Nonlinear Functional Analysis and Applications
Vol. 19, No. 4 (2014), pp. 479-487

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IMPRIMITIVITY FINSLER C^*-BIMODULES

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Abstract. In this paper, we introduce the notion of Finsler C^*-bimodule. We generalize some significant properties of Hilbert C^*-bimodules in the framework of Finsler C^*-bimodules and show that if E is an imprimitivity Finsler $\mathcal{A}$-$\mathcal{B}$-bimodule of C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ such that the corresponding maps ${}_{\mathcal{A}}\rho$ and $\rho_{\mathcal{B}}$ satisfy the parallelogram law on E, then E is a Hilbert $\mathcal{A}$-$\mathcal{B}$-bimodule.

1. Introduction

The notion of Finsler module is an interesting generalization of Hilbert C^*-module.
In 1995, Phillips and Weaver [8] introduced the notion of Finsler C^*-module and showed that if a C^*-algebra $\mathcal{A}$ has no nonzero commutative ideal, then any Finsler $\mathcal{A}$-module is a Hilbert $\mathcal{A}$-module. In this paper, we introduce the notion of Finsler C^*-bimodules and prove some properties of Finsler bimodules over commutative C^*-algebras and show that if E is an imprimitivity Finsler $\mathcal{A}$-$\mathcal{B}$-bimodule of C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ such that the corresponding maps ${}_{\mathcal{A}}\rho$ and $\rho_{\mathcal{B}}$ satisfy the parallelogram law on E, then E is a Hilbert $\mathcal{A}$-$\mathcal{B}$-bimodule.

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[0]Received February 18, 2014. Revised July 8, 2014.
[0]2010 Mathematics Subject Classification: 46L08, 46L05.
[0]Keywords: $\mathcal{A}$-$\mathcal{B}$-bimodule, Finsler C^*-bimodules, α-homomorphism, associated ideal.
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2. Preliminaries

Let us recall the definition of a Finlser module [2, 8].

Definition 2.1. Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{A}^+$ be the set of all positive elements of $\mathcal{A}$. Let E be a left module over $\mathcal{A}$ and the map ${}_{\mathcal{A}}\rho : E \to \mathcal{A}^+$ satisfy the following conditions:

(i) The map $\|.\|_E \ : \ x \mapsto \|{}_{\mathcal{A}}\rho(x)\|$ makes E into a Banach space;
(ii) ${}_{\mathcal{A}}\rho(ax)^2 = a{}_{\mathcal{A}}\rho(x)^2 a^*$, for all $a \in \mathcal{A}$ and $x \in E$.

Then E is called a left Finsler module over $\mathcal{A}$ under the map ${}_{\mathcal{A}}\rho$. A right Finsler module is defined similarly.
A left Finsler module over a C^*-algebra $\mathcal{A}$ is said to be full if the linear span $\left\{{}_{\mathcal{A}}\rho(x)^2 : x \in E\right\}$ denoted by $\mathcal{F}(E)$ is dense in $\mathcal{A}$.

Example 2.2. If E is a left (full) Hilbert C^*-module over $\mathcal{A}$, then E together with ${}_{\mathcal{A}}\rho(x) = \langle x, x\rangle^{\frac{1}{2}}$ is a left (full) Finsler module over $\mathcal{A}$, since ${}_{\mathcal{A}}\rho(ax)^2 = \langle ax, ax\rangle = a\langle x, x\rangle a^* = a{}_{\mathcal{A}}\rho(x)^2 a^*$.

3. Finsler C^*-bimodules

In this section, we state the notions of Finsler C^*-bimodule and imprimitivity Finsler bimodule. We then investigate some properties of Finsler C^*-bimodule and compare them with the Hilbert C^*-bimodule. By a pre-Hilbert bimodule ${}_{\mathcal{A}}E_{\mathcal{B}}$ over two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ we mean a left pre-Hilbert $\mathcal{A}$-module and a right pre-Hilbert $\mathcal{B}$-module such that

$$\begin{aligned}(ax)b &= a(xb),\\ \langle x, ax\rangle_{\mathcal{B}} &= \langle a^*x, y\rangle_{\mathcal{B}},\\ {}_{\mathcal{A}}\langle xb, y\rangle &= {}_{\mathcal{A}}\langle x, yb^*\rangle\end{aligned}$$

for all $a \in \mathcal{A}, b \in \mathcal{B}$ and $x \in_{\mathcal{A}} E_{\mathcal{B}}$. See [4, Definition 2.13].

Definition 3.1. A Finsler C^*-bimodule ${}_{\mathcal{A}}E_{\mathcal{B}}$ over a pair of C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ is a left Finsler module over $\mathcal{A}$ under the map ${}_{\mathcal{A}}\rho$ and a right Finsler module over $\mathcal{B}$ under the map $\rho_{\mathcal{B}}$ such that the following conditions are satisfied:

(i) ${}_{\mathcal{A}}\rho(x)^2 x = x\rho_{\mathcal{B}}(x)^2$;
(ii) ${}_{\mathcal{A}}\rho(xb)^2 = {}_{\mathcal{A}}\rho(xb^*)^2$ and $\rho_{\mathcal{B}}(ax)^2 = \rho_{\mathcal{B}}(a^*x)^2$,

where $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $x \in {}_{\mathcal{A}}E_{\mathcal{B}}$.

Recall that a Finsler C^*-bimodule ${}_{\mathcal{A}}E_{\mathcal{B}}$ has two norms, usually different, as follows ${}_{E}\|x\| = \|{}_{\mathcal{A}}\rho(x)\|$ and $\|x\|_E = \|\rho_{\mathcal{B}}(x)\|$. We however have the following result.

Lemma 3.2. *Let ${}_{\mathcal{A}}E_{\mathcal{B}}$ be a Finsler $\mathcal{A}$-$\mathcal{B}$-bimodule and $x \in {}_{\mathcal{A}}E_{\mathcal{B}}$. If ${}_{\mathcal{A}}\rho(xb)^2 \le \|b\|^2{}_{\mathcal{A}}\rho(x)^2$ and $\rho_{\mathcal{B}}(ax)^2 \le \|a\|^2\rho_{\mathcal{B}}(x)^2$ for each $a \in \mathcal{A}$ and $b \in \mathcal{B}$, then $\|{}_{\mathcal{A}}\rho(x)\| = \|\rho_{\mathcal{B}}(x)\|$.*

Proof. Suppose that $x \in E$ and $a = {}_{\mathcal{A}}\rho(x)^2$ and $b = \rho_{\mathcal{B}}(x)^2$. Then $ax = xb$ and

$$\begin{aligned} a^4 &= ({}_{\mathcal{A}}\rho(x)^2)^4 = {}_{\mathcal{A}}\rho(ax)^2_{\mathcal{A}}\rho(x)^2 = {}_{\mathcal{A}}\rho(xb)^2_{\mathcal{A}}\rho(x)^2 \\ &\le \|b\|^2_{\mathcal{A}}\rho(x)^2_{\mathcal{A}}\rho(x)^2 = \|b\|^2a^2. \end{aligned}$$

Hence $\|a\|^4 = \|a^4\| \le \|b\|^2\|a\|^2$, so $\|a\| \le \|b\|$ or $\|{}_{\mathcal{A}}\rho(x)\| \le \|\rho_{\mathcal{B}}(x)\|$. Similarly we have $\|\rho_{\mathcal{B}}(x)\| \le \|{}_{\mathcal{A}}\rho(x)\|$. □

Definition 3.3. A Finsler $\mathcal{A}$-$\mathcal{B}$-bimodule E is called an imprimitivity bimodule if it is full both as a left and as a right Finsler moddule over $\mathcal{A}$ and $\mathcal{B}$, respectively.

Example 3.4. Every C^*-algebra $\mathcal{A}$ is a imprimitivity Finsler $\mathcal{A}$-$\mathcal{A}$-bimodule over $\mathcal{A}$ under the mappings $\rho_{\mathcal{A}}(x) = (x^*x)^{\frac{1}{2}}$ and ${}_{\mathcal{A}}\rho(x) = (xx^*)^{\frac{1}{2}}$, $x \in \mathcal{A}$.

Example 3.5. Let $\mathcal{A}$ be a C^*-subalgebra of a C^*-algebra $\mathcal{B}$ and $\mathcal{E} : \mathcal{B} \to \mathcal{A}$ be a conditional expectation (i.e. a positive map of norm one satisfying the following conditions:

$$\mathcal{E}(ab) = a\mathcal{E}(b) \ , \quad \mathcal{E}(ba) = \mathcal{E}(b)a \ , \quad \mathcal{E}(a) = a \ ,$$

for each $a \in \mathcal{A}$ and $b \in \mathcal{B}$). Then $\mathcal{B}$ is a Finsler $\mathcal{A}$-$\mathcal{A}$-bimodule with respect to the mappings ${}_{\mathcal{A}}\rho(x) = (\mathcal{E}(xx^*))^{\frac{1}{2}}$ and $\rho_{\mathcal{A}}(x) = (\mathcal{E}(x^*x))^{\frac{1}{2}}$.

Theorem 3.6. *Suppose that E is an imprimitivity Finsler $\mathcal{A}$-$\mathcal{B}$-bimodule of C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ with the maps ${}_{\mathcal{A}}\rho$ and $\rho_{\mathcal{B}}$. If ${}_{\mathcal{A}}\rho$ and $\rho_{\mathcal{B}}$ fulfill the parallelogram law on E, then E is a Hilbert $\mathcal{A}$-$\mathcal{B}$-bimodule.*

Proof. Let ${}_{\mathcal{A}}\rho$ and $\rho_{\mathcal{B}}$ satisfy the parallelogram law on E. Then we have

$$\rho_{\mathcal{B}}(x+y)^2 + \rho_{\mathcal{B}}(x-y)^2 = 2\rho_{\mathcal{B}}(x)^2 + 2\rho_{\mathcal{B}}(y)^2$$

and

$${}_{\mathcal{A}}\rho(x+y)^2 +_{\mathcal{A}} \rho(x-y)^2 = 2{}_{\mathcal{A}}\rho(x)^2 + 2{}_{\mathcal{A}}\rho(y)^2$$

for each $x, y \in E$. By [8, Lemma 13], E is a Hilbert left $\mathcal{A}$-module and a Hilbert right $\mathcal{B}$-module, with the following inner products

$${}_{\mathcal{A}}\langle x, y\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k_{\mathcal{A}}\rho(x+i^ky)^2$$

and

$$\langle x, y\rangle_{\mathcal{B}} = \frac{1}{4}\sum_{k=0}^{3} i^k \rho_{\mathcal{B}}(x + i^k y)^2.$$

Also

$$\begin{aligned}\langle x, ax\rangle_{\mathcal{B}} &= \frac{1}{4}\sum_{k=0}^{3} i^k \rho_{\mathcal{B}}(x + i^k ax)^2 = \frac{1}{4}\sum_{k=0}^{3} i^k \rho_{\mathcal{B}}((1 + i^k a)x)^2 \\ &= \frac{1}{4}\sum_{k=0}^{3} i^k \rho_{\mathcal{B}}((1 + i^k a)^* x)^2 = \frac{1}{4}\sum_{k=0}^{3} i^k \rho_{\mathcal{B}}((1 + i^{-k} a^*)x)^2 \\ &= \frac{1}{4}\sum_{k=0}^{3} i^k \rho_{\mathcal{B}}(a^* x + i^k x)^2 = \langle a^* x, x\rangle_{\mathcal{B}}.\end{aligned}$$

Let $\alpha \in \mathbb{C}$. Replacing x by $x + \alpha y$ in $\langle x, ax\rangle_{\mathcal{B}} = \langle a^* x, x\rangle_{\mathcal{B}}$, we get

$$\langle x + \alpha y, a(x + \alpha y)\rangle_{\mathcal{B}} = \langle a^*(x + \alpha y), x + \alpha y\rangle_{\mathcal{B}},$$

whence

$$\begin{aligned}&\langle x, ax\rangle_{\mathcal{B}} + \alpha\langle x, ay\rangle_{\mathcal{B}} + \bar{\alpha}\langle y, ax\rangle_{\mathcal{B}} + \alpha\bar{\alpha}\langle y, ay\rangle_{\mathcal{B}} \\ &= \langle a^* x, x\rangle_{\mathcal{B}} + \alpha\langle a^* x, y\rangle_{\mathcal{B}} + \bar{\alpha}\langle a^* y, x\rangle_{\mathcal{B}} + \alpha\bar{\alpha}\langle a^* y, y\rangle_{\mathcal{B}}.\end{aligned}$$

Hence

$$\alpha\langle x, ay\rangle_{\mathcal{B}} + \bar{\alpha}\langle y, ax\rangle_{\mathcal{B}} = \alpha\langle a^* x, y\rangle_{\mathcal{B}} + \bar{\alpha}\langle a^* y, x\rangle_{\mathcal{B}}.$$

Choose $\alpha = 1$ to get

$$\langle x, ay\rangle_{\mathcal{B}} + \langle y, ax\rangle_{\mathcal{B}} = \langle a^* x, y\rangle_{\mathcal{B}} + \langle a^* y, x\rangle_{\mathcal{B}}.$$

Also $\alpha = i$ gives

$$\langle x, ay\rangle_{\mathcal{B}} - \langle y, ax\rangle_{\mathcal{B}} = \langle a^* x, y\rangle_{\mathcal{B}} - \langle a^* y, x\rangle_{\mathcal{B}}.$$

Therefore $\langle x, ay\rangle_{\mathcal{B}} = \langle a^* x, y\rangle_{\mathcal{B}}$.
Similarly ${}_{\mathcal{A}}\langle xb, y\rangle ={}_{\mathcal{A}} \langle x, yb^*\rangle$. Hence by [4, Definition 2.13], E is a Hilbert $\mathcal{A}$-$\mathcal{B}$-bimodule. □

Theorem 3.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be two commutative C^*-algebras and ${}_{\mathcal{A}}E_{\mathcal{B}}$ be an imprimitivity Finsler bimodule and there exist a map $\varphi : \mathcal{A} \to \mathcal{B}$ such that*

$$ax = x\varphi(a), \tag{3.1}$$

$$\varphi({}_{\mathcal{A}}\rho\,(x)^2) = \rho_{\mathcal{B}}(x)^2, \tag{3.2}$$

where $a \in A$ and $x \in {}_{\mathcal{A}}E_{\mathcal{B}}$. Then φ_E is a $*$-isomorphism.

Proof. The proof is similar to that of [1, Main Theorem]. □

Let E be a Finsler module over C^*-algebra $\mathcal{A}$ and $\mathcal{I}$ be a closed two-sided ideal. Let $\mathcal{I}E$ be the closed linear span of the set $\{ax; a \in \mathcal{I},\ x \in E\}$. Clearly $\mathcal{I}E$ is a closed submodule of E and by applying the Cohen-Hewitt factorization theorem ([7, Theorem4.1], and [9, Proposition 2.31]) it is easy to see that ${}_{\mathcal{I}}E = \mathcal{I}E = \{ax; a \in \mathcal{I},\ x \in E\}$.

Theorem 3.8. ([8]) *Let E be a Finsler module over a C^*-algebra $\mathcal{A}$, $\mathcal{I}$ be an ideal of $\mathcal{A}$ and $\pi : \mathcal{A} \to \frac{\mathcal{A}}{\mathcal{I}}$ be the quotient map and let $\rho = \pi \circ {}_{\mathcal{A}}\rho$. Then $\frac{E}{\mathcal{I}E}$ is a $\frac{\mathcal{A}}{\mathcal{I}}$-module and ρ descends to a $\frac{\mathcal{A}}{\mathcal{I}}$-valued Finsler norm on $\frac{E}{\mathcal{I}E}$.*

Lemma 3.9. *Let E be a full Finsler module over a C^*-algebra $\mathcal{A}$ and $\mathcal{I}$ be an ideal of $\mathcal{A}$. Then $\frac{E}{\mathcal{I}E}$ is a full Finsler module over C^*-algebra $\frac{\mathcal{A}}{\mathcal{I}}$.*

Proof. By Lemma 3.8, $\frac{E}{\mathcal{I}E}$ is a Finsler module over C^*-algebra $\frac{\mathcal{A}}{\mathcal{I}}$. Let $b \in \frac{\mathcal{A}}{\mathcal{I}}$ be arbitrary. Then there exists $a \in \mathcal{A}$ such that $b = a + \mathcal{I}$. Since E is full, there exists $\{u_n\}$ in $\mathcal{F}(E)$ such that $a = \lim_n u_n$. Each u_n is of the form $u_n = \sum_{i=1}^{k_n} \lambda_{i,n\mathcal{A}}\rho(x_{i,n})^2$ in which $x_{i,n} \in E$ and $\lambda_{i,n} \in \mathbb{C}$. Hence

$$b = \left(\lim_n \sum_{i=1}^{k_n} \lambda_{i,n\mathcal{A}}\rho(x_{i,n})^2\right) + \mathcal{I} = \left(\lim_n \sum_{1}^{k_n} \left(\lambda_{i,n\mathcal{A}}\rho(x_{i,n})^2 + \mathcal{I}\right)\right).$$

Therefore the linear span of $\left\{{}_{\mathcal{A}}\rho(x)^2 + \mathcal{I} : x \in E\right\}$ is equivalent to the linear span of $\left\{\rho(x + \mathcal{I}E)^2 : x \in E\right\}$ which is dense in $\frac{\mathcal{A}}{\mathcal{I}}$ as well as $\frac{E}{\mathcal{I}E}$ is a full Finsler module over $\frac{\mathcal{A}}{\mathcal{I}}$. □

Theorem 3.10. *Let E be an imprimitivity Finsler bimodule over commutative C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ and $\mathcal{I}$ be an ideal of $\mathcal{A}$. Then $\frac{E}{\mathcal{I}E}$ is an imprimitivity Finsler bimodule over $\frac{\mathcal{A}}{\mathcal{I}}$ and $\frac{\mathcal{B}}{\varphi(\mathcal{I})}$, when φ is the $*$-isomorphism in Theorem* 3.7.

Proof. Suppose E is an imprimitivity Finsler bimodule over the commutative C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ and $\varphi : \mathcal{A} \to \mathcal{B}$ is the $*$-isomorphism in Theorem 3.7. Then $\varphi(\mathcal{I})$ is an ideal in $\mathcal{B}$ and by (3.1), $\mathcal{I}E = E\varphi(\mathcal{I})$. We know that $\frac{E}{\mathcal{I}E}$ is a left module over $\frac{\mathcal{A}}{\mathcal{I}}$, via $(a + \mathcal{I})(x + \mathcal{I}E) = ax + \mathcal{I}E$ and is a right module over $\frac{\mathcal{B}}{\varphi(\mathcal{I})}$, via $(x + \mathcal{I}E)(b + \varphi(\mathcal{I})) = xb + \mathcal{I}E$, for all $x \in E, a \in \mathcal{A}$ and $b \in \mathcal{B}$. By Lemmas 3.8 and 3.9, $\frac{E}{\mathcal{I}E}$ is a left full Finsler module over $\frac{\mathcal{A}}{\mathcal{I}}$ and a right full Finsler module over $\frac{\mathcal{B}}{\varphi(\mathcal{I})}$, where $\pi : \mathcal{A} \to \frac{\mathcal{A}}{\mathcal{I}}$ is the quotient map and $\rho : \frac{E}{\mathcal{I}E} \to (\frac{\mathcal{A}}{\mathcal{I}})^+$ is defined by $\rho(x + \mathcal{I}E) = (\pi \circ_{\mathcal{A}} \rho)(x)$.
Also $\pi' : \mathcal{B} \to \frac{\mathcal{B}}{\varphi(\mathcal{I})}$ is the quotient map and $\rho' : \frac{E}{\mathcal{I}E} \to (\frac{\mathcal{B}}{\varphi(\mathcal{I})})^+$ is defined by

$\rho'(x+\mathcal{I}E) = \pi' \circ \rho_{\mathcal{B}}(x)$. Since E is a Finsler $\mathcal{A}$-$\mathcal{B}$-bimodule, by (i) of Definition 3.1, we have ${}_{\mathcal{A}}\rho(x)^2 x = x\rho_{\mathcal{B}}(x)^2$. Hence

$$\begin{aligned} \rho(x+\mathcal{I}E)^2(x+\mathcal{I}E) &= (\pi \circ_{\mathcal{A}} \rho)(x)^2(x+\mathcal{I}E) \\ &= \left({}_{\mathcal{A}}\rho(x)^2 + \mathcal{I}\right)(x+\mathcal{I}E) \\ &= {}_{\mathcal{A}}\rho(x)^2 x + \mathcal{I}E \\ &= x\rho_{\mathcal{B}}(x)^2 + \mathcal{I}E \\ &= (x+\mathcal{I}E)(\rho_{\mathcal{B}}(x)^2 + \varphi_E(\mathcal{I})) \\ &= (x+\mathcal{I}E)(\pi' \circ \rho_{\mathcal{B}})(x)^2 \\ &= (x+\mathcal{I}E)\rho'(x+\mathcal{I}E)^2. \end{aligned}$$

Also

$$\begin{aligned} \rho\left((x+\mathcal{I}E)(b+\varphi_E(\mathcal{I}))\right)^2 &= \rho(xb+\mathcal{I}E)^2 \\ &= (\pi \circ_{\mathcal{A}} \rho)(xb)^2 \\ &= (\pi \circ_{\mathcal{A}} \rho)(xb^*)^2 \\ &= \rho(xb^*+\mathcal{I}E)^2 \\ &= \rho\left((x+\mathcal{I}E)(b+\varphi_E(\mathcal{I}))^*\right)^2 \end{aligned}$$

for each $x \in_{\mathcal{A}} E_{\mathcal{B}}$ and $b \in B$.
Similarly we can show that $\rho'\left((a+\mathcal{I})(x+\mathcal{I}E)\right)^2 = \rho'\left((a+\mathcal{I})^*(x+\mathcal{I}E)\right)^2$. Therefore $\frac{E}{\mathcal{I}E}$ is an imprimitivity Finsler bimodule over $\frac{\mathcal{A}}{\mathcal{I}}$ and $\frac{\mathcal{B}}{\varphi_E(\mathcal{I})}$. □

Definition 3.11. Let E be a Finsler $\mathcal{A}$-$\mathcal{B}$-bimodule, $\mathcal{I}$ and $\mathcal{J}$ be ideals in $\mathcal{A}$ and $\mathcal{B}$, respectively. The ideal subbimodule ${}_{\mathcal{I}}E_{\mathcal{J}}$ of E associated to $\mathcal{I}$ and $\mathcal{J}$ is defined by

$${}_{\mathcal{I}}E_{\mathcal{J}} = \overline{\text{span}}\{axb : x \in E, a \in \mathcal{I}, b \in \mathcal{J}\}.$$

Clearly, ${}_{\mathcal{I}}E_{\mathcal{J}}$ is a closed subbimodule of E. It can be also regarded as a Finsler bimodule over $\mathcal{I}$ and $\mathcal{J}$.

Theorem 3.12. *Let E be a Finsler $\mathcal{A}$-$\mathcal{B}$-bimodule , $\mathcal{I}$ and $\mathcal{J}$ are ideals in $\mathcal{A}$ and $\mathcal{B}$, respectively. Then*

$${}_{\mathcal{I}}E_{\mathcal{J}} = \mathcal{I}E\mathcal{J} = \{axb : x \in E, a \in A, b \in B\}.$$

Proof. The proof is similar to that of [3, Proposition 1.2] and we remove it. □

Remark 3.13. Let E be a Finsler bimodule over commutative C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ and there exists a $*$-isomorphism $\varphi_E : \mathcal{A} \to \mathcal{B}$ as in Theorem 3.7. If $\mathcal{I}$ and $\mathcal{J}$ are ideals of $\mathcal{A}$ and $\mathcal{B}$, respectively, and ${}_{\mathcal{I}}E_{\mathcal{J}}$ is the associated ideal subbimodule, then $\frac{E}{{}_{\mathcal{I}}E_{\mathcal{J}}}$ is a $\frac{\mathcal{A}}{\mathcal{I}}$-$\frac{\mathcal{B}}{\mathcal{J}}$-bimodule, where $q : E \to \frac{E}{{}_{\mathcal{I}}E_{\mathcal{J}}}$ and

$\pi : \mathcal{A} \to \frac{\mathcal{A}}{\mathcal{I}}$ and $\pi' : \mathcal{B} \to \frac{\mathcal{B}}{\mathcal{J}}$ are the quotient maps and the left action of $\frac{\mathcal{A}}{\mathcal{I}}$ and the right action of $\frac{\mathcal{B}}{\mathcal{J}}$ over linear space $\frac{E}{\mathcal{I}E_{\mathcal{J}}}$ are defined by $\pi(a)q(x) = q(ax)$ and $q(x)\pi'(b) = q(xb)$, respectively. By Theorem 3.10, $\frac{E}{\mathcal{I}E_{\mathcal{J}}}$ is a Finsler $\frac{\mathcal{A}}{\mathcal{I}}$-$\frac{\mathcal{B}}{\mathcal{J}}$ -bimodule, where ${}_{\frac{\mathcal{A}}{\mathcal{I}}}\rho(q(x)) = \pi({}_{\mathcal{A}}\rho(x))$ and $\rho_{\frac{\mathcal{B}}{\mathcal{J}}}(q(x)) = \pi'(\rho_{\mathcal{B}}(x))$.
In addition, $\frac{E}{\mathcal{I}E_{\mathcal{J}}}$ is an imprimitivity Finsler $\frac{\mathcal{A}}{\mathcal{I}}$-$\frac{\mathcal{B}}{\mathcal{J}}$-bimodule if and only if E is an imprimitivity Finsler $\mathcal{A}$-$\mathcal{B}$-bimodule. This follows at once from the evident equalities $({}_{\frac{\mathcal{A}}{\mathcal{I}}}\rho(q(E))) = \pi({}_{\mathcal{A}}\rho(E))$ and $(\rho_{\frac{\mathcal{B}}{\mathcal{J}}}(q(E))) = \pi'(\rho_{\mathcal{B}}(E))$.

Recall that an ideal $\mathcal{I}$ of a C^*-algebra $\mathcal{A}$ is essential, if $\mathcal{I}^{\perp} = \{a \in \mathcal{A} : a\mathcal{I} = 0\} = \{0\}$.

Lemma 3.14. *Let $\mathcal{I}$ be an ideal in a C^*-algebra $\mathcal{A}$ and $\mathcal{I}^+$ be the set of all positive elements of $\mathcal{I}$. The following condition are mutually equivalent:*

(a) *$\mathcal{I}$ is an essential ideal in $\mathcal{A}$;*
(b) $\|a\| = \sup\limits_{b\in\mathcal{I}^+, \|b\|\leqslant 1} (\|ab\|)$;
(c) $\|a\| = \sup\limits_{b\in\mathcal{I}^+, \|b\|\leqslant 1} (\|ba\|)$ *for each $a \in \mathcal{A}$; and*
(d) $\|a\| = \sup\limits_{b\in\mathcal{I}^+, \|b\|\leqslant 1} (\|bab\|)$ *for each $a \in \mathcal{A}^+$.*

Proof. The proof is similar to that of [3, Lemma 1.10], by replacing $\mathcal{I}$ with $\mathcal{I}^+$. □

Theorem 3.15. *Let E be a Finsler $\mathcal{A}$-$\mathcal{B}$-bimodule and $\mathcal{I}$, $\mathcal{J}$ be the essential ideals of $\mathcal{A}$ and $\mathcal{B}$, respectively. Then*

$$\|x\| = \sup_{b\in\mathcal{I}^+, \|b\|\leqslant 1} (\|bx\|) = \sup_{b\in\mathcal{J}^+, \|b\|\leqslant 1} (\|xb\|)$$

for each $x \in E$. Conversely, if E is an imprimitivity Finsler $\mathcal{A}$-$\mathcal{B}$-bimodule and for each $x \in E$,

$$\|x\| = \sup_{b\in\mathcal{I}^+, \|b\|\leqslant 1} (\|bx\|) = \sup_{b\in\mathcal{J}^+, \|b\|\leqslant 1} (\|xb\|),$$

then $\mathcal{I}$ and $\mathcal{J}$ are essential ideals in $\mathcal{A}$ and $\mathcal{B}$, respectively.

Proof. Since E is a left Finsler module over C^*-algebra $\mathcal{A}$, we have

$$\begin{aligned}\|x\|^2 = \|{}_{\mathcal{A}}\rho(x)\|^2 = \|{}_{\mathcal{A}}\rho(x)^2\| &= \sup_{b\in\mathcal{I}^+, \|b\|\leqslant 1} (\|b{}_{\mathcal{A}}\rho(x)^2 b\|) \\ &= \sup_{b\in\mathcal{I}^+, \|b\|\leqslant 1} (\|bx\|)^2,\end{aligned}$$

for each $x \in E$. Since E is a right Finsler module over C^*-algebra $\mathcal{B}$, we have

$$\begin{aligned}\|x\|^2 = \|\rho_{\mathcal{B}}(x)\|^2 = \|\rho_{\mathcal{B}}(x)^2\| &= \sup_{b\in\mathcal{J}^+,\|b\|\leqslant 1}(\|b\rho_{\mathcal{B}}(x)^2 b\|)\\ &= \sup_{b\in\mathcal{J}^+,\|b\|\leqslant 1}(\|xb\|)^2,\end{aligned}$$

for each $x \in E$.

Conversely, let E be an imprimitivity Finsler bimodule. If $\mathcal{I}$ and $\mathcal{J}$ are not essential, then $\mathcal{I}^\perp \neq \{0\}$ and $\mathcal{J}^\perp \neq \{0\}$. Hence there exist nonzero elements $c_1 \in \mathcal{I}^\perp$ and $c_2 \in \mathcal{J}^\perp$. By [2, Theorem 3.2(iii)], there exist $x_1, x_2 \in E$ such that $c_1x_1 \neq 0$ and $x_2c_2 \neq 0$. By the assumption, we have

$$\|c_1x_1\| = \sup_{b\in\mathcal{I}^+,\|b\|\leqslant 1}(\|b(c_1x_1)\|) = \sup_{b\in\mathcal{I}^+,\|b\|\leqslant 1}(\|(bc_1)x_1\|) = 0$$

and

$$\|x_2c_2\| = \sup_{b\in\mathcal{J}^+,\|b\|\leqslant 1}(\|(x_2c_2)b\|) = \sup_{b\in\mathcal{J}^+,\|b\|\leqslant 1}(\|x_2(c_2b)\|) = 0.$$

So $c_1x_1 = 0$ and $x_2c_2 = 0$, which is a contradiction. Therefore $\mathcal{I}$ and $\mathcal{J}$ are essential ideals of $\mathcal{A}$ and $\mathcal{B}$, respectively. □

Corollary 3.16. *Suppose that E is an imprimitivity Finsler $\mathcal{A}$-$\mathcal{B}$-bimodule of commutative C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ with the maps ${}_{\mathcal{A}}\rho$ and $\rho_{\mathcal{B}}$. Suppose that $\mathcal{I}$ and $\mathcal{J}$ are essential ideals of $\mathcal{A}$ and $\mathcal{B}$, respectively. If ${}_{\mathcal{A}}\rho$ and $\rho_{\mathcal{B}}$ satisfy the parallelogram law on ${}_{\mathcal{I}}E$ and $E_{\mathcal{J}}$, respectively, then E is a Hilbert $\mathcal{A}$-$\mathcal{B}$-bimodule.*

Proof. Since E is an imprimitivity Finsler $\mathcal{A}$-$\mathcal{B}$-bimodule with the maps ${}_{\mathcal{A}}\rho$ and $\rho_{\mathcal{B}}$, the essential ideal submodules ${}_{\mathcal{I}}E$ and $E_{\mathcal{J}}$ are Finsler modules with the restriction mappings ${}_{\mathcal{A}}\rho|_{{}_{\mathcal{I}}E}$ and $\rho_{\mathcal{B}}|_{E_{\mathcal{J}}}$, respectively. Since ${}_{\mathcal{A}}\rho$ and $\rho_{\mathcal{B}}$ satisfy the parallelogram law on ${}_{\mathcal{I}}E$ and $E_{\mathcal{J}}$, respectively. Hence for each $x, y \in E$ and $a \in \mathcal{I}$ and $b \in \mathcal{J}$ such that $\|a\| \leq 1$ and $\|b\| \leq 1$, we have

$$\begin{aligned}{}_{\mathcal{A}}\rho(ax+ay)^2 + {}_{\mathcal{A}}\rho(ax-ay)^2 - 2{}_{\mathcal{A}}\rho(ax)^2 - 2{}_{\mathcal{A}}\rho(ay)^2 &= 0,\\ a({}_{\mathcal{A}}\rho(x+y)^2 + {}_{\mathcal{A}}\rho(x-y)^2 - 2{}_{\mathcal{A}}\rho(x)^2 - 2{}_{\mathcal{A}}\rho(y)^2)a^* &= 0,\\ ({}_{\mathcal{A}}\rho(x+y)^2 + {}_{\mathcal{A}}\rho(x-y)^2 - 2{}_{\mathcal{A}}\rho(x)^2 - 2{}_{\mathcal{A}}\rho(y)^2)a^*a &= 0.\end{aligned}$$

It follows from Lemma 3.14(b)

$$\|{}_{\mathcal{A}}\rho(x+y)^2 + {}_{\mathcal{A}}\rho(x-y)^2 - 2{}_{\mathcal{A}}\rho(x)^2 - 2{}_{\mathcal{A}}\rho(y)^2\| = 0.$$

Hence

$${}_{\mathcal{A}}\rho(x+y)^2 + {}_{\mathcal{A}}\rho(x-y)^2 = 2{}_{\mathcal{A}}\rho(x)^2 + 2{}_{\mathcal{A}}\rho(y)^2.$$

Similarly

$$\rho_{\mathcal{B}}(x+y)^2 + \rho_{\mathcal{B}}(x-y)^2 = 2\rho_{\mathcal{B}}(x)^2 + 2\rho_{\mathcal{B}}(y)^2.$$

Hence ${}_{\mathcal{A}}\rho$ and $\rho_{\mathcal{B}}$ satisfy the parallelogram law on E. By Theorem 3.6, E is a Hilbert $\mathcal{A}$-$\mathcal{B}$-bimodule. □

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