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HYBRID FIXED POINT THEORY WITH PPF DEPENDENCE IN BANACH ALGEBRAS WITH APPLICATIONS TO NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, a couple of hybrid fixed point theorems with PPF dependence are proved in a Banach algebra involving three systems of operators and they are then applied to some nonlinear hybrid functional differential equations of delay and neutral type for proving the existence of PPF dependent solutions under certain mixed Lipschitz and compactness type conditions. Our abstract results as well as considered functional differential equations are new to the literature in the subject of nonlinear analysis.

1. Introduction

In recent papers [1, 7], the authors proved some fundamental fixed point theorems for nonlinear operators in a Banach space satisfying the conditions of linear contractions, wherein the domain and range of the operators are not same. The fixed point theorems of this kind are called PPF dependent fixed point theorems and are useful for proving the existence (and uniqueness) of solutions of nonlinear functional differential and integral equations which may depend upon the past, present and future. The properties of a special minimal or Razumikhin or $\mathcal{D}$-*class* of functions are employed in the development of existence theory of PPF dependent solutions for certain nonlinear equations in abstract spaces. In this paper, we prove some new hybrid fixed point theorems with PPF dependence involving three systems of nonlinear operators in a

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Banach algebra and discuss some of their applications to nonlinear functional hybrid differential equations for proving the existence of PPF dependent solutions. Our hybrid fixed point theorems as well as existence theorems include some known results concerning the PPF dependence as special cases.

Given a Banach space E with norm $\|\cdot\|_E$ and given a closed and bounded interval $I = [a, b]$ in $\mathbb{R}$, the set of real numbers, let $E_0 = C(I, E)$ denote the Banach space of continuous E-valued continuous functions defined on I. We equip the space E_0 with the supremum norm $\|\cdot\|_{E_0}$ defined as

$$\|\phi\|_{E_0} = \sup_{t\in I} \|\phi(t)\|_E. \tag{1.1}$$

Let $c \in I$ be arbitrarily fixed. The *minimal* or *Razumikhin class* or $\mathcal{D}$*-class* of functions (cf. [1, 7]) is defined as

$$\mathcal{M}_c = \big\{\phi \in E_0 \mid \|\phi\|_{E_0} = \|\phi(c)\|_E\big\}. \tag{1.2}$$

A Razumikhin class of functions $\mathcal{M}_c$ is said to be algebraically closed w.r.t. difference if $\phi - \xi \in \mathcal{M}_c$ whenever $\phi, \xi \in \mathcal{M}_c$. Similarly, $\mathcal{M}_c$ is topologically closed if it is closed in the topology of of E_0 generated by the norm $\|\cdot\|_{E_0}$. Similarly, other notions such as compactness and connectedness for $\mathcal{M}_c$ may be defined.

Let $T : E_0 \to E$. A point $\phi^* \in E_0$ is called a PPF dependent fixed point of T if $T\phi^* = \phi^*(c)$ for some $c \in I$ and any statement that guarantees the existence of PPF dependent fixed point is called a fixed point theorem with PPF dependence for the mappings T.

As mentioned in Bernfield *et al.* [1], the *minimal class* of functions plays a significant role in proving the existence of PPF dependent fixed points with different domain and range of the operators. Very recently, generalizing a fixed point theorem of Bernfield *et al.* [1], the present author in Dhage [4] proved first some fixed point theorems with PPF dependence in the setting of nonlinear contractions of the operators in Banach spaces.

Definition 1.1. A nonlinear operator $T : E_0 \to E$ is called a nonlinear $(\mathcal{B}, \mathcal{W})$- contraction if there exists an upper continuous function from the right $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|T\phi - T\xi\|_E \leq \psi(\|\phi - \xi\|_{E_0}) \tag{1.3}$$

for all $\phi, \xi \in E_0$, where $\psi(r) < r$, $r > 0$. T is called $\mathcal{B}$-contraction if there exists a nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ which is continuous from right and satisfies (1.3). Finally, T is called $\mathcal{M}$-contraction if there exists a nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ that satisfies satisfies (1.3), where $\lim_{n\to\infty} \psi^n(t) = 0$,

$t > 0$. We say T is nonlinear contraction if it is either a nonlinear $(\mathcal{D}, \mathcal{W})$ or $\mathcal{B}$ or $\mathcal{M}$-contraction on E_0 into E.

Note that every contraction is a nonlinear $(\mathcal{D}, \mathcal{W})$-contraction and every nonlinear $\mathcal{B}$-contraction is $\mathcal{M}$-contraction. However, the converse of the above statements may not be true. The details of different types of contractions appear in the monographs of Boyd and Wong [2], Browder [3], Granas and Dugundji [8], Krasnoselskii [9] and Mathowski [10]. The following fixed point theorem is a slight generalization of a fixed point theorem proved in Dhage [4] with PPF dependence.

Theorem 1.1. *Suppose that $T : E_0 \to E$ is a nonlinear contraction. Then the following statements hold in E_0.*

(a) *If $\mathcal{M}_c$ is algebraically closed with respect to difference, then every sequence $\{\phi_n\}$ of successive iterates of T at each point $\phi_0 \in E_0$ converges to a PPF dependent fixed point of T.*
(b) *If $\mathcal{M}_c$ is topologically closed, then ϕ^* is the only fixed point of T in $\mathcal{M}_c$.*

Proof. The proof is similar to Theorem 2.3 of Dhage [4] and can be obtained with appropriate modifications. Hence we omit the details. □

In this paper, we prove some hybrid fixed point theorems with PPF dependence in a Banach algebra using mixed arguments form analysis and topology and apply them to hybrid differential equations of functional differential equations of delay and neutral type for proving the existence of solutions with PPF dependence.

2. PPF dependent hybrid fixed point theory

Throughout subsequent part of this paper, unless otherwise specified, let E denote a Banach algebra with norm $\|\cdot\|_E$. Then $E_0 = C(I, E)$ becomes a Banach algebra with respect to the norm (1.1) and the multiplication " $\cdot$ " defined by

$$(x \cdot y)(t) = x(t) \cdot y(t) = x(t)y(t)$$

for all $t \in I$, whenever $x, y \in E_0$. When there is no confusion, we simply write xy instead of $x \cdot y$.

While working on fixed point theorems in abstract algebras, the present author introduced a class of $\mathcal{D}$-functions to define the growth of the operators in question. We mention that $\mathcal{D}$-functions are in line with the the growth functions mentioned in Definition 1.1 and are useful in practical applications to nonlinear differential equations. Here also we employ same notations and terminologies in what follows.

Definition 2.1. A mapping $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a *dominating function* or, in short, *$\mathcal{D}$-function* if it is upper semi-continuous and nondecreasing function satisfying $\psi(0) = 0$. A mapping $Q : E_0 \to E$ is called *strong $\mathcal{D}$-Lipschitz* if there is a $\mathcal{D}$-function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$\|Q\phi - Q\xi\|_E \leq \psi(\|\phi(c) - \xi(c)\|_E) \tag{2.1}$$

for all $\phi, \xi \in E$. The function ψ is called a $\mathcal{D}$-function of Q on E. If $\psi(r) = k\,r$, $k > 0$, then Q is called *strong Lipschitz* with the Lipschitz constant k. In particular, if $k < 1$, then Q is called a *strong contraction* on X with the contraction constant k. Further, if $\psi(r) < r$ for $r > 0$, then Q is called *strong nonlinear $\mathcal{D}$-contraction* and the function ψ is called $\mathcal{D}$-function of Q on X.

There do exist $\mathcal{D}$-functions and the commonly used $\mathcal{D}$-functions are $\psi(r) = k\,r$ and $\psi(r) = \dfrac{r}{1+r}$, etc. These $\mathcal{D}$-functions have been used in the theory of nonlinear differential and integral equations for proving the existence results via fixed point methods. Another notion that we need in the sequel is the following definition.

Definition 2.2. An operator Q on a Banach space E into itself is called compact if $Q(E)$ is a relatively compact subset of E. Q is called totally bounded if for any bounded subset S of E, $Q(S)$ is a relatively compact subset of E. If Q is continuous and totally bounded, then it is called completely continuous on E.

Note every compact operator is totally bounded but the converse may not be true. Our main hybrid fixed point theorem with PPF dependence is the following result in a Banach algebra E.

Theorem 2.1. *Let E be a Banach algebra and let $A_i, C_j : E_0 \to E$ and $B_i : E \to E$ for $1 \leq i \leq k$ and $1 \leq j \leq l$ be three systems of operators such that for each i and j,*

(a) *A_i is bounded and strong $\mathcal{D}$-Lipschitz with the $\mathcal{D}$-function ψ_{A_i},*
(b) *C_j is strong $\mathcal{D}$-Lipschitz with the $\mathcal{D}$-function ψ_{C_j},*
(c) *B_i is continuous and compact, and*
(d) $\displaystyle\sum_{i=1}^{k} M_i \psi_{A_i}(r) + \sum_{i=j}^{l} \psi_{C_j}(r) < r$ *if* $r > 0$,
where $M_i = \|B_i(E)\| = \sup\{\|B_i x\| : x \in E\}$.

Further, if the minimal class of functions $\mathcal{M}_c$ is topologically and algebraically closed with respect to difference, then for a given $c \in [a, b]$ the operator equation

$$\sum_{i=1}^{k} A_i\phi\, B_i\phi(c) + \sum_{i=j}^{l} C_j\phi = \phi(c) \tag{2.2}$$

has a PPF dependent solution.

Proof. Let $\xi \in E_0$ be fixed and let $c \in [a,b]$ be given. Define an operator $T_{\xi(c)} : E_0 \to E$ by

$$T_{\xi(c)}(\phi) = \sum_{i=1}^{k} A_i\phi\, B_i\xi(c) + \sum_{i=j}^{l} C_j\phi. \tag{2.3}$$

Clearly, $T_{\xi(c)}$ is a strong nonlinear $\mathcal{B}$-contraction on E_0. To see this, let $\phi_1, \phi_2 \in E_0$. Then,

$$\begin{aligned}
&\|T_{\xi(c)}(\phi_1) - T_{\xi(c)}(\phi_2)\|_E \\
&\leq \sum_{i=1}^{k} \|A_i\phi_1 - A_i\phi_2\|_E\, \|B_i\xi(c)\|_E + \sum_{i=j}^{l} \|C_j\phi_1 - C_j\phi_2\|_E \\
&\leq \sum_{i=1}^{k} \|B_i(E)\|_E\, \psi_{A_i}(\|\phi_1(c) - \phi_2(c)\|_E) \qquad (2.4)\\
&\qquad + \sum_{j=1}^{l} \psi_{C_j}(\|\phi_1(c) - \phi_2(c)\|_E) \\
&\leq \sum_{i=1}^{k} M_i\, \psi_{A_i}(\|\phi_1(c) - \phi_2(c)\|_E) + \sum_{i=j}^{l} \psi_{C_j}(\|\phi_1(c) - \phi_2(c)\|_E).
\end{aligned}$$

This shows that $T_{\xi(c)}$ is a strong nonlinear $\mathcal{D}$-contraction and hence nonlinear $\mathcal{D}$-contraction on E_0. By Theorem 1.1, there is a unique PPF dependent fixed point $\phi^* \in E_0$ such that

$$T_{\xi(c)}(\phi^*) = \phi^*(c) \quad \text{or} \quad \sum_{i=1}^{k} A_i\phi^*\, B_i\xi(c) + \sum_{j=1}^{l} C_j\phi^*(c) = \phi^*(c). \tag{2.5}$$

Next, we define a mapping $Q : E \to E$ by

$$Q\xi(c) = \phi^*(c) = \sum_{i=1}^{k} A_i\phi^*\, B_i\xi(c) + \sum_{j=1}^{l} C_j\phi^*. \tag{2.6}$$

It then follows that

$$\begin{aligned}
&\|Q\xi_1(c) - Q\xi_2(c)\|_E \\
&= \sum_{i=1}^{k} \|A_i\phi_1^*\, B_i\xi_1(c) - A_i\phi_2^*\, B_i\xi_2(c)\|_E + \sum_{j=1}^{l} \|C_j\phi_1^* - C_j\phi_2^*\|_E \\
&\leq \sum_{i=1}^{k} \|A_i\phi_1^* - A_i\phi_2^*\|_E\, \|B_i\xi_1\|_E + \sum_{i=1}^{k} \|A_i\phi_2\|_E\, \|B_i\xi_1(c) - B_i\xi_2(c)\|_E
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{l}\|C_j\phi_1^*-C_j\phi_2^*\|_E\\
&\le\sum_{i=1}^{k}M_i\,\psi_{A_i}(\|\phi_1^*(c)-\phi_2^*(c)\|_E)+\sum_{i=1}^{k}K_i\,\|B_i\xi_1(c)-B_i\xi_2(c)\|_E\\
&\quad+\sum_{j=1}^{l}\psi_{C_j}(\|\phi_1^*(c)-\phi_2^*(c)\|_E)\\
&\le\sum_{i=1}^{k}M_i\,\psi_{A_i}(\|\phi_1^*(c)-\phi_2^*(c)\|_E)+\sum_{j=1}^{l}\psi_{C_j}(\|\phi_1^*(c)-\phi_2^*(c)\|_E)\\
&\quad+\sum_{i=1}^{k}K_i\,\|B_i\xi_1(c)-B_i\xi_2(c)\|_E
\end{aligned}
\tag{2.7}
$$

where K_i is a bound of A_i on E_0. Since each B_i is compact, if $\{B_i\xi_n(c)\}$ is any sequence in E, then $\{B_i\xi_n(c)\}$ has a convergent subsequence. Without loss of generality, we may assume that $\{B_i\xi_n(c)\}$ is convergent. Hence, $\{B_i\xi_n(c)\}$ is a Cauchy sequence. From inequality (2.7), we obtain

$$
\begin{aligned}
&\|Q\xi_m(c)-Q\xi_n(c)\|_E\\
&\le\sum_{i=1}^{k}M_i\,\psi_{A_i}(\|\phi_m^*(c)-\phi_n^*(c)\|_E)+\sum_{j=1}^{l}\psi_{C_j}(\|\phi_m^*(c)-\phi_n^*(c)\|_E)\\
&\quad+\sum_{i=1}^{k}K_i\,\|B_i\xi_m(c)-B_i\xi_n(c)\|_E.
\end{aligned}
$$

Taking the limit superior in above inequality yields

$$
\begin{aligned}
&\limsup_{m,n\to\infty}\|Q\xi_m(c)-Q\xi_n(c)\|_E\\
&\le\sum_{i=1}^{k}M_i\,\limsup_{m,n\to\infty}\psi_{A_i}(\|Q\xi_m^*(c)-Q\xi_n^*(c)\|_E)\\
&\quad+\sum_{j=1}^{l}\limsup_{m,n\to\infty}\psi_{C_j}(\|Q\xi_m^*(c)-Q\xi_n^*(c)\|_E)\\
&\quad+\sum_{i=1}^{k}K_i\,\limsup_{m,n\to\infty}\|B_i\xi_m(c)-B_i\xi_n(c)\|_E
\end{aligned}
$$

$$\leq \sum_{i=1}^{k} M_i\,\psi_{A_i}\left(\limsup_{m,n\to\infty}\|Q\xi_m^*(c)-Q\xi_n^*(c)\|_E\right) + \sum_{j=1}^{l}\psi_{C_j}\left(\limsup_{m,n\to\infty}\|Q\xi_m^*(c)-Q\xi_n^*(c)\|_E\right).$$

Hence,

$$\lim_{m,n\to\infty}\|Q\xi_m(c)-Q\xi_n(c)\|_E = 0.$$

As a result, $\{Q\xi_n(c)\}$ is a Cauchy sequence. Since E is complete, $\{Q\xi_n(c)\}$ has a convergent subsequence. Now a direct application of Schauder fixed point principle yields that there is a point $\xi \in E_0$ such that $Q\xi^*(c) = \xi^*(c)$. Consequently $\sum_{i=1}^{k} A_i\xi^*\,B_i\xi^*(c) + \sum_{j=1}^{l} C_j\xi^* = \xi^*(c)$. This completes the proof of Theorem 2.1. □

Theorem 2.2. *Let E be a Banach algebra and let $A_i : E_0 \to E$ and $B_i, C_j : E \to E$ for $1 \leq i \leq k$ and $1 \leq j \leq l$, be three systems of operators such that*

(a) *A_i is bounded and strong $\mathcal{D}$-Lipschitz with the $\mathcal{D}$-function ψ_{A_i},*
(b) *B_i is continuous and compact,*
(c) *C_j is continuous and compact, and*
(d) *$\sum_{i=1}^{k} M_i\psi_{A_i}(r) < r \quad$ if $r > 0$,*
where $M_i = \|B_i(E)\| = \sup\{\|B_i x\| : x \in E\}$.

Further, if the minimal class of functions $\mathcal{M}_c$ is algebraically closed with respect to difference and topologically closed, then for a given $c \in [a,b]$ the operator equation

$$\sum_{i=1}^{k} A_i\phi\,B_i\phi(c) + \sum_{j=1}^{l} C_j\phi(c) = \phi(c) \tag{2.8}$$

has a PPF dependent solution.

Proof. The proof is similar to Theorem 2.2 with appropriate modifications. □

Remark 2.1. If we consider Theorems 2.1 and 2.2 in a closed, convex and bounded subset of the Banach algebra E, then condition of the boundedness of the operator A_i is not required because in that case the boundedness of A_i follows immediately from the strong Lipschitz condition.

Remark 2.2. If we take $\psi_{\mathcal{A}_i}(r) = \frac{L_i r}{K_i + r}$ and $\psi_{\mathcal{C}_|}(r) = q_j\, r$, then hypothesis (d) of the above hybrid fixed point theorem takes the form $\sum_{i=1}^{k} \frac{L_i}{K_i+r} + \sum_{j=1}^{l} q_j < 1$ for each real number $r > 0$. Similarly, if $\psi_{\mathcal{A}_i}(r) = L_i\, r$, and $\psi_{\mathcal{C}_j}(r) = \frac{q_j r}{N_j + r}$, then hypothesis (d) of the above hybrid fixed point theorem takes the form $\sum_{i=1}^{k} L_i + \sum_{j=1}^{l} \frac{q_j}{N_j+r} < 1$ for each real number $r > 0$.

In view of above remark, we obtain the following special cases of Theorems 2.1 and 2.2 as corollaries which are applicable to various nonlinear equations in the subject of nonlinear analysis.

Corollary 2.1. *Let E be a Banach algebra and let $A_i, C_j : E_0 \to E$ and $B_i : E \to E$ for $1 \leq i \leq k$ and $1 \leq j \leq l$, be three systems of operators such that for each i and j,*

(a) *A_i is bounded and strong Lipschitz with the Lipschitz constant ℓ_i,*
(b) *C_j is strong Lipschitz with the Lipschitz constant q_j,*
(c) *B_i is continuous and compact, and*
(d) $\displaystyle\sum_{i=1}^{k} M_i \ell_i + \sum_{j=1}^{l} q_j < 1$, *where* $M_i = \|B_i(E)\| = \sup\{\|B_i x\| : x \in E\}$.

Further, if the minimal class of functions $\mathcal{M}_c$ is topologically and algebraically closed with respect to difference, then for a given $c \in [a, b]$ the operator equation (2.2) *has a PPF dependent solution.*

Corollary 2.2. *Let E be a Banach algebra and let $A_i : E_0 \to E$ and $B_i, C_j : E \to E$ for $1 \leq i \leq k$ and $1 \leq j \leq l$, be three systems of operators such that for each i and j,*

(a) *A_i is bounded and strong Lipschitz with the Lipschitz constant ℓ_i,*
(b) *B_i is continuous and compact,*
(c) *C_j is continuous and compact, and*
(d) $\displaystyle\sum_{i=1}^{k} M_i \ell_i < 1$, *where* $M_i = \|B_i(E)\| = \sup\{\|B_i x\| : x \in E\}$.

Further, if the minimal class of functions $\mathcal{M}_c$ is algebraically closed with respect to difference and topologically closed, then for a given $c \in [a, b]$ the operator equation (2.6) *has a PPF dependent solution.*

Notice that Theorem 3.1 includes the following interesting fixed point results involving a couple of systems of operators satisfying the mixed conditions and which are useful in applications to nonlinear perturbed differential and integral equations. We mention that these results are also new to the literature on abstract fixed point theory on the lines of Dhage [4, 5] and Krasnoselskii [9].

Theorem 2.3. *Let E be a Banach space and let $A_i : E_0 \to E$ and $B_j : E \to E$ for $1 \leq i \leq k$ and $1 \leq j \leq l$ be two systems of operators such that for each i and j,*

(a) *A_i is bounded and strong $\mathcal{D}$-Lipschitz with the $\mathcal{D}$-function ψ_{A_i},*
(b) *B_j is continuous and compact, and*
(c) $\sum_{i=1}^{k} \psi_{A_i}(r) < r$ *if* $r > 0$.

Further, if the minimal class of functions $\mathcal{M}_c$ is topologically and algebraically closed with respect to difference, then for a given $c \in [a, b]$ the operator equation

$$\sum_{i=1}^{k} A_i \phi + \sum_{i=j}^{l} B_j \phi(c) = \phi(c) \tag{2.9}$$

has a PPF dependent solution.

Theorem 2.4. *Let E be a Banach algebra and let $A_i : E_0 \to E$ and $B_i : E \to E$ for $1 \leq i \leq k$ be two systems of operators such that for each i,*

(a) *A_i is bounded and strong $\mathcal{D}$-Lipschitz with the $\mathcal{D}$-function ψ_{A_i},*
(b) *B_j is continuous and compact, and*
(c) $\sum_{i=1}^{k} M_i \psi_{A_i}(r) < r$ *if* $r > 0$,
where $M_i = \|B_i(E)\| = \sup\{\|B_i x\| : x \in E\}$.

Further, if the minimal class of functions $\mathcal{M}_c$ is topologically and algebraically closed with respect to difference, then for a given $c \in [a, b]$ the operator equation

$$\sum_{i=1}^{k} A_i \phi \, B_i \phi(c) = \phi(c) \tag{2.10}$$

has a PPF dependent solution.

3. Applications

In this section, we apply the abstract results of the previous section to functional differential equations for proving the existence of solutions under a weaker Lipschitz condition. Given a closed interval $I_0 = [-r, 0]$ in $\mathbb{R}$ for some real number $r > 0$, let $\mathcal{C}$ denote the space of continuous real-valued functions defined on I_0. We equip the space $\mathcal{C}$ with supremum norm $\|\cdot\|_{\mathcal{C}}$ defined by

$$\|\phi\|_{\mathcal{C}} = \sup_{\theta \in I_0} |\phi(\theta)|. \tag{3.1}$$

It is clear that $\mathcal{C}$ is a Banach space with respect to this norm called the history space of the problems under consideration.

Given the closed and bounded interval $J = [-r, T]$ in $\mathbb{R}$, let $C(J, \mathbb{R})$ denote the Banach space of continuous and real-valued functions defined on J with the usual supremum norm $\| \cdot \|$. Given a function $x \in C(J, \mathbb{R})$, for each $t \in I = [0, T]$, define a function $t \to x_t \in \mathcal{C}$ by

$$x_t(\theta) = x(t + \theta), \quad \theta \in I_0, \tag{3.2}$$

where the argument θ represents the delay in the argument of solutions.

Now we are well equipped with the necessary details to study the nonlinear problems of functional differential equations.

3.1. **Functional differential equation of delay type.** Given a function $\phi \in \mathcal{C}$, consider the perturbed or a hybrid differential equation of functional differential equations of delay type (in short HDE),

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{x(t) - k(t, x(t))}{f(t, x(t))}\right] &= \sum_{i=1}^{k} g_i(t, x_t) \\ x_0 &= \phi \end{aligned}\right\} \tag{3.3}$$

for all $t \in I$, where $f : I \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ and $g : I \times \mathcal{C} \to \mathbb{R}$ are continuous.

By a solution x of the HDE (3.3) we mean a function $x \in C(J, \mathbb{R})$ that satisfies

(i) the function $t \mapsto \dfrac{x - k(t, x)}{f(t, x)}$ is continuous in I for each $x \in \mathbb{R}$, and

(ii) x satisfies the equations in (3.3) on J,

where $C(J, \mathbb{R})$ is the space of continuous real-valued functions defined on $J = I_0 \cup I$.

The HDE (3.3) is a generalization of the much studied functional differential equation

$$\left.\begin{aligned} x'(t) &= \sum_{i=1}^{k} g_i(t, x_t) \\ x_0 &= \phi \end{aligned}\right\} \tag{3.4}$$

and includes the new HDEs

$$\left.\begin{aligned} \frac{d}{dt}\left[x(t) - k(t, x(t))\right] &= \sum_{i=1}^{k} g_i(t, x_t) \\ x_0 &= \phi \end{aligned}\right\} \tag{3.5}$$

and

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{x(t)}{f(t,x(t))}\right] &= \sum_{i=1}^{k} g_i(t,x_t) \\ x_0 &= \phi. \end{aligned}\right\} \tag{3.6}$$

as special cases. Therefore, the existence results of this paper include the existence results for the above hybrid functional differential equations.

We consider the following hypotheses in what follows.

(H_1) There exist real numbers $L_i > 0$ and $K_i > 0$ such that

$$|g_i(t,x) - g_i(t,y)| \leq \frac{L_i|x(0) - y(0)|}{K_i + |x(0) - y(0)|}$$

for all $t \in I$ and $x, y \in \mathcal{C}$.

(H_2) The function f is uniformly continuous and there exists a real number $M_f > 0$ such that

$$0 < |f(t,x)| \leq M_f$$

for all $t \in I$ and $x \in \mathbb{R}$.

(H_3) The function h is uniformly continuous and there exists a real number $M_h > 0$ such that

$$|h(t,x)| \leq M_h$$

for all $t \in I$ and $x \in \mathbb{R}$.

Remark 3.1. If $L_i < K_i$ in hypothesis (H_1), then it reduces to the usual Lipschitz condition of g, namely,

$$|g_i(t,x) - g_i(t,y)| \leq (L_i/K_i)|x(0) - y(0)|$$

for all $t \in I$ and $x, y \in \mathcal{C}$.

Theorem 3.1. *Assume that the hypotheses* (H_1) *through* (H_3) *hold. Furthermore, if*

$$\sum_{i=1}^{k} \frac{L_i T \max\{M_f, 1\}\, r}{K_i + r} < r, \quad r > 0,$$

then the HDE (3.3) *has a solution defined on* J.

Proof. Set $E = C(J,\mathbb{R})$. Then E is a Banach algebra with respect to the usual supremum norm $\|\cdot\|_E$ and the multiplication " $\cdot$ " defined by

$$(x \cdot y)(t) = x(t) \cdot y(t) = x(t)y(t)$$

for all $t \in I$, whenever $x, y \in E$.

Define a set of functions

$$\widehat{E} = \{\hat{x} = (x_t)_{t\in I} : x_t \in \mathcal{C}, x \in C(I,\mathbb{R}) \text{ and } x_0 = \phi\}. \tag{3.7}$$

Define a norm $\|\hat{x}\|_{\widehat{E}}$ in $\widehat{E}$ by

$$\|\hat{x}\|_{\widehat{E}} = \sup_{t\in I} \|x_t\|_{\mathcal{C}}. \tag{3.8}$$

Clearly, $\hat{x} \in C(I_0,\mathbb{R}) = \mathcal{C}$. Next we show that $\widehat{E}$ is a Banach space. Consider a Cauchy sequence $\{\hat{x}_n\}$ in $\hat{E}$. Then, $\{(x_t^n)_{t\in I}\}$ is a Cauchy sequence in $\mathcal{C}$ for each $t \in I$. This further implies that $\{x_t^n(s)\}$ is a Cauchy sequence in $\mathbb{R}$ for each $s \in [-r, 0]$. Then $\{x_t^n(s)\}$ converges to $x_t(s)$ for each $t \in I_0$. Since $\{x_t^n\}$ is a sequence of uniformly continuous functions for a fixed $t \in I$, $x_t(s)$ is also continuous in $s \in [-r, 0]$. Hence the sequence $\{\hat{x}_n\}$ converges to $\hat{x} \in \hat{E}$. As a result, $\widehat{E}$ is Banach space.

Now the HDE (3.3) is equivalent to the nonlinear hybrid integral equation (in short HIE)

$$x(t) = \begin{cases} h(t,x(t)) + \big[f(t,x(t))\big]\bigg(\dfrac{\phi(0) - k(0,\phi(0)}{f(0,\phi(0))} \\ \qquad\qquad + \displaystyle\sum_{i=1}^{k}\int_0^t g_i(s,x_s)\,ds\bigg), & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0. \end{cases} \tag{3.9}$$

Consider the operators $A : \widehat{E} \to \mathbb{R}$, $B : C(J,\mathbb{R}) \to \mathbb{R}$ and $C : C(J,\mathbb{R}) \to \mathbb{R}$ defined by

$$A_i\hat{x} = A_i(x_t)_{t\in I} = \begin{cases} \dfrac{\phi(0) - k(0,\phi(0)}{f(0,\phi(0))} + \displaystyle\int_0^t g_i(s,x_s)\,ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0, \end{cases} \tag{3.10}$$

$$Bx(t) = \begin{cases} f(t,x(t)), & \text{if } t \in I, \\ 1, & \text{if } t \in I_0, \end{cases} \tag{3.11}$$

and

$$Cx(t) = \begin{cases} k(t,x(t)), & \text{if } t \in I, \\ 0, & \text{if } t \in I_0. \end{cases} \tag{3.12}$$

Then the HIE (3.9) is equivalent to the operator equation

$$\sum_{i=1}^{k} A_i\hat{x}\,B\hat{x}(0) + C\hat{x}(0) = \hat{x}(0). \tag{3.13}$$

We shall show that the operators A_i, B and C satisfy all the conditions of Theorem 2.2. First we show that A_i is a bounded operator on $\widehat{E}$ into E. Now for any $\hat{x} \in \widehat{E}$, one has

$$\begin{aligned}\|A_i\hat{x}\|_E &\le \|A_i(0)\|_E + \|A_i(x_t)_{t\in I} - A_i(0)\|_E \\ &\le \|A_i(0)\|_E + \left|\int_0^t g_i(s,x_s)\,ds - \int_0^t g_i(s,0)\,ds\right| \\ &\le \|A_i(0)\|_E + \int_0^t \frac{L_i|x_s(0)-0|}{K_i + \|x_s(0)-0\|_{\mathcal{C}}}\,ds \\ &\le \|A_i(0)\|_E + \int_0^t \frac{L_i\|\hat{x}(0)\|_{\widehat{E}}}{K_i + \|\hat{x}(0)\|_{\widehat{E}}}\,ds \\ &\le \|A_i(0)\|_E + L_iT,\end{aligned}$$

which shows that A_i is a bounded operator on $\widehat{E}$ with bound $\|A_i(0)\|_E + L_iT$.

Next, we prove that A is a strong $\mathcal{D}$-Lipschitz on $\widehat{E}$. Then,

$$\begin{aligned}\|A_i\hat{x} - A_i\hat{y}\|_E &= \|A_i(x_t)_{t\in I} - A_i(y_t)_{t\in I}\| \\ &= \left|\int_0^t g_i(s,x_s)\,ds - \int_0^t g_i(s,y_s)\,ds\right| \\ &\le \int_0^t \frac{L_i|x_s(0)-y_s(0)|}{K_i + |x_s(0)-y_s(0)|}\,ds \\ &\le \int_0^t \frac{L_i\|\hat{x}(0)-\hat{y}(0)\|_{\widehat{E}}}{K_i + \|\hat{x}(0)-\hat{y}(0)\|_{\widehat{E}}}\,ds \\ &= \psi_{A_i}(\|\hat{x}(0)-\hat{y}(0)\|_E)\end{aligned}$$

for all $\hat{x}, \hat{y} \in \widehat{E}$, where $\psi_{A_i}(r) = \dfrac{L_iTr}{K_i + r}$. Hence, A_i is a strong $\mathcal{D}$-Lipschitz on $\widehat{E}$ with $\mathcal{D}$-function ψ_{A_i}.

Next, we show that B is compact and continuous operator on $C(J,\mathbb{R})$. Let $\{x_n\}$ be a sequence in $C(J,\mathbb{R})$ such that $x_n \to x$ as $n \to \infty$. Then by continuity of f,

$$\lim_{n\to\infty} Bx_n(t) = \lim_{n\to\infty} f(s,x_n(s)) = f(s,x(s)) = Bx(t)$$

for all $t \in I$. Similarly, if $t \in I_0$, then $\lim_{n\to\infty} Bx_n(t) = 1 = Bx(t)$. This shows that $\{Bx_n(t)\}$ converges to $Bx(t)$ point-wise on J. But $\{Bx_n(t)\}$ is a sequence of uniformly continuous functions on J, so $Bx_n \to Bx$ uniformly. Hence, B is a continuous operator on E into itself.

Secondly, we show that B is compact. To finish, it is enough to show that $B(E)$ is uniformly bounded and equi-continuous set in E. Let $x \in E$ be arbitrary. Then,

$$|Bx(t)| \le |f(s,x(s))| \le M_f$$

for all $t \in J$, and $|Bx(t)| \le 1$ for all $t \in I_0$. From this it follows that

$$|Bx(t)| \le \max\{M_f\,,\,1\} = M^*$$

for all $t \in J$, whence B is uniformly bounded on E.

To show equi-continuity, let $t, \tau \in I$. Then, from the uniform continuity of f it follows that

$$|Bx(t) - Bx(\tau)| \le |f(t,x(t)) - f(\tau,x(\tau))| < \epsilon$$

uniformly for all $x \in C(J,\mathbb{R})$. If $\tau \in I_0$ and $t \in I$, then $\tau \to 0$ and $t \to 0$ whenever, $|\tau - t| \to 0$. Whence it follows that

$$|Bx(t) - Bx(\tau)| \le |Bx(\tau) - Bx(0)| + |Bx(t) - Bx(0)| \to 0 \text{ as } t \to \tau$$

uniformly for all $x \in C(J,\mathbb{R})$. From this, it follows that $B(E)$ is an equi-continuous set in E. Now an application of Arzella-Ascoli theorem yields that B is a compact operator on E into itself. Similarly, it can be shown that the operators C is also a compact and continuous operator on E into itself.

Finally,

$$\sum_{i=1}^{k} M\psi_{A_i}(r) = \sum_{i=1}^{k} \frac{L_i T \max\{M_f, 1\}\, r}{K_i + r} < r$$

for all $r > 0$ and so, all the conditions of Theorem 2.1 are satisfied. Moreover, here the minimal class $\mathcal{M}_0$, $0 \in [-r, T]$ is $C([0,T],\mathbb{R})$ which is topologically and algebraically closed with respect to difference. Hence, an application of Theorem 2.2 yields that integral equation (3.9) has a solution on J with PPF dependence. This further implies that the HDE (3.3) has a PPF dependent solution defined on J. This completes the proof. □

3.2. **Functional differential equation of neutral type.** Given a function $\phi \in \mathcal{C}$, consider the perturbed or a hybrid functional differential equation of neutral type (in short HDE)

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{x(t) - \sum_{j=1}^{l} h_j(t,x_t)}{\sum_{i=1}^{k} f_i(t,x_t)}\right] &= g(t,x(t)) \\ x_0 &= \phi \end{aligned}\right\} \tag{3.14}$$

for all $t \in I$, where $f_i : I \times \mathcal{C} \to \mathbb{R} \setminus \{0\}$, $h_j : I \times \mathcal{C} \to \mathbb{R}$ and $g : I \times \mathbb{R} \to \mathbb{R}$ are continuous.

By a solution x of the FDE (3.14) we mean a function $x \in C(J,\mathbb{R})$ that satisfies

(i) the function $t \mapsto \dfrac{y - \sum_{j=1}^{l} h_j(t,y)}{\sum_{i=1}^{k} f_i(t,y)}$ is continuous in I for all $y \in \mathcal{C}$, and

(ii) x satisfies the equations in (3.14) on J,

where $C(J,\mathbb{R})$ is the space of continuous real-valued functions defined on $J = I_0 \cup I$.

The HDE (3.14) is again a generalization of the functional differential equation of neutral type

$$\left.\begin{aligned} \frac{d}{dt}\left[x(t) - h(t,x_t)\right] = g(t,x(t)) \\ x_0 = \phi \end{aligned}\right\} \tag{3.15}$$

and contains the following HDE of neutral type

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{x(t)}{\sum_{i=1}^{k} f_i(t,x_t)}\right] = g(t,x(t)) \\ x_0 = \phi \end{aligned}\right\} \tag{3.16}$$

as special cases which is also new to the literature.

We consider the following hypotheses in what follows.

(H_4) There exist real numbers $L_i > 0$ and $K_i > 0$ such that

$$|f_i(t,x) - f_i(t,y)| \le \frac{L_i|x(0) - y(0)|}{K_i + |x(0) - y(0)|}$$

for all $x, y \in \mathcal{C}$.

(H_5) There exists a real number $M_g > 0$ such that

$$|g(t,x)| \le M_g$$

for all $t \in I$ and $x \in \mathbb{R}$.

(H_6) There exist real numbers $q_j > 0$ and $N_j > 0$ such that

$$|h_j(t,x) - h_j(t,y)| \le \frac{q_j|x(0) - y(0)|}{N_j + |x(0) - y(0)|}$$

for all $x, y \in \mathcal{C}$.

Theorem 3.2. *Assume that the hypotheses* (H_4) *through* (H_6) *hold. Furthermore, if*

$$\sum_{i=1}^{k} \frac{L_i\left[\left\|\frac{\phi - \sum_{j=1}^{l} h(0,\phi)}{\sum_{i=1}^{k} f(0,\phi)}\right\|_{\mathcal{C}} + M_g T\right] r}{K_i + r} + \sum_{j=1}^{l} \frac{q_j r}{N_j + r} < r,$$

then the HDE (3.14) *has a solution defined on* J.

Proof. Set $E = C(J,\mathbb{R})$. Clearly, E is a Banach algebra with respect to the norm and the multiplication as defined in the proof of Theorem 3.1. Define a set of functions $\widehat{E}$ by (3.7) which is equipped with the norm $\|\hat{x}\|_{\widehat{E}}$ defined by (3.8) Clearly, $\hat{x} \in C(I_0,\mathbb{R}) = \mathcal{C}$. It can be shown as in Theorem 3.1 that $\widehat{E}$ is Banach space.

Now the HDE (3.13) is equivalent to the nonlinear hybrid integral equation (in short HIE)

$$x(t) = \begin{cases} \displaystyle\sum_{j=1}^{l} h_j(t,x_t) + \sum_{i=1}^{k} \big[f_i(t,x_t)\big]\bigg(\frac{\phi - \sum_{j=1}^{l} h(0,\phi)}{\sum_{i=1}^{k} f(0,\phi)} \\ \qquad\qquad\qquad\qquad + \displaystyle\int_0^t g(s,x(s))\,ds\bigg), & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0. \end{cases} \tag{3.17}$$

Consider three operators $A_i, B : \widehat{E} \to \mathbb{R}$, $B : C(J,\mathbb{R}) \to \mathbb{R}$ and $C_j : C(J,\mathbb{R}) \to \mathbb{R}$ defined by

$$A_i\hat{x} = A_i(x_t)_{t\in I} = \begin{cases} f_i(t,x_t), & \text{if } t \in I, \\ 1, & \text{if } t \in I_0, \end{cases} \tag{3.18}$$

$$Bx(t) = \begin{cases} \dfrac{\phi - \sum_{j=1}^{l} h_j(0,\phi)}{\sum_{i=1}^{k} f_i(0,\phi)} + \displaystyle\int_0^t g(s,x(s))\,ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0, \end{cases} \tag{3.19}$$

and

$$C_j\hat{x} = C_j(x_t)_{t\in I} = \begin{cases} h_j(t,x_t), & \text{if } t \in I, \\ 0, & \text{if } t \in I_0. \end{cases} \tag{3.20}$$

Then the HIE (3.14) is equivalent to the operator equation

$$\sum_{i=1}^{k} A_i\hat{x}\, B\hat{x}(0) + \sum_{j=1}^{l} C_j\hat{x} = \hat{x}(0). \tag{3.21}$$

We shall show that the operators A_i, B_i and C_j satisfy all the condition of Theorem 2.1. First we show that A_i is bounded on $\widehat{E}$.

$$
\begin{aligned}
\|A_i\hat{x}\| &\le \|A_i(0)\|_E + \|A_i(x_t)_{t\in I} - A_i(0)\| \\
&\le |f_i(t,0)| + |f_i(t,x_t) - f_i(t,0)| \\
&\le F_i + \frac{L_i|x_t(0)-0|}{K_i + |x_t(0)-0|} \\
&\le F_i + \frac{L_i\|\hat{x}(0)\|_{\widehat{E}}}{K_i + \|\hat{x}(0)\|_{\widehat{E}}} = F_i + L_i,
\end{aligned}
$$

for all $\hat{x} \in \widehat{E}$, where $F_i = \sup_{t\in I}|f_i(t,0)|$. Hence, A_i is bounded on $\widehat{E}$ with bound $F_i + L_i$.

Next, we show that A_i is a strong $\mathcal{D}$-Lipschitz on $\widehat{E}$. Then,

$$
\begin{aligned}
\|A_i\hat{x} - A_i\hat{y}\|_E &= \|A_i(x_t)_{t\in I} - A_i(y_t)_{t\in I}\| = |f_i(t,x_t) - f_i(t,y_t)| \\
&\le \frac{L_i|x_t(0)-y_t(0)|}{K_i + |x_t(0)-y_t(0)|} \le \frac{L_i\|\hat{x}(0)-\hat{y}(0)\|_{\widehat{E}}}{K_i + \|\hat{x}(0)-\hat{y}(0)\|_{\widehat{E}}} \\
&= \psi_{A_i}(\|\hat{x}(0)-\hat{y}(0)\|_E)
\end{aligned}
$$

for all $\hat{x}, \hat{y} \in \widehat{E}$, where $\psi_{A_i}(r) = \dfrac{L_i r}{K_i + r}$. Hence, A_i is a strong $\mathcal{D}$-Lipschitz on $\widehat{E}$ with $\mathcal{D}$-function ψ_{A_i}. Similarly, it can be shown that C_j is also a strong $\mathcal{D}$-Lipschitz on $\widehat{E}$ with $\mathcal{D}$-function $\psi_{C_j}(r) = \dfrac{q_j r}{N_j + r}$.

Next, we show that B is compact and continuous operator on $C(J,\mathbb{R})$. Let $\{x_n\}$ be a sequence in $C(J,\mathbb{R})$ such that $x_n \to x$ as $n \to \infty$. Then by Lebesgue dominated convergence theorem,

$$
\begin{aligned}
\lim_{n\to\infty} Bx_n(t) &= \frac{\phi - \sum_{j=1}^{l} h(0,\phi)}{\sum_{i=1}^{k} f(0,\phi)} + \lim_{n\to\infty}\int_0^t g(s,x_n(s))\,ds \\
&= \frac{\phi - \sum_{j=1}^{l} h(0,\phi)}{\sum_{i=1}^{k} f(0,\phi)} + \int_0^t \lim_{n\to\infty} g(s,x_n(s))\,ds \\
&= Bx(t)
\end{aligned}
$$

for all $t \in I$. Similarly, if $t \in I_0$, then $\lim\limits_{n\to\infty} Bx_n(t) = \phi(t) = Bx(t)$. This shows that $\{Bx_n(t)\}$ converges to $Bx(t)$ point-wise on J. But $\{Bx_n(t)\}$ is a sequence of uniformly continuous functions on J, So $Bx_n \to Bx$ uniformly. Hence, B is a continuous operator on E into itself.

Secondly, we show that B is compact. To finish, it is enough to show that $B(E)$ is uniformly bounded and equi-continuous set in E. Let $x \in E$ be arbitrary. Then,

$$\begin{aligned}|Bx(t)| &\leq \left\| \frac{\phi - \sum_{j=1}^{l} h(0,\phi)}{\sum_{i=1}^{k} f(0,\phi)} \right\|_{\mathcal{C}} + \int_0^t |g(s,x(s))|\,ds \\ &\leq \left\| \frac{\phi - \sum_{j=1}^{l} h(0,\phi)}{\sum_{i=1}^{k} f(0,\phi)} \right\|_{\mathcal{C}} + M_g T\end{aligned}$$

for all $t \in J$ which shows that $B(E)$ is uniformly bounded set in E. To show equi-continuity, let $t, \tau \in I$. Then,

$$|Bx(t) - Bx(\tau)| \leq \left| \int_\tau^t |g(s,x(s))|\,ds \right| \leq M_g\,|t-\tau|.$$

If $\tau \in I_0$ and $t \in I$, then $\tau \to 0$ and $t \to 0$ whenever, $|\tau - t| \to 0$. Whence it follows that

$$|Bx(t) - Bx(\tau)| \leq |Bx(\tau) - Bx(0)| + |Bx(t) - Bx(0)| \leq M_g\,|t-\tau|.$$

From the above inequalities it follows that $B(E)$ is an equi-continuous set in E. Now an application of Arzelá-Ascoli theorem yields that B is a compact operator on E into itself. Finally,

$$\begin{aligned}&\sum_{i=1}^{k} M_i \psi_{A_i}(r) + \sum_{j=1}^{l} \psi_{C_j}(r) \\ &= \sum_{i=1}^{k} \frac{L_i \left[\left\| \frac{\phi - \sum_{j=1}^{l} h(0,\phi)}{\sum_{i=1}^{k} f(0,\phi)} \right\|_{\mathcal{C}} + M_g T \right] r}{K_i + r} + \sum_{j=1}^{l} \frac{q_j r}{N_j + r} \\ &< r\end{aligned}$$

for all $r > 0$ and so, all the conditions of Theorem 2.1 are satisfied. Again, here the minimal class $\mathcal{M}_0$, $0 \in [-r, T]$ is $C([0,T],\mathbb{R})$ which is topologically and algebraically closed with respect to difference. Hence, an application of Theorem 2.1 yields that the integral equation (3.17) has a solution on J with PPF dependence. This further implies that the HDE (3.14) has a PPF dependent solution defined on J. This completes the proof. □

Remark 3.2. In this paper, we discussed the functional differential equations involving the past and present data only, however similar results may be obtained for the functional differential equations with the given past, present and future data of the unknown function. Furthermore, the PPF dependence theory is useful in the recruitment or selection theory of various organizations like, firm, industry and other institutions. The study along this line is definitely useful to individual as well as society and some of the results in this direction will be reported elsewhere.

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