}, \\ \bar{R}(x) &= R(x)+\sum_{i=1}^p R(\sigma_i(x)) \\ &= \frac{2\pi}{1-k_1}\left[e^{-\gamma|x|_1^2-|x|_1}+\sum_{i=1}^p e^{-\gamma\bar{\sigma}_i^2|x|_1^2-\bar{\sigma}_i|x|_1}\right] \\ &\le \frac{2(1+p)\pi}{1-k_1}e^{-\gamma\theta|x|_1^2}, \\ \theta &= \min_{1\le i\le p}\bar{\sigma}_i^2, \qquad \bar{R}(0) = \frac{2(1+p)\pi}{1-k_1}, \end{aligned}$$

$$\begin{aligned}
\bar{R}(x)\exp\left(\bar{R}(0)x_1x_2\right) &\le \frac{2(1+p)\pi}{1-k_1}e^{-\gamma\theta|x|_1^2}\exp\left(\bar{R}(0)\frac{1}{4}|x|_1^2\right)\\
&= \frac{2(1+p)\pi}{1-k_1}e^{-\gamma\theta|x|_1^2}\exp\left(\frac{(1+p)\pi}{2(1-k_1)}|x|_1^2\right)\\
&= \frac{2(1+p)\pi}{1-k_1}\exp\left[-\left(\gamma-\frac{(1+p)\pi}{2(1-k_1)\theta}\right)\theta|x|_1^2\right]\\
&\to 0, \quad \text{as } |x|_1 \to +\infty,
\end{aligned}$$

since $\gamma - \frac{(1+p)\pi}{2(1-k_1)\theta} > 0$. The result is $\bar{R}(x)\exp\left(\bar{R}(0)x_1x_2\right)\int_{B_x}\bar{a}(y)dy \to 0$ as $|x|_1 \to +\infty$, then (3.7) follows. Theorem 3.1 holds for (5.1). For more details, it is not difficult to show that the following equation

$$\xi(t) = V\left(x, \xi(x), \int_{B_x} V_1\left(x, y, \xi(\sigma_1(y)), ..., \xi(\sigma_p(y))\right)dy\right), \quad x \in \mathbb{R}_+^2$$

has a unique solution ξ defined by

$$\xi : \mathbb{R}_+^2 \to E, \quad \xi(x)(t) = \xi(x,t) = \frac{2}{t+e^{|x|_1}}, \quad \forall\, t \in [0,1], \tag{5.2}$$

and

$$u_* : \mathbb{R}_+^2 \to E, \quad u_*(x)(t) = u_*(x,t) = \frac{1}{t+e^{|x|_1}}, \quad \forall\, t \in [0,1], \tag{5.3}$$

is the solution of (5.1). Furthermore

$$\lim_{|x|_1\to\infty}\|u_*(x)-\xi(x)\| = \lim_{|x|_1\to\infty} e^{-|x|_1} = 0.$$

Consequently, ξ and x_* as in (5.2), (5.3) are asymptotically stable solutions of (5.1).

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References

[1] M.A. Abdou, A.A. Badr and M.M. El-Kojok, *On the solution of a mixed nonlinear integral equation*, Appl. Math. and Comput., **217**(12) (2011), 5466–5475.

[2] C. Avramescu and C. Vladimirescu, *An existence result of asymptotically stable solutions for an integral equation of mixed type*, Electronic J. Qualitative Theory of Diff. Equat., **25** (2005), 1–6.

[3] C. Corduneanu, *Integral equations and applications*, Cambridge University Press, New York, 1991.

[4] K. Deimling, *Nonlinear Functional Analysis*, Springer, New York, 1985.

[5] M.M. El-Borai, M.A. Abdou and M.M. El-Kojok, *On a discussion of nonlinear integral equation of type Volterra-Hammerstein*, J. Korea Soc. Math. Educ., Ser. B, Pure Appl. Math., **15**(1) (2008), 1–17.

[6] Z. Liu, S.M. Kang and J.S. Ume, *Solvability and asymptotic stability of a nonlinear functional-integral equation*, Appl. Math. Letters, **24**(6) (2011), 911–917.

[7] S. Lang, *Analysis II,* Addison-Wesley, Reading, Mass., California London, 1969.

[8] N. Lungu and I.A. Rus, *On a functional Volterra–Fredholm integral equation via Picard operator*, Jou. of Math. Inequal., **3**(4) (2009), 519–527.

[9] L.T.P. Ngoc and N.T. Long, *On a fixed point theorem of Krasnosel'skii type and application to integral equations*, Fixed Point Theory and Appl., **2006**(2006), Article ID 30847, 24 pages.

[10] L.T.P. Ngoc and N.T. Long, *Applying a fixed point theorem of Krasnosel'skii type to the existence of asymptotically stable solutions for a Volterra–Hammerstein integral equation,* Nonlinear Anal. TMA., **74**(11) (2011), 3769–3774.

[11] L.T.P. Ngoc and N.T. Long, *On a nonlinear Volterra–Hammerstein integral equation in two variables* Acta Math. Scientia, **33B**(2) (2013), 484–494.

[12] B.G. Pachpatte, *On Fredholm type integral equation in two variables*, Differential Equ. & App., **1**(1) (2009), 27–39.

[13] B.G. Pachpatte, *Volterra integral and integrodifferential equations in two variables*, J. Inequal. Pure and Appl. Math., **10**(4) (2009), Art. 108, 10 pp.

[14] I.K. Purnaras, *A note on the existence of solutions to some nonlinear functional integral equations*, Electronic J. Qualitative Theory of Diff. Equ., **17** (2006), 1–24.