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STRONG CONVERGENCE OF MODIFIED ISHIKAWA ITERATION FOR TWO ASYMPTOTICALLY ϕ-NONEXPANSIVE MAPPINGS IN A BANACH SPACE

Ying Liu[1] and Guochun Zhang[2]

[1]College of Mathematics and Computer, Hebei University
Baoding 071002, P.R. China
e-mail: ly_cyh2013@163.com

[2]College of Mathematics and Computer, Hebei University
Baoding 071002, P.R. China
e-mail: zhangguochun@cmc.hbu.cn

Abstract. In this paper, we propose an iteration sequence by using the modified Ishikawa iteration method in a Banach space. Furthermore, we prove the iteration sequence converges strongly a concrete common fixed point of two asymptotically ϕ-nonexpansive mappings.

1. Introduction

Let E be a Banach space, E^* be the dual space of E. $\langle \cdot, \cdot \rangle$ denotes the duality pairing of E and E^*. The function $\phi : E \times E \to R$ is defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2,$$

for all $x, y \in E$, where J is the normalized duality mapping from E to E^*. Let C be a closed convex subset of E, and let T be a mapping from C into itself. We denote by $F(T)$ the set of fixed points of T. A point p in C is said to be an asymptotic fixed point of T [10] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty}(x_n - Tx_n) = 0$. The set of asymptotic fixed points of T will be denoted by $\hat{F}(T)$. From [2, 14], we can find the following definitions:

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The mapping T is said to be closed if for any sequence $\{x_n\} \subset C$ such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} Tx_n = y_0$, then $Tx_0 = y_0$. The mapping T is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$ and quasi-nonexpansive if $F(T) \neq \emptyset$ and $\|x - Ty\| \leq \|x - y\|$ for all $x \in F(T)$ and $y \in C$. T is said to be asymptotically nonexpansive if there exists a sequence $k_n \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that $\|T^n x - T^n y\| \leq k_n \|x - y\|$, $\forall x, y \in C, \forall n \geq 1$. T is said to be asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $k_n \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that $\|x - T^n y\| \leq k_n \|x - y\|$, $\forall x \in F(T)$, $y \in C$, $\forall n \geq 1$. T is said to be relatively nonexpansive if $F(T) = \hat{F}(T)$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. T is said to be relatively asymptotically nonexpansive, if $F(T) = \hat{F}(T) \neq \emptyset$ and there exists a sequence $k_n \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that $\phi(p, T^n x) \leq k_n \phi(p, x)$ for all $x \in C, p \in F(T)$ and $n \geq 1$. T is said to be $\phi-$nonexpansive if $\phi(Tx, Ty) \leq \phi(x, y)$ for all $x, y \in C$. T is said to be quasi-$\phi-$nonexpansive if $F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. T is said to be asymptotically $\phi-$nonexpansive if there exists a sequence $k_n \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that $\phi(T^n x, T^n y) \leq k_n \phi(x, y)$ for all $x, y \in C$. T is said to be asymptotically quasi-ϕ-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, +\infty)$ with $k_n \to 1$ as $n \to \infty$ such that $\phi(p, T^n x) \leq k_n \phi(p, x)$ for all $x \in C$, $p \in F(T)$ and $n \geq 1$.

Remark 1.1. ([14]) The class of (asymptotically) quasi-ϕ-nonexpansive mappings is more general than the class of relatively (asymptotically) nonexpansive mappings which requires the restriction: $F(T) = \hat{F}(T)$.

Remark 1.2. In the framework of Hilbert spaces, (asymptotically) ϕ-nonexpansive mappings are reduced to (asymptotically) nonexpansive mappings.

Two classical iteration processes are often used to approximate a fixed point of a nonexpansive mapping. The first one is introduced in 1953 by Mann [8] which is well-known as Mann's iteration process and is defined as follows:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad n \geq 0, \end{cases} \tag{1.1}$$

where the sequence $\{\alpha_n\}$ is chosen in $[0, 1]$. Twenty-one years later, Ishikawa [6] enlarged and improved Mann's iteration (1.1) to the new iteration method, it is often cited as Ishikawa iteration process which is defined recursively by

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = \beta_n x_n + (1 - \beta_n) T x_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T y_n, \quad n \geq 0, \end{cases} \tag{1.2}$$

where α_n and β_n are sequences in the interval $[0, 1]$.

Both iterations processes (1.1) and (1.2) have only weak convergence, in general Banach space (see [4], for more details). As a matter of fact, process (1.1) may fail to converge while process (1.2) can still converge for a Lipschitz pseudo-contractive mapping in a Hilbert space [3].

Some attempts to modify the Mann iteration method so that strong convergence is guaranteed have recently been made. Nakajo and Takahashi [12] proposed the following modification of the Mann iteration method for a single nonexpansive mapping T in a Hilbert space H:

$$\begin{cases} x_0 &= x \in C, \\ y_n &= \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n &= \{z \in C : \|z - y_n\| \le \|z - x_n\|\}, \\ Q_n &= \{z \in C : \langle x_n - z, x - x_n\rangle \ge 0\}, \\ x_{n+1} &= P_{C_n \bigcap Q_n} x, \quad n = 0, 1, 2, \cdots, \end{cases} \tag{1.3}$$

where P_K denotes the metric projection from H onto a closed convex subset K of H. They proved that if the sequence $\{\alpha_n\}$ is bounded above from one, then $\{x_n\}$ defined by (1.3) converges strongly to $P_{F(T)}x$.

In 2006, Martinez-Yanes and Xu [9] has adapted Nakajo and Takahashi's [12] idea to modify the process (1.2) for a single nonexpansive mapping T in a Hilbert space H :

$$\begin{cases} x_0 &\in C, \\ z_n &= \beta_n x_n + (1-\beta_n)Tx_n, \\ y_n &= \alpha_n x_n + (1-\alpha_n)Tz_n, \\ C_n &= \{v \in C : \|y_n - v\|^2 \le \|x_n - v\|^2 + (1-\alpha_n)(\|z_n\|^2 \\ & \quad -\|x_n\|^2 + 2\langle x_n - z_n, v\rangle)\}, \\ Q_n &= \{v \in C : \langle x_n - v, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} &= P_{C_n \bigcap Q_n} x_0. \end{cases} \tag{1.4}$$

They proved that if $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ such that $\alpha_n \le 1-\delta$ for some $\delta \in (0,1]$ and $\beta_n \to 1$, then the sequence $\{x_n\}$ generated by (1.4) converges strongly to $P_{F(T)}x_0$.

In 2007, Plubtieng and Ungchittrakool [13] have again modified the process (1.2) for two asymptotically nonexpansive mappings. More precisely, they proved the following theorem.

Theorem PU. Let C be a bounded closed convex subset of a Hilbert space H and let $S, T : C \to C$ be two asymptotically nonexpansive mappings with sequences $\{s_n\}$ and $\{t_n\}$ respectively. Assume that $\alpha_n \le a$ for all n and for some $0 < a < 1$ and $\beta_n \in [b, c]$ for all n and $0 < b < c < 1$. If $F := F(S) \bigcap F(T) \ne \emptyset$, then the sequence $\{x_n\}$ generated by

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)T^n z_n, \\ z_n = \beta_n x_n + (1-\beta_n)S^n x_n, \\ C_n = \{v \in C : \|y_n - v\|^2 \leq \|x_n - v\|^2 + \theta_n\}, \\ Q_n = \{v \in C : \langle x_n - v, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \bigcap Q_n} x_0, \end{cases} \tag{1.5}$$

where $\theta_n = (1-\alpha_n)[(t_n^2 - 1) + (1-\beta_n)t_n^2(s_n^2 - 1)](diamC)^2 \to 0$ as $n \to \infty$, converges in norm to $P_F x_0$.

The ideas to generalize the processes (1.3)-(1.5) from Hilbert space to Banach space have recently been made. By using available properties on uniformly convex and uniformly smooth Banach space, Matsushita and Takahashi [10] presented their ideas as the following method for a single relatively nonexpansive mapping T in a Banach space E :

$$\begin{cases} x_0 &= x \in C, \\ y_n &= J^{-1}(\alpha_n J x_n + (1-\alpha_n)JTx_n), \\ H_n &= \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ W_n &= \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \geq 0\}, \\ x_{n+1} &= \Pi_{H_n \bigcap W_n} x, n = 0, 1, 2, \cdots, \end{cases} \tag{1.6}$$

where $\alpha_n \subset [0, 1)$, $\limsup_{n\to\infty} \alpha_n < 1$, and $\Pi_{H_n \bigcap W_n}$ is the generalized projection from C into $H_n \bigcap W_n$. They proved $\{x_n\}$ converges strongly $\Pi_{F(T)} x_0$.

Qin and Su [15] proposed the following modified Ishikawa iteration process for a single relatively nonexpansive mapping T in a Banach space E :

$$\begin{cases} x_0 &\in C, \\ z_n &= J^{-1}(\beta_n J x_n + (1-\beta_n)JTx_n), \\ y_n &= J^{-1}(\alpha_n J x_n + (1-\alpha_n)JTz_n), \\ C_n &= \{v \in C : \phi(v, y_n) \leq \alpha_n \phi(v, x_n) + (1-\alpha_n)\phi(v, z_n)\}, \\ Q_n &= \{v \in C : \langle x_n - v, Jx_0 - Jx_n\rangle \geq 0\}, \\ x_{n+1} &= \Pi_{C_n \bigcap Q_n} x_0, \end{cases} \tag{1.7}$$

where $\alpha_n \subset [0, 1)$, $\limsup_{n\to\infty} \alpha_n < 1$, $\beta_n \to 1$. They proved if T is uniformly continuous, then $\{x_n\}$ converges strongly to $\Pi_{F(T)} x_0$.

In 2009, Liu et al. [7] generalized the modified Ishikawa iteration process (1.7) for two relatively nonexpansive mappings T and S in a Banach space without assuming the uniform continuity on T or S.

Very recently, Qin et al. [14] proposed the following modified Mann iteration process for a single closed, asymptotically quasi-ϕ-nonexpansive mapping T in a uniformly smooth and strictly convex Banach space E which enjoys the Kadec-Klee property:

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily,} \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T^n x_n), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n) + (k_n - 1) M_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases} \tag{1.8}$$

where $M_n = \sup\{\phi(z, x_n) : z \in F(T)\}$ for each $n \geq 1$, $\alpha_n \subset [0,1)$, $\limsup_{n\to\infty} \alpha_n < 1$. They proved if T is asymptotically regular and $F(T)$ is bounded, then $\{x_n\}$ converges strongly to $\Pi_{F(T)} x_0$.

Inspired and motivated by these facts, our purpose in this paper is to generalize the modified Mann iteration process (1.8) to modified Ishikawa iteration process for two closed, asymptotically ϕ-nonexpansive mappings T and S in a uniformly smooth and strictly convex Banach space E which enjoys the Kadec-Klee property without assuming asymptotically regularity on T or S.

2. Preliminaries

Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. A Banach space E is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in U$ and $x \neq y$. It is said to be uniformly convex if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}, \{y_n\}$ in U and $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 1$. E is said to be smooth provided $\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$ exists for each $x, y \in U$. It is said to be uniformly smooth if the limit is attained uniformly for $x, y \in U$. A Banach space E is said to have the Kadec-Klee property if a sequence $\{x_n\}$ of E satisfying that $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, then $x_n \to x$. It is known that if E is uniformly convex, then E has the Kadec-Klee property.

When $\{x_n\}$ is a sequence in E, we denote strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$.

We denote by $J : E \to 2^{E^*}$ the normalized duality mapping from E to 2^{E^*}, defined by

$$J(x) := \{v \in E^* : \langle v, x\rangle = \|v\|^2 = \|x\|^2\}, \quad \forall\, x \in E.$$

The following properties for the duality mapping J can be found in [2]:

(i) If E is an arbitrary Banach space, then J is monotone and bounded.

(ii) If E is smooth, then J is single-valued and demi-continuous, i.e., J is continuous from the strong topology of E to the weak star topology of E^*.
(iii) If E is strictly convex, then J is strictly monotone.
(iv) If E is reflexive, then J is surjective.
(v) If E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.
(vi) If E is a reflexive and strictly convex Banach space with a strictly convex dual E^* and $J^* : E^* \to E$ is the normalized duality mapping in E^*, then $J^{-1} = J^*$, $JJ^* = I_{E^*}$ and $J^*J = I_E$.
(vii) If E is a smooth, strictly convex and reflexive Banach space, then J is single-valued, one-to-one and onto.
(viii) It is well known that a Banach space E is uniformly smooth if and only if E^* is uniformly convex. If E is uniformly smooth, then it is smooth and reflexive.

Let E be a smooth Banach space. The function $\phi : E \times E \to R$ is defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2$$

for all $x, y \in E$. It is obvious from the definition of the function ϕ that

(A1) $(\|x\| - \|y\|)^2 \le \phi(y, x) \le (\|x\| + \|y\|)^2$.
(A2) $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle$.
(A3) $\phi(x, y) = \langle x, Jx - Jy\rangle + \langle y - x, Jy\rangle \le \|x\|\|Jx - Jy\| + \|y - x\|\|y\|$.

Remark 2.1. From the Remark 2.1 of reference [10], we can know that if E is a strictly convex and smooth Banach space, then for $x, y \in E, \phi(y, x) = 0$ if and only if $x = y$.

Let C be a nonempty closed convex subset of E. Suppose that E is reflexive, strictly convex and smooth. Then, for any $x \in E$, there exists a unique point $x_0 \in C$ such that

$$\phi(x_0, x) = \min_{y \in C} \phi(y, x).$$

The mapping $\Pi_C : E \to C$ defined by $\Pi_C x = x_0$ is called the generalized projection [1, 10]. In a Hilbert space, $\Pi_C = P_C$(metric projection).

Let $\{C_n\}$ be a sequence of nonempty closed convex subsets of a reflexive Banach space E. We define two subsets $s - Li_n C_n$ and $w - Ls_n C_n$ as follows: $x \in s - Li_n C_n$ if and only if there exists $\{x_n\} \subset E$ such that $\{x_n\}$ converges strongly to x and such that $x_n \in C_n$ for all $n \ge 1$. Similarly, $y \in w - Ls_n C_n$ if and only if there exists a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{y_{n_i}\} \subset E$ such that $\{y_{n_i}\}$ converges weakly to y and such that $y_{n_i} \in C_{n_i}$ for all $i \ge 1$. We define the Mosco convergence [16] of C_n as follows: If C_0 satisfies that

$C_0 = s - Li_n C_n = w - Ls_n C_n$, it is said that $\{C_n\}$ converges to C_0 in the sense of Mosco, and we write $C_0 = M - \lim_{n\to\infty} C_n$. For more details, see [11].

Lemma 2.2. ([5]) *Let E be a smooth, reflexive and strictly convex Banach space having the Kadec-klee property. Let $\{K_n\}$ be a sequence of nonempty closed convex subsets of E. If $K_0 = M - \lim_{n\to\infty} K_n$ exists and is nonempty, then $\{\Pi_{K_n} x\}$ converges strongly to $\Pi_{K_0} x$ for each $x \in E$.*

Lemma 2.3. ([2, 14]) *Let E be a uniformly smooth and strictly convex Banach space which enjoys the Kadec-Klee property, let C be a nonempty closed convex subset of E, and let T be a closed and asymptotically quasi-ϕ-nonexpansive mapping from C into itself. Then $F(T)$ is closed and convex.*

Lemma 2.4. ([1]) *Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \le \phi(y, x)$$

for all $y \in C$.

Lemma 2.5. ([17]) *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a continuous strictly increasing convex function $g : [0, 2r] \to R$ such that $g(0) = 0$ and*

$$\|tx + (1-t)y\|^2 \le t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g(\|x - y\|),$$

for all $x, y \in B_r$ and $t \in [0, 1]$, where $B_r = \{z \in E : \|z\| \le r\}$.

Lemma 2.6. *Let E be a reflexive, smooth and strictly convex Banach space such that E and E^* have the Kadec-Klee property. Let $\{x_n\}$, $\{y_n\}$ be two sequences of E and $x_n \to \bar{x}$. If $\phi(x_n, y_n) \to 0$, then $y_n \to \bar{x}$ and $\|x_n - y_n\| \to 0$, as $n \to \infty$.*

Proof. Since $\phi(x_n, y_n) \to 0$, from (A1), we know that $\|y_n\| \to \|\bar{x}\|$. It follows that $\|Jy_n\| \to \|J\bar{x}\|$. This implies that $\{Jy_n\}$ is bounded. We may assume that $Jy_n \rightharpoonup y^* \in E^*$. By the reflexivity of E, we see that $JE = E^*$. This shows that there exists a $y \in E$ such that $Jy = y^*$. It follows that

$$\begin{aligned} \phi(\bar{x}, y) &= \|\bar{x}\|^2 - 2\langle \bar{x}, Jy\rangle + \|Jy\|^2 \\ &\le \liminf_{n\to\infty} (\|x_n\|^2 - 2\langle x_n, Jy_n\rangle + \|Jy_n\|^2) \\ &= \liminf_{n\to\infty} \phi(x_n, y_n) = 0, \end{aligned}$$

which implies that $\bar{x} = y$. This is $Jy_n \rightharpoonup J\bar{x}$. Since E^* satisfies the Kadec-Klee property, we have $Jy_n \to J\bar{x}$. Note that $J^{-1} : E^* \to E$ is demi-continuous, it follows that $y_n \rightharpoonup \bar{x}$. Since E satisfies the Kadec-Klee property, we have $y_n \to \bar{x}$. Since $\|x_n - y_n\| \le \|x_n - \bar{x}\| + \|\bar{x} - y_n\|$, we also have $\lim_{n\to\infty} \|x_n - y_n\| = 0$. □

3. Main results

In this section, we prove a strong convergence theorem of a common fixed point for two closed and asymptotically ϕ-nonexpansive mappings from C into itself.

Theorem 3.1. *Let E be a uniformly smooth and strictly convex Banach space which enjoys the Kadec-Klee property, let C be a nonempty closed convex subset of E, let T, S be two closed and asymptotically quasi-ϕ-nonexpansive mappings from C into itself with sequences $\{t_n\}$ and $\{s_n\}$ respectively such that $F = F(T)\bigcap F(S) \neq \emptyset$ and F is bounded. Let the sequence $\{x_n\}$ be generated by*

$$\begin{cases} x_0 & \in & E \quad \textit{chosen arbitrarily}, \\ C_1 & = & C, \\ x_1 & = & \Pi_{C_1} x_0, \\ u_n & = & J^{-1}(\beta_n J x_n + (1-\beta_n) J S^n x_n), \\ z_n & = & \Pi_C u_n, \\ y_n & = & J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T^n z_n), \\ C_{n+1} & = & \{z \in C_n : \phi(z, y_n) \le \phi(z, x_n) + \theta_n\}, \\ x_{n+1} & = & \Pi_{C_{n+1}} x_1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ satisfy:

$$0 \le \alpha_n < 1, \quad \limsup_{n\to\infty} \alpha_n < 1,$$

$$0 < \beta_n < 1, \quad \liminf_{n\to\infty} \beta_n(1-\beta_n) > 0,$$

$$\theta_n = (1-\alpha_n)[(t_n - 1) + (1-\beta_n) t_n (s_n - 1)] \sup\{\phi(z, x_n) : z \in F\}.$$

Then $\lim_{n\to\infty} x_n = q$, where $q = \Pi_{C_0} x_1, C_0 = \bigcap_{n=1}^{\infty} C_n$ and $\lim_{n\to\infty} \|x_n - S^n x_n\| = \|z_n - T^n z_n\| = 0$. Further, if T and S are two closed, asymptotically ϕ-nonexpansive mappings from C into itself, then $q = \Pi_F x_1$.

Proof. The proof will be split into five steps.

Step 1. We show that C_n is closed and convex for each $n \ge 1$.

It is obvious that $C_1 = C$ is closed and convex. Suppose that C_k is closed and convex for some k. For $z \in C_k$, we see that $\phi(z, y_k) \le \phi(z, x_k) + \theta_k$ is equivalent to

$$2\langle z, Jx_k - Jy_k\rangle \le \|x_k\|^2 - \|y_k\|^2 + \theta_k.$$

Hence C_{k+1} is closed and convex. Then, for each $n \ge 1$, C_n is closed and convex.

Step 2. We show that $F \subset C_n$ for all $n \ge 1$.

It is easy to see that $F \subset C_1 = C$. Suppose that $F \subset C_k$ for some k. Then for any $p \in F \subset C_k$, we have

$$\begin{aligned}
\phi(p, z_k) &\leq \phi(p, u_k) = \phi(p, J^{-1}(\beta_k Jx_k + (1-\beta_k)JS^k x_k)) \\
&\leq \|p\|^2 - 2\beta_k\langle p, Jx_k\rangle - 2(1-\beta_k)\langle p, JS^k x_k\rangle \\
&\quad +\beta_k\|x_k\|^2 + (1-\beta_k)\|S^k x_k\|^2 \\
&= \beta_k\phi(p, x_k) + (1-\beta_k)\phi(p, S^k x_k) \\
&\leq \beta_k\phi(p, x_k) + (1-\beta_k)s_k\phi(p, x_k) \\
&= \phi(p, x_k) + (1-\beta_k)(s_k - 1)\phi(p, x_k)
\end{aligned}$$

and then

$$\begin{aligned}
\phi(p, y_k) &= \phi(p, J^{-1}(\alpha_k Jx_k + (1-\alpha_k)JT^k z_k)) \\
&\leq \|p\|^2 - 2\alpha_k\langle p, Jx_k\rangle - 2(1-\alpha_k)\langle p, JT^k z_k\rangle \\
&\quad +\alpha_k\|x_k\|^2 + (1-\alpha_k)\|T^k z_k\|^2 \\
&= \alpha_k\phi(p, x_k) + (1-\alpha_k)\phi(p, T^k z_k) \\
&\leq \alpha_k\phi(p, x_k) + (1-\alpha_k)t_k\phi(p, z_k) \\
&= \phi(p, x_k) + (1-\alpha_k)(t_k\phi(p, z_k) - \phi(p, x_k)) \\
&\leq \phi(p, x_k) + (1-\alpha_k)[t_k\phi(p, x_k) \\
&\quad +t_k(1-\beta_k)(s_k - 1)\phi(p, x_k) - \phi(p, x_k)] \\
&\leq \phi(p, x_k) + \theta_k.
\end{aligned}$$

Thus, we have $p \in C_{k+1}$. Therefore we obtain $F \subset C_n$ for each $n \geq 1$.

Step 3. We show that $\lim_{n\to\infty} x_n = \Pi_{C_0} x_1 = q$.

Since $\{C_n\}$ is a decreasing sequence of closed convex subsets of E such that $F \subset C_0 = \bigcap_{n=1}^{\infty} C_n$ is nonempty, it follows that $M-\lim_{n\to\infty} C_n = C_0 = \bigcap_{n=1}^{\infty} C_n \neq \emptyset$. By Lemma 2.2, $\{x_n\} = \{\Pi_{C_n} x_1\}$ converges strongly to $q = \Pi_{C_0} x_1$.

Step 4. We show that $\lim_{n\to\infty} \|x_n - S^n x_n\| = \|z_n - T^n z_n\| = 0$.

Since $x_n \to q$, we obtain

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{3.2}$$

In view of $x_{n+1} \in C_{n+1}$ and (3.2), we arrive at $\phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, x_n) +\theta_n \to 0$ as $n \to \infty$. From Lemma 2.6, we have $y_n \to q$ and

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.3}$$

Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|Jx_n - Jy_n\| = 0. \tag{3.4}$$

Notice that $\|Jy_n - Jx_n\| = (1-\alpha_n)(JT^nz_n - Jx_n)$. From the assumption on $\{\alpha_n\}$ and (3.4), we see that

$$\lim_{n\to\infty} \|JT^nz_n - Jx_n\| = 0. \tag{3.5}$$

Since $Jx_n \to Jq$, we have $JT^nz_n \to Jq$. The demi-continuity of $J^{-1} : E^* \to E$ implies that $T^nz_n \rightharpoonup q$. Notice that

$$|\|T^nz_n\| - \|q\|| = |\|JT^nz_n\| - \|Jq\|| \leq \|JT^nz_n - Jq\| \to 0.$$

It follows from the Kadec-Klee property of E, we obtain

$$T^nz_n \to q, \text{ as } n \to \infty. \tag{3.6}$$

Since $\|T^nz_n - x_n\| \leq \|T^nz_n - q\| + \|q - x_n\|$, from (3.6), we have

$$\lim_{n\to\infty} \|T^nz_n - x_n\| = 0. \tag{3.7}$$

Since $\{x_n\}$ is bounded, $\phi(p, S^nx_n) \leq s_n\phi(p, x_n)$, where $p \in F$, we also obtain $\{Jx_n\}$, $\{JS^nx_n\}$ are bounded, then there exists $r > 0$ such that $\{Jx_n\}$, $\{JS^nx_n\} \subset B_r$. Therefore Lemma 2.5 is applicable and we observe that

$$\begin{aligned}
\phi(p, z_n) &\leq \phi(p, u_n) \\
&= \|p\|^2 - 2\langle p, \beta_nJx_n + (1-\beta_n)JS^nx_n\rangle + \beta_n\|Jx_n\|^2 \\
&\quad +(1-\beta_n)\|JS^nx_n\|^2 - \beta_n(1-\beta_n)g(\|Jx_n - JS^nx_n\|) \\
&\leq \phi(p, x_n) + (1-\beta_n)(s_n - 1)\phi(p, x_n) \\
&\quad -\beta_n(1-\beta_n)g(\|Jx_n - JS^nx_n\|)
\end{aligned} \tag{3.8}$$

and hence

$$\begin{aligned}
\phi(p, y_n) &\leq \alpha_n\phi(p, x_n) + (1-\alpha_n)t_n\phi(p, z_n) \\
&= \phi(p, x_n) + (1-\alpha_n)(t_n\phi(p, z_n) - \phi(p, x_n)) \\
&\leq \phi(p, x_n) + \theta_n - (1-\alpha_n)t_n\beta_n(1-\beta_n)g(\|Jx_n - JS^nx_n\|).
\end{aligned} \tag{3.9}$$

That is

$$(1-\alpha_n)t_n\beta_n(1-\beta_n)g(\|Jx_n - JS^nx_n\|) \leq \phi(p, x_n) - \phi(p, y_n) + \theta_n. \tag{3.10}$$

From (3.3), (3.4), we have

$$\begin{aligned}
\phi(p, x_n) - \phi(p, y_n) &= 2\langle p, Jy_n - Jx_n\rangle + \|x_n\|^2 - \|y_n\|^2 \\
&= 2\langle p, Jy_n - Jx_n\rangle + (\|x_n\| + \|y_n\|)(\|x_n\| - \|y_n\|) \\
&\to 0.
\end{aligned}$$

By $\limsup_{n\to\infty} \alpha_n < 1$, $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, $\theta_n \to 0$ and (3.10), we have

$$g(\|Jx_n - JS^nx_n\|) \to 0.$$

From the properties of the function g, we obtain

$$\lim_{n\to\infty} \|Jx_n - JS^nx_n\| = 0. \tag{3.11}$$

Since $Jx_n \to Jq$, we have $JS^nx_n \to Jq$. Since J^{-1} is demi-continuous, we obtain $S^nx_n \rightharpoonup q$. Since $\|S^nx_n\| = \|JS^nx_n\| \to \|Jq\| = \|q\|$, by the Kadec-klee property of E, we obtain that $S^nx_n \to q$. Since $\|S^nx_n - x_n\| \leq \|S^nx_n - q\| + \|q - x_n\|$, we have

$$\lim_{n\to\infty} \|S^nx_n - x_n\| = 0. \tag{3.12}$$

Since

$$\|Ju_n - Jx_n\| = (1-\beta_n)\|JS^nx_n - Jx_n\| \to 0, \tag{3.13}$$

therefore, $Ju_n \to Jq$. Hence $\|u_n\| \to \|q\|$. By the demi-continuity of J^{-1}, we have $u_n \rightharpoonup q$. It follows from the Kadec-klee property of E that $u_n \to q$. Since $\|u_n - x_n\| \leq \|u_n - q\| + \|q - x_n\|$, we have

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{3.14}$$

From (A3), (3.13) and (3.14), we have

$$\lim_{n\to\infty} \phi(x_n, u_n) = 0. \tag{3.15}$$

Since $\phi(x_n, z_n) = \phi(x_n, \Pi_C u_n) \leq \phi(x_n, u_n)$, from (3.15), we have $\phi(x_n, z_n) \to 0$. From Lemma 2.6, we have

$$\lim_{n\to\infty} z_n = q \text{ and } \lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{3.16}$$

Since $\|z_n - T^nz_n\| \leq \|z_n - x_n\| + \|x_n - T^nz_n\|$, from (3.16) and (3.7), we have

$$\lim_{n\to\infty} \|z_n - T^nz_n\| = 0. \tag{3.17}$$

Step 5. If T, S are two closed and asymptotically ϕ-nonexpansive mappings from C into itself, we show that $q = \Pi_F x_1$.

We first show that $q \in F$. From (3.11), (3.12) and (A3), we obtain

$$\lim_{n\to\infty} \phi(x_n, S^nx_n) = 0. \tag{3.18}$$

Since J is uniformly norm-to-norm continuous on bounded sets, from (3.17) we have

$$\lim_{n\to\infty} \|Jz_n - JT^nz_n\| = 0. \tag{3.19}$$

It follows from (A3), (3.17) and (3.19) that

$$\lim_{n\to\infty} \phi(T^nz_n, z_n) = 0. \tag{3.20}$$

From (3.16), the continuity of J and the definition of ϕ, we have

$$\lim_{n\to\infty} \phi(z_{n+1}, z_n) = 0. \tag{3.21}$$

It follows from (A2) that

$$\begin{aligned}
&\phi(z_n, Tz_n)\\
&= \phi(z_n, z_{n+1}) + \phi(z_{n+1}, Tz_n) + 2\langle z_n - z_{n+1}, Jz_{n+1} - JTz_n\rangle\\
&= \phi(z_n, z_{n+1}) + \phi(z_{n+1}, T^{n+1}z_{n+1}) + \phi(T^{n+1}z_{n+1}, Tz_n)\\
&\quad +2\langle z_{n+1} - T^{n+1}z_{n+1}, JT^{n+1}z_{n+1} - JTz_n\rangle\\
&\quad +2\langle z_n - z_{n+1}, Jz_{n+1} - JTz_n\rangle.
\end{aligned} \tag{3.22}$$

By $\phi(T^{n+1}z_{n+1}, T^{n+1}z_n) \le t_{n+1}\phi(z_{n+1}, z_n)$, $\phi(T^{n+1}z_n, Tz_n) \le t_1\phi(T^n z_n, z_n)$, (3.21) and (3.20), we have

$$\lim_{n\to\infty} \phi(T^{n+1}z_{n+1}, T^{n+1}z_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} \phi(T^{n+1}z_n, Tz_n) = 0. \tag{3.23}$$

It follows from (3.6), (3.23) and Lemma 2.6 that

$$\lim_{n\to\infty} \|T^{n+1}z_{n+1} - T^{n+1}z_n\| = 0. \tag{3.24}$$

By (A2), we have

$$\begin{aligned}
\phi(T^{n+1}z_{n+1}, Tz_n) &= \phi(T^{n+1}z_{n+1}, T^{n+1}z_n) + \phi(T^{n+1}z_n, Tz_n)\\
&\quad +2\langle T^{n+1}z_{n+1} - T^{n+1}z_n, JT^{n+1}z_n - JTz_n\rangle.
\end{aligned} \tag{3.25}$$

Combining (3.25), (3.24), (3.23), (3.17) with (3.22), we have

$$\lim_{n\to\infty} \phi(z_n, Tz_n) = 0. \tag{3.26}$$

Since $z_n \to q$, by Lemma 2.6, we have

$$\lim_{n\to\infty} \|z_n - Tz_n\| = 0. \tag{3.27}$$

Similarly, from $x_n \to q$, (3.2), (3.12), (3.18) and Lemma 2.6, we can obtain

$$\lim_{n\to\infty} \|Sx_n - x_n\| = 0. \tag{3.28}$$

Since $x_n \to q$ and $z_n \to q$, by the closedness of S and T and (3.27), (3.28), we have $q \in F$.

Next, we show that $q = \Pi_F x_1$. Since $q = \Pi_{C_0} x_1 \in F$ and F is a nonempty closed convex subset of $C_0 = \bigcap_{n=1}^{\infty} C_n$, we conclude that $q = \Pi_F x_1$. □

Remark 3.2. Theorem 3.1 is a version of Theorem PU in Banach space. The hybrid projection algorithm considered in Theorem 3.1 is simpler than that of Theorem PU, because we can remove the set "Q_n". In addition, we do not assume that C is bounded as in Theorem PU, but the common fixed point set of T and S is bounded instead.

Remark 3.3. Theorem 3.1 is different from Theorem 2.1 of the [14] in the following senses:

(a) We develop Theorem 2.1 of the [14] from a single closed, asymptotically quasi-ϕ-nonexpansive mapping to two closed asymptotically ϕ-nonexpansive mappings.
(b) We remove the asymptotically regularity on T or S as in Theorem 2.1 of the [14].

Corollary 3.4. *Let E be a uniformly smooth and strictly convex Banach space with the Kadec-Klee property, let C be a nonempty closed convex subset of E, let T, S be two closed and quasi-ϕ-nonexpansive mappings from C into itself such that $F = F(T) \bigcap F(S) \neq \emptyset$. Then the sequence $\{x_n\}$ generated by*

$$\begin{cases} x_0 & \in & E \quad \text{chosen arbitrarily}, \\ C_1 & = & C, \\ x_1 & = & \Pi_{C_1} x_0, \\ u_n & = & J^{-1}(\beta_n J x_n + (1-\beta_n) J S x_n), \\ z_n & = & \Pi_C u_n, \\ y_n & = & J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T z_n), \\ C_{n+1} & = & \{z \in C_n : \phi(z, y_n) \le \phi(z, x_n)\}, \\ x_{n+1} & = & \Pi_{C_{n+1}} x_1, \end{cases} \tag{3.28}$$

where $\{\alpha_n\}$,$\{\beta_n\}$ satisfy:

$$0 \le \alpha_n < 1, \quad \limsup_{n\to\infty} \alpha_n < 1,$$

$$0 < \beta_n < 1, \quad \liminf_{n\to\infty} \beta_n(1-\beta_n) > 0,$$

converges strongly to $\Pi_F x_1$.

Proof. From the definition of quasi-ϕ-nonexpansive mappings, we see that every quasi-ϕ-nonexpansive mapping is asymptotically quasi-ϕ-nonexpansive with the constant sequence $\{1\}$. From the proof of Theorem 3.1, we have $F \subset C_n$ for all $n \ge 1$, $\lim_{n\to\infty} x_n = \Pi_{C_0} x_1 = q$, where $C_0 = \bigcap_{n=1}^{\infty} C_n$ and $\lim_{n\to\infty} \|x_n - Sx_n\| = \|z_n - T^n z_n\| = 0$. By the closedness of S and T, we have $q \in F$. Since $q = \Pi_{C_0} x_1 \in F$ and F is a nonempty closed convex subset of $C_0 = \bigcap_{n=1}^{\infty} C_n$, we conclude that $q = \Pi_F x_1$. □

Remark 3.5. Corollary 3.4 improves Theorem 3.1 of [7] in the following senses:

(a) Since T and S are two quasi-ϕ-nonexpansive mappings, we remove the restrictions $F(T) = \hat{F}(T)$ and $F(S) = \hat{F}(S)$.
(b) The hybrid projection algorithm considered in Corollary 3.1 is simpler than that of Theorem 3.1 of [7], because we can remove the set "Q_n".

(c) The uniformly smooth and strictly convex Banach spaces with the Kadec-Klee property considered by Corollary 3.4 are more general than the uniformly smooth and uniformly convex Banach spaces considered by Theorem 3.1 of [7].

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